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LONG-TIME ASYMPTOTICS OF SOLUTIONS OF THE THIRD-ORDER NONLINEAR EVOLUTION EQUATION GOVERNING WAVE PROPAGATION IN RELAXING MEDIA

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Abstract. A classical Cauchy problem for a third-order nonlinear evolution equation is considered. This equation describes the propagation of weakly nonlinear waves in relaxing media. The global existence and uniqueness of its solutions is proved and the solution is constructed in the form of a series in a small parameter present in the initial conditions. Its long-time asymptotics is calculated, which shows the presence of two solitary wave pulses traveling in opposite directions and diffusing in space. Each of them is governed by Burgers' equation with a transfer.

1. Introduction. In recent years, interest has grown in the study of nonlinear wave motions in media with dispersion and absorption. A thorough examination of the corresponding linear equations often serves as a starting point for such investigations. The problems arising here are not only of interest as applications, but are also of theoretical interest since they are, as a rule, unusually distinctive and nonclassical.

An interesting linear third-order hyperbolic equation governing wave propagation in relaxing media is presented in the monograph [13]

$$\tau_0\partial_t(\partial_t^2 u - c_f^2\partial_x^2 u) + \partial_t^2 u - c_e^2\partial_x^2 u = 0. \tag{1.1}$$

Here $u(x,t)$ is a pressure, τ_0 is a relaxation time, c_e is the equilibrium sound speed, and c_f is the so-called "frozen" sound speed. In a relaxing medium the propagation of a wave disturbs the state of a thermodynamical equilibrium. Having been disturbed, such a medium tends towards a state of equilibrium, but with new values of parameters. If the relaxation time is much smaller than the period of oscillations, the propagation of sound occurs with the same speed c_e as in the absence of relaxation. If the inverse relation holds, the relaxation process is "frozen" (not fast enough to follow the oscillations) and the sound propagates with the "frozen" sound speed $c_f > c_e$. In fact, for the majority of relaxing media (mixtures of gases, chemically reacting fluids, water with bubbles, etc.) the ratio c_e^2/c_f^2 is rather close to one [13]. Similar processes occur in "standard"

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viscoelastic materials (see [4], [11]) and in cracked and porous media [10], where wave propagation disturbs the state of the mechanical equilibrium.

In dimensionless variables $\tilde{t} = t/\tau_0$, $\tilde{x} = x/(c_f\tau_0)$ Eq. (1.1) can be rewritten as (tilde will be omitted in the sequel):

$$(u_{tt} - u_{xx})_t + u_{tt} - \alpha u_{xx} = 0, \tag{1.2}$$

where $\alpha = c_c^2/c_f^2$, $0 < \alpha < 1$. It has been studied from various points of view in the papers [7, 11, 12, 16–18]. Fundamental solutions and Cauchy problems for (1.2) have been examined in [11, 16, 17] and initial boundary value problems in [7, 12, 18]. Note that in [18] a singular perturbation problem for Eq. (1.2) with variable-in-space coefficients has been considered and some aspects of its derivation have been discussed.

It would be interesting to examine Eq. (1.2) with a nonlinear term $\beta(u^2)_{xx}$, $\beta = \text{const}$, identical to the one present in the classical Boussinesq equation [2, 8]

$$u_{tt} - \alpha u_{xx} = -\gamma u_{xxxx} + \beta(u^2)_{xx}, \tag{1.3}$$

where it is responsible for the shape of the wave. Thus, we shall consider

$$(u_{tt} - u_{xx})_t + u_{tt} - \alpha u_{xx} = \beta(u^2)_{xx}. \tag{1.4}$$

Strictly speaking, this equation was not derived in connection with nonlinear acoustics or viscoelasticity and therefore can only be considered potentially of applied interest. Note that (1.4) can be rewritten as $(\partial_t + \alpha)(u_{tt} - u_{xx}) + (1 - \alpha)u_{tt} = \beta(u^2)_{xx}$ and, after integrating with respect to t, reduced to the nonlinear integrodifferential equation (with the exponential integral kernel) similar to the ones that describe wave propagation in materials with exponentially fading memory. Equations of such a type (in a more general form, but with different nonlinearities) were studied in [4, 5, 14] from the point of view of establishing global existence and uniqueness of solutions and deducing some energy estimates.

In the present paper, long-time behavior of solutions of a classical Cauchy problem for (1.4) is considered. It is well known that one of the powerful tools for studying initial-value problems for nonlinear evolution equations is the inverse scattering transform [1,3]. However, many important (for physics) nonlinear dissipative equations are not completely integrable and this method does not work for them. In his work [6], T.-P. Liu has used another approach. He has studied a system of viscous conservation laws

$$u_t + f(u)_x = (B(u)u_x)_x.$$

It is not clear what is the best way of putting (1.4) in this form; moreover, reducing a higher-order hyperbolic equation to the corresponding system, we can sometimes lose in our analysis. As far as the author knows, there is no general theory of long-time behavior of systems of conservation laws with dispersive and dissipative effects. Therefore, we shall examine (1.4) directly, without this reduction, developing in the process a certain method for studying higher-order nonlinear dissipative equations. In order to explain our approach we have to give a bit of history.

In the monograph [9], nonlinear dissipative equations of the first order in time have been treated. By virtue of the successive application of the spectral and perturbation theories, the authors succeeded in constructing small solutions of the classical Cauchy problems and in calculating their long-time asymptotic representations. The major term of the asymptotics contained a series in a small parameter present in the initial conditions and the algorithm for calculating the coefficients of this series involved a special diagram technique of the perturbation theory, which was rather unwieldy. In [19–22] this method was developed further and adapted for equations of the second order in time. As well as in [9], nonnegative real parts of the eigenvalues of the linear operator of the equation were used for obtaining preliminary time estimates of the solutions in question, while their imaginary parts played the major role in separating the solitary wave pulses and in estimating their interaction. The major term of the asymptotics was obtained in the explicit form convenient for the qualitative description of the physical phenomena. It must be emphasized that the long-time approximation obtained in [20] for the solution of the damped Boussinesq equation for the one-dimensional case in space was essentially nonlinear and described two solitary wave pulses traveling in opposite directions and diffusing in space. Each of them was governed by Burgers' equation with a transfer. Application of this method to solving periodic problems for the damped Boussinesq equation can be found in [21, 22], where the major terms of the long-time asymptotics were expressed in terms of the Airy functions of a negative argument.

In the present note a small solution of the classical Cauchy problem for Eq. (1.4) is constructed and its long-time asymptotic expansion is obtained. Although the equation studied below is of the third order in time and from the first glance looks quite different from the damped Boussinesq one, the long-time behavior of its solutions is very similar to the case of [20]. The major term of the asymptotics also contains two solitary wave pulses governed by Burgers' equation with a transfer (this effect is described in [6] as well). The reason for this similarity is explained in the discussion section, where the intuitive explanation of the method applied is given. So the reader who is more interested in understanding the basic ideas can skip the rigorous proof of the main results and go straight to the last section.

We must point out that the way of constructing the classical solution of the problem in question and calculating its long-time asymptotics requires more subtle considerations than in [20], and, in our opinion, represents a step forward in the development of the method described above.

2. Statement of the problem and the main theorem. We pose the following Cauchy problem:

$$(u_{tt} - u_{xx})_t + u_{tt} - \alpha u_{xx} = \beta(u^2)_{xx}, \quad x \in \mathbf{R}^1, \ t > 0,$$
$$u(x,0) = \varepsilon\varphi(x), \quad u_t(x,0) = \varepsilon\psi(x), \quad u_{tt}(x,0) = \varepsilon\chi(x), \quad x \in \mathbf{R}^1, \tag{2.1}$$

where $\alpha, \beta, \varepsilon = \text{const} \in \mathbf{R}^1$, $0 < \alpha < 1$, $\varepsilon > 0$; $\varphi(x), \psi(x)$, and $\chi(x)$ are real functions.

In the sequel we shall denote $\int_{\mathbf{R}^1}$ by $\int$, a Fourier transform of the function $u(x,t)$ by

$$\hat{u}(p,t) = \int e^{-ipx} u(x,t)\, dx,$$

an inverse transform by

$$u(x,t) = (2\pi)^{-1} \int e^{ipx} \hat{u}(p,t)\, dp = F^{-1}(\hat{u}(p,t)),$$

and a convolution in p by

$$f(p,t) * g(p,t) = \int f(p-q,t)g(q,t)\,dq.$$

We shall also use the notation

$$\|f(t)\|_r = \|f(t)\|_{L_r(\mathbf{R}^1)} = \left(\int |f(p,t)|^r\,dp\right)^{1/r},$$
$$\|f(t)\| = \operatorname{vrai}\sup_{p\in\mathbf{R}^1} |f(p,t)|$$

and denote by c different positive constants not dependent on x, t, p, and ε. They may depend on the coefficients of the equation and the initial data.

DEFINITION. A function $f(x)$ belongs to the class $\widetilde{C}^n(\mathbf{R}^1)$ if $f(x), f'(x), \dots, f^{(n-1)}(x) \to 0$ as $|x| \to \infty$, and $f^{(n)}(x) \in L_1(\mathbf{R}^1)$.

THEOREM. Let

A) $1/3 \le \alpha < 1$;

B) $\varphi(x) \in \widetilde{C}^5(\mathbf{R}^1)$, $\psi(x) \in \widetilde{C}^4(\mathbf{R}^1)$, $\chi(x) \in \widetilde{C}^3(\mathbf{R}^1)$;

C) $\hat{\psi}(0) + \hat{\chi}(0) = 0$, where $\hat{\psi}(0) = \int \psi(x)\,dx$, $\hat{\chi}(0) = \int \chi(x)\,dx$;

$$\int (1+|x|)|\varphi(x)|\,dx, \quad \int (1+|x|)^2|\psi(x)|\,dx, \quad \int (1+|x|^2)|\chi(x)|\,dx < +\infty.$$

Then there is such ε_0 that for $0 < \varepsilon \le \varepsilon_0$ there exists a unique classical solution of the problem (2.1) that can be represented in the form

$$u(x,t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} u^{(N)}(x,t), \tag{2.2}$$

where the functions $u^{(N)}(x,t)$ will be defined in the proof (see (3.14), (3.20)). This solution has the following asymptotics as $t \to +\infty$:

$$\begin{aligned}
u(x,t) &= u_-(x,t) + u_+(x,t) + O(t^{-(1+\delta)/2}), \quad 0 < \delta < 1/2,\\
u_-(x,t) &= \frac{2}{\beta}\sqrt{\frac{\alpha b}{\pi t}}\,\frac{(1-g_1)\exp(-\frac{(x+\sqrt{\alpha}t)^2}{4bt})}{1+g_1+(g_1-1)\operatorname{erf}(\frac{x+\sqrt{\alpha}t}{2\sqrt{bt}})},\\
u_+(x,t) &= \frac{2}{\beta}\sqrt{\frac{\alpha b}{\pi t}}\,\frac{(1-g_2)\exp(-\frac{(x-\sqrt{\alpha}t)^2}{4bt})}{1+g_2-(g_2-1)\operatorname{erf}(\frac{x-\sqrt{\alpha}t}{2\sqrt{bt}})},\\
g_1 &= \exp\left\{-\frac{\varepsilon\beta}{4b\sqrt{\alpha}}\left[\hat{\varphi}(0)+\hat{\psi}(0)-\frac{\hat{\psi}_1+\hat{\chi}_1}{\sqrt{\alpha}}\right]\right\},\\
g_2 &= \left\{-\frac{\varepsilon\beta}{4b\sqrt{\alpha}}\left[\hat{\varphi}(0)+\hat{\psi}(0)+\frac{\hat{\psi}_1+\hat{\chi}_1}{\sqrt{\alpha}}\right]\right\},\\
b &= \frac{1-\alpha}{2}, \quad \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-\zeta^2)\,d\zeta,\\
\hat{\varphi}(0) &= \int \varphi(x)\,dx, \quad \hat{\psi}_1 = \int x\psi(x)\,dx, \quad \hat{\chi}_1 = \int x\chi(x)\,dx,
\end{aligned} \tag{2.3}$$

and the estimate of the remainder is uniform in $x \in \mathbf{R}^1$.

REMARK 1. Assumption A) guarantees the absence of multiple eigenvalues of the linear operator in (2.1) and thus simplifies considerations. When $\alpha = 1$, the linear operator in (2.1) factorizes and becomes $(\partial/\partial t + 1)(\partial^2/\partial t^2 - \partial^2/\partial x^2)$. This case is much simpler than the one considered above, but must be treated separately. The linear equation has two purely oscillating solutions and one with exponential decay. Dissipation plays here a secondary role, and the method used for constructing solutions is based on using their dissipative properties. However, this case is not interesting for applications.

REMARK 2. Assumptions B) secure the fast decay of the Fourier images of the initial data as $|p| \to +\infty$, namely $\hat{\varphi}(p) = O(|p|^{-5})$, $\hat{\psi}(p) = O(|p|^{-4})$, $\hat{\chi}(p) = O(|p|^{-3})$, and are needed to provide the necessary smoothness of the constructed solution (see (2.2)). Note that we do not calculate the asymptotic form as $\varepsilon \to 0$ of the solution in question. This small parameter is needed for the convergence of the series in (2.2) and its derivatives are included in the equation. The way of estimating ε_0 will be given in the proof.

3. Proof of global existence and construction of small solutions. Applying the Fourier transform in x to the problem (2.1) we get

$$\hat{u}_{ttt} + \hat{u}_{tt} + p^2\hat{u}_t + \alpha p^2\hat{u} = -\beta p^2\hat{u} * \hat{u}, \quad t > 0,$$
$$\hat{u}(p,0) = \varepsilon\hat{\varphi}(p), \quad \hat{u}_t(p,0) = \varepsilon\hat{\psi}(p), \quad \hat{u}_{tt}(p,0) = \varepsilon\hat{\chi}(p). \tag{3.1}$$

Integrating by parts in the representations of $\hat{\varphi}(p), \hat{\psi}(p)$, and $\hat{\chi}(p)$ and using condition B) we find that

$$|\hat{\varphi}(p)| \le c|p|^{-5}, \quad |\hat{\psi}(p)| \le c|p|^{-4}, \quad |\hat{\chi}(p)| \le |p|^{-3}, \quad p \in \mathbf{R}^1. \tag{3.2}$$

Choosing the fundamental system of solutions of the homogeneous equation in (3.1) in the form $\exp(-\lambda t)$ we obtain the following characteristic equation:

$$-\lambda^3 + \lambda^2 = p^2(\lambda - \alpha). \tag{3.3}$$

It is easy to establish that for $1/3 \le \alpha < 1$ this equation has one positive root $\lambda_1(p)$ and two complex conjugate roots $\lambda_{2,3}(p)$. Rewriting (3.3) as $1 - \lambda = p^2(1-\alpha)/(\lambda^2 + p^2)$, we conclude that $0 < \lambda_1(p) \le 1$ and $\lambda_1(0) = 1$. Representing (3.3) as $\lambda - \alpha - (1-\lambda)\lambda^2/p^2$, we find that $\lambda_1(p) \ge \alpha$ and $\lambda_1(\infty) = \alpha$. Thus $\alpha \le \lambda_1(p) \le 1$.

Next we obtain some bounds for the complex roots of (3.3). Setting $\lambda = \lambda_{\rm r} + i\lambda_{\rm im}$ we insert it into (3.3) and get a system of equations

$$\lambda_{\rm r}^3 - \lambda_{\rm im}^2(3\lambda_{\rm r} - 1) - \lambda_{\rm r}^2 + p^2\lambda_{\rm r} - p^2\alpha = 0,$$
$$\lambda_{\rm im}(-3\lambda_{\rm r}^2 + \lambda_{\rm im}^2 + 2\lambda_{\rm r} - p^2) = 0.$$

Hence $\lambda_{\rm im} = 0$ together with the first of these equations gives us the case of real roots discussed above. The second possibility is $\lambda_{\rm im}^2 = 3\lambda_{\rm r}^2 - 2\lambda_{\rm r} + p^2$. After substituting it into the first equation we find that $\lambda_{\rm r}$ must satisfy

$$8\lambda_{\rm r}^3 - 8\lambda_{\rm r}^2 + 2(p^2+1)\lambda_{\rm r} - p^2(1-\alpha) = 0$$

whence

$$\lambda_{\rm r}[(2\lambda_{\rm r} - 1)^2 + p^2] = bp^2, \quad b = (1-\alpha)/2. \tag{3.4}$$

We deduce from (3.4) that $0 \leq \lambda_{\rm r} \leq b$ and using this bound obtain from (3.4) that $\lambda_{\rm r} \geq bp^2/(1+p^2)$. Finally, we have

$$bp^2/(1+p^2) \leq \lambda_{\rm r} \leq b. \tag{3.5}$$

In what follows we shall also need some asymptotic expansions of the roots of (3.3). By means of the method of Newton's diagrams [15] we find for $|p| \leq p_0$ with some $p_0 \leq 1$

$$\begin{aligned} \lambda_1 &= 1 - 2bp^2 + O(p^3), \\ \lambda_{2,3} &= \mp i\sqrt{\alpha}p + bp^2 + O(p^3). \end{aligned} \tag{3.6}$$

By the same method we get for $|p| \to \infty$

$$\begin{aligned} \lambda_1 &= \alpha + 2b\alpha^2 p^{-2} + O(p^{-4}), \\ \lambda_{2,3} &= \mp ip + b + O(p^{-1}). \end{aligned} \tag{3.7}$$

Setting $\sigma(p) = |\lambda_{\rm im}(p)|$, so that $\lambda_{2,3} = \lambda_{\rm r} \mp i\sigma$, we notice that $\sigma(p) = (3\lambda_{\rm r}^2 - 2\lambda_{\rm r} + p^2)^{1/2}$ and consequently $\sigma(p) \leq c|p|$ for all $p \in \mathbf{R}^1$. We shall denote the major terms of the expansions (3.6) of the complex roots by

$$\tilde{\lambda}_{2,3}(p) = \tilde{\lambda}_{\rm r}(p) \mp i\tilde{\sigma}(p), \quad \tilde{\lambda}_{\rm r}(p) = bp^2, \quad \tilde{\sigma}(p) = \sqrt{\alpha}p. \tag{3.8}$$

We return to the problem (3.1). Integrating the equation in t we find that

$$\begin{aligned} \hat{u}(p,t) &= \varepsilon Z(p,t) - \beta p^2 \int_0^t K(p,t,\tau)\hat{u}(p,\tau) * \hat{u}(p,\tau)\, d\tau, \\ Z(p,t) &= A_1(p)\exp(-\lambda_1(p)t) + A_2(p)\exp(-\lambda_2(p)t) + A_3(p)\exp(-\lambda_3(p)t), \\ K(p,t,\tau) &= f_1(p)\exp[-\lambda_1(p)(t-\tau)] + f_2(p)\exp[-\lambda_2(p)(t-\tau)] \\ &\quad + f_3(p)\exp[-\lambda_3(p)(t-\tau)], \\ A_1(p) &= (\lambda_3 - \lambda_2)[\lambda_2\lambda_3\hat{\varphi}(p) + (\lambda_3+\lambda_2)\hat{\psi}(p) + \hat{\chi}(p)]/\Delta(p), \\ A_2(p) &= (\lambda_1 - \lambda_3)[\lambda_1\lambda_3\hat{\varphi}(p) + (\lambda_1+\lambda_3)\hat{\psi}(p) + \hat{\chi}(p)]/\Delta(p), \\ A_3(p) &= (\lambda_2 - \lambda_1)[\lambda_1\lambda_2\hat{\varphi}(p) + (\lambda_1+\lambda_2)\hat{\psi}(p) + \hat{\chi}(p)]/\Delta(p), \\ f_1(p) &= (\lambda_3-\lambda_2)/\Delta(p), \quad f_2(p) = (\lambda_1-\lambda_3)/\Delta(p), \quad f_3(p) = (\lambda_2-\lambda_1)/\Delta(p), \\ \Delta(p) &= \lambda_1^2(\lambda_3-\lambda_2) + \lambda_2^2(\lambda_1-\lambda_3) + \lambda_3^2(\lambda_2-\lambda_1). \end{aligned} \tag{3.9}$$

We introduce the functions

$$m(p) = \begin{cases} |p|, & |p| \leq p_0, \\ 1, & |p| > p_0, \end{cases} \qquad M(p) = \begin{cases} 1, & |p| < p_0, \\ |p|, & |p| \geq p_0, \end{cases}$$

where p_0 is the same as in (3.9). Note that $|p| = m(p)M(p)$.

From (3.2) and (3.5) we find that

$$|A_i(p)| \leq cM^{-5}(p), \quad i = 1,2,3, \ p \in \mathbf{R}^1. \tag{3.10}$$

Setting $2\kappa(p) = bp^2/(1+p^2)$ in (3.8) we note that

$$2\kappa(p) \geq c_0(\alpha)p^2 \quad \text{as } |p| \leq 1 \tag{3.11}$$

with $c_0(\alpha) = (1-\alpha)/4 \to 0$ as $\alpha \to 1$. For the integral kernel in (3.9) we can easily deduce that for t, $\tau > 0$, $p \in \mathbf{R}^1$,

$$|K(p,t,\tau)| \leq cm^{-1}(p)M^{-2}(p)\exp[-2\kappa(p)(t-\tau)] \tag{3.12}$$

since $\lambda_1(p) \geq \alpha \geq (1-\alpha)/2 \geq \lambda_{\rm r}$ for $\alpha \geq 1/3$.

For solving the integral equation (3.9) we shall use perturbation theory. Representing $\hat{u}(p,t)$ as a formal series in ε,

$$\hat{u}(p,t) = \sum_{N=0}^{\infty} \varepsilon^{N+1}\hat{v}^{(N)}(p,t), \tag{3.13}$$

we substitute this expression into (3.9) and compare coefficients of like powers in ε. As a result, we have

$$\begin{aligned}
\hat{v}^{(0)}(p,t) &= Z(p,t);\\
\hat{v}^{(N)}(p,t) &= -\beta p^2 \int_0^t K(p,t-\tau)\sum_{j=1}^{\infty} \hat{v}^{(j-1)}(p,\tau) * \hat{v}^{(j-1)}(p,\tau), \quad N \geq 1,\\
Z(p,t) &= [(\lambda_{\rm r}^2+\sigma^2)\hat{\varphi}(p) + 2\lambda_{\rm r}\hat{\psi}(p) + \hat{\chi}(p)]\exp(-\lambda_1(p)t)/R(p)\\
&\quad + \left\{\lambda_1\left\{\frac{[\lambda_{\rm r}(\lambda_1-\lambda_{\rm r})]+\sigma^2}{R(p)}\frac{\sin(\sigma t)}{\sigma} + \frac{\lambda_1 - 2\lambda_{\rm r}}{R(p)}\cos(\sigma t)\right\}\hat{\varphi}(p)\right.\\
&\quad + \left[\frac{(\lambda_1^2-\lambda_{\rm r}^2+\sigma^2)}{R(p)}\frac{\sin(\sigma t)}{\sigma} - \frac{2\lambda_{\rm r}\cos(\sigma t)}{R(p)}\right]\hat{\psi}(p)\\
&\quad \left. + \left[\frac{(\lambda_1-\lambda_{\rm r})}{R(p)}\frac{\sin(\sigma t)}{\sigma} - \frac{\cos(\sigma t)}{R(p)}\right]\hat{\chi}(p)\right\}\exp(-\lambda_{\rm r}(p)t),
\end{aligned} \tag{3.14}$$

$$R(p) = (\lambda_1-\lambda_{\rm r})^2+\sigma^2, \qquad K(p,t,\tau) = K_1(p,t,\tau) + K_2(p,t,\tau),$$

$$\begin{aligned}
K_1(p,t,\tau) &= \frac{\lambda_1-\lambda_{\rm r}}{R(p)}\frac{\sin(\sigma t)}{\sigma}\exp[-\lambda_{\rm r}(t-\tau)],\\
K_2(p,t,\tau) &= \{\exp[-\lambda_1(t-\tau)] - \cos[\sigma(t-\tau)]\exp[-\lambda_{\rm r}(t-\tau)]\}/R(p).
\end{aligned}$$

We shall establish the following estimates for $N \geq 0$, $p \in \mathbf{R}^1$, $t > 0$:

$$|\hat{v}^{(N)}(p,t)| \leq c^N(N+1)^{-7}M^{-5}(p)\exp[-\kappa(p/(N+1))], \tag{3.15}$$

where $\kappa(p)$ is defined by (3.11). For $N = 0$, (3.15) follows directly from (3.9), (3.10), (3.14), and condition C) of the theorem. Indeed, by Taylor's theorem we have as $p \to 0$

$$\begin{aligned}
\hat{\varphi}(p) &= \hat{\varphi}(0) + O(p),\\
\hat{\psi}(p) &= \hat{\psi}(0) - ip\hat{\psi}_1 + O(p^2),\\
\hat{\chi}(p) &= \hat{\chi}(0) - ip\hat{\chi}_1 + O(p^2),
\end{aligned} \tag{3.16}$$

where the integrals representing the terms in the right-hand sides of these expansions and the constants in the residual terms converge absolutely, according to C). The assumption $\hat{\psi}(0) + \hat{\chi}(0) = 0$ allows us to eliminate the singularity as $p \to 0$ in the term $\lambda_1[\lambda_1\hat{\psi}(p) + \hat{\chi}(p)] \times \sin(\sigma t)/[\sigma R(p)]$. Next we use induction on N. Suppose that (3.15) holds for all $\hat{v}^{(s)}(p,t)$ with $0 \leq s \leq N-1$. We shall show that it is valid for $s = N$. Since $\kappa(p)$ is a monotonically nondecreasing nonnegative function of p, according to [9], we have for $1 \le j \le N$ and $p, q \in \mathbf{R}^1$,

$$\kappa((p-q)/j) + \kappa(q/(N+1-j)) \ge \kappa(p/(N+1)), \tag{3.17}$$
$$M^{-5}(p-q)M^{-5}(q) \le 2^5 M^{-5}(p)[M^{-5}(q) + M^{-5}(p-q)],$$
$$j^{-7}(N+1-j)^{-7} \le 2^7 (N+1)^{-7}[j^{-7} + (N+1-j)^{-7}].$$

Using (3.11) and changing the order of integration in (3.14), we find that

$$|\hat{v}^{(N)}(p,t)| \le cm(p)(N+1)^{-7}M^{-7}(p)\sum_{j=1}^{N} c^{j-1}c^{N-j}[(N+1-j)^{-7} + j^{-7}]$$
$$\times \int [M^{-5}(q) + M^{-5}(p-q)]S_N(p,q,t)\,dq,$$
$$S_N(p,q,t) = \int_0^t \exp[-2\kappa(p)t + L_N(p,q)\tau]\,d\tau,$$
$$L_N(p,q) = 2\kappa(p) - \kappa\left(\frac{p-q}{j}\right) - \kappa\left(\frac{q}{N+1-j}\right), \tag{3.18}$$

since $|p|^2 m^{-1}(p)M^{-2}(p) = m(p)$.

Now we shall find bounds for a typical term I_N in the right-hand side of (3.18). Note that the following estimate holds [20]:

$$S_N(p,q,t) \le \frac{\exp[-\kappa(p/(N+1))t]}{\max\{\kappa(p), |L_N(p,q)|\}}.$$

Thanks to (3.11) we get the following estimate for the denominator of the last fraction as $|q| \le 1$:

$$\max\{\kappa, |L_N|\} \ge \frac{1}{4}\max\left\{\kappa(p), \kappa\left(\frac{q}{N+1-j}\right)\right\}$$
$$\ge \frac{c_0(\alpha)\max\{\kappa(p),\kappa(q)\}}{4(N+1-j)^2} \ge \frac{c_0(\alpha)(p^2+q^2)}{8(N+1-j)^2}$$

and for $|q| \ge 1$,

$$\max\{\kappa, |L_N|\} \ge \frac{1}{8}\left[\kappa(p) + \kappa\left(\frac{q}{N+1-j}\right)\right] \ge \frac{1}{8}\kappa\left(\frac{q}{N+1-j}\right).$$

Therefore,

$$I_N \le c^{N-1}(N+1)^{-7}M^{-5}\exp\left[-\kappa\left(\frac{p}{N+1}\right)t\right]\sum_{j=1}^{N}(N+1-j)^{-7}m(p)G^{(N)}(p),$$
$$G^{(N)}(p) = \int_0^1 [|p|^2 + |q|^2/(N+1-j)^2]^{-1}d|q| + \int_1^\infty |q|^{-4}[\kappa(q/(N+1-j))]\,d|q|. \tag{3.19}$$

Since $\int_0^1 (p^2+q^2)^{-1}\,dq \le c/|p|$ for $p \in \mathbf{R}^1$, (3.18) holds for I_N with $c = c(\alpha) \to \infty$ as $\alpha \to 1$. The estimates in question are proved.

Applying an inverse Fourier transform to (3.13), we find that

$$u(x,t) = (2\pi)^{-1} \int \exp(ipx)\hat{u}(p,t)\, dp = \sum_{N=0}^{\infty} \varepsilon^{N+1} u^{(N)}(x,t),$$

$$u^{(N)}(x,t) = (2\pi)^{-1} \int \exp(ipx)\hat{v}^{(N)}(p,t)\, dp, \tag{3.20}$$

where the functions $\hat{v}^{(N)}(p,t)$ are defined by (3.14). Here we have integrated the series in (3.20) termwise, which is possible due to its absolute and uniform convergence for $\varepsilon < 1/c$, which, in turn, follows from (3.15). By means of (3.15) it can be verified straightforwardly that (3.20) (and correspondingly (2.2)) is a classical solution of the problem (2.1) with sufficiently small ε.

4. Proof of uniqueness. We shall prove the uniqueness of the constructed solution. We assume the contrary, that is, that there exist two solutions $u_1(x,t)$ and $u_2(x,t)$, and set $w = u_1 - u_2$. Thanks to (3.9) and (3.12) its Fourier image $\hat{w}(p,t)$ satisfies the following inequality:

$$|\hat{w}(p,t)| \leq c \int_0^t \exp[-2\kappa(p)(t-\tau)]\, d\tau \int |\hat{u}_1(p,q,\tau) - \hat{u}_2(p-q,\tau)|\, |\hat{u}_1(q,\tau) + u_2(q,\tau)|\, dq$$
$$\leq c \int_0^t \exp[-2\kappa(p)(t-\tau)] |\hat{w}(p,\tau)| * U(p,\tau)\, d\tau,$$

where $U(p,\tau) = |\hat{u}_1(p,\tau)| + |\hat{u}_2(p,\tau)|$. Squaring both sides of the last inequality, integrating in p over $\mathbf{R}^1$, using Cauchy-Schwarz's inequality and changing the order of integration, we have

$$\|\hat{w}(p,t)\|_2^2 \leq c \int dp \left(\int_0^t \exp[-2\kappa(p)(t-\tau)] |\hat{w}(p,\tau)| * U(p,\tau)\, d\tau \right)^2$$
$$\leq c \left\| \int_0^t \exp[-4\kappa(p)(t-\tau)]\, d\tau \right\|_\infty \int_0^t H(\tau)\, d\tau,$$

where $H(\tau) = \int (\int |\hat{w}(p-q),\tau| U(q,\tau)\, dq)^2\, dp$. We can write that

$$H(\tau) = \int dp \int |\hat{w}(p-q_1,\tau)| U(q_1,\tau)\, dq_1 \int |\hat{w}(p-q_2,\tau)| U(q_2,\tau)\, dq_2$$
$$= \int dq_1 U(q_1,\tau) \int dq_2 U(q_2,\tau) \int dp |\hat{w}(p-q_1,\tau)|\, |\hat{w}(p-q_2,\tau)|$$
$$\leq \|\hat{w}(\tau)\|_2^2 \int U(q_1,\tau)\, dq_1 \int U(q_2,\tau)\, dq_2$$
$$\leq \|\hat{w}(\tau)\|_2^2 \left(\int U(q,\tau)\, dq \right)^2.$$

The convergence of the integral $\int U(q,\tau)\, dq$ follows from (3.13), (3.16). Thus, we conclude that for some $T_1 > 0$ and $t \in [0, T_1]$,

$$\|\hat{w}(t)\|_2^2 \leq c \|[1 - \exp(-4\kappa(p)t)]/\kappa(p)\|_\infty \sup_{t \in [0,T_1]} \|\hat{w}(t)\|_2^2 \int_0^t \left(\int U(q,\tau)\, dq \right)^2 d\tau,$$

and therefore,

$$\sup_{t\in[0,T_1]} \|\hat{w}(t)\|_2^2 \le c(T_1) \sup_{t\in[0,T_1]} \|\hat{w}(t)\|_2^2,$$

where the constant $c(T_1) \to 0$ as $T_1 \to 0$. Making $c(T_1)$ less than one by means of the appropriate choice of T_1, we come to the contradiction, which allows us to complete the proof of the uniqueness of solutions for $t \in [0, T_1]$. Continuing this process for the segments $[T_1, T_2], [T_2, T_3], \dots$, with $\{T_k\} \to +\infty$ we obtain the same result for each $t > 0$.

5. Long-time asymptotics. We pass to the calculation of the asymptotics as $t \to +\infty$ of the constructed solution (see (3.13), (3.20)). For this purpose we shall expand $\hat{v}^{(0)}(p,t)$ and single out the major terms of $\hat{v}^{(N)}(p,t)$, $N \ge 1$, as $p \to 0$ in (3.13).

(a) *Estimate of the linear part of the solution.* First we shall calculate the long-time asymptotics of $\varepsilon u^{(0)}(x,t)$, which corresponds to the solution of the linear Cauchy problem. We can write that

$$u^{(0)}(x,t) = \sum_{j=1}^{5} J_j(x,t),$$

$$J_1 = (2\pi)^{-1} \int \exp(ipx) A_1(p) \exp(-\lambda_1(p)t)\, dp,$$

$$J_2 = (2\pi)^{-1} \int_{|p|\le p_0} \exp(ipx) \sum_{k=2}^{3} A_k(0) \exp(-\tilde{\lambda}_k(p)t)\, dp,$$

$$J_3 = (2\pi)^{-1} \int_{|p|\le p_0} \exp(ipx) \sum_{k=2}^{3} A_k(p)[\exp(-\lambda_k(p)t) - \exp(-\tilde{\lambda}_k(p)t)]\, dp,$$

$$J_4 = (2\pi)^{-1} \int_{|p|\le p_0} \exp(ipx) \sum_{k=2}^{3} [A_k(p) - A_k(0)] \exp(-\tilde{\lambda}_k(p)t)\, dp,$$

$$J_5 = (2\pi)^{-1} \int_{|p|\ge p_0} \exp(ipx) \sum_{k=2}^{3} A_k(p) \exp(-\lambda_k(p)t)\, dp,$$

where

$$A_2(0) = \lim_{p\to 0} A_2(p) = \frac{1}{2}\left[\hat{\varphi}(0) + \hat{\psi}(0) - \frac{\hat{\psi}_1 + \hat{\chi}_1}{\sqrt{\alpha}}\right],$$

$$A_3(0) = \lim_{p\to 0} A_3(p) = \frac{1}{2}\left[\hat{\varphi}(0) + \hat{\psi}(0) + \frac{\hat{\psi}_1 + \hat{\chi}_1}{\sqrt{\alpha}}\right].$$

Since $\lambda_1(p) \ge \alpha \ge 1/3$, we find easily that for all $t > 0$, $|J_1| \le c\exp(-\alpha t)$. The integral J_2 makes its contribution to the major term of the asymptotic formula (3.3), while all the other integrals contribute only to the remainder. Noting that $J_2 = O(t^{-1/2})$ as $t \to +\infty$ we leave it temporarily in order to combine it later with the main part of the series $\sum_{N=1}^{\infty} \varepsilon^{N+1} u^{(N)}(x,t)$.

For estimating J_3 we shall use the inequalities

$$\begin{aligned} &|\exp(-\Phi+i\Psi)-\exp(-\widetilde{\Phi}+i\widetilde{\Psi})| \\ &\quad\le 2\{\exp[-(\Phi+\widetilde{\Phi})/2]|\sinh[(\Phi-\widetilde{\Phi})/2]|+\exp(-\widetilde{\Phi})|\sin[(\Psi-\widetilde{\Psi})/2]|\} \\ &\quad\le c\{\exp[-(\Phi+\widetilde{\Phi})/2]|\Phi-\widetilde{\Phi}|+\exp(-\widetilde{\Phi})|\Psi-\widetilde{\Psi}|\} \text{ as } |\Phi-\widetilde{\Phi}|\le c<\infty; \\ &\qquad\qquad \Phi,\widetilde{\Phi},\Psi,\widetilde{\Psi}\in\mathbf{R}^1. \end{aligned}$$

Since $|\lambda_{\mathrm{r}}(p)-\tilde{\lambda}_{\mathrm{r}}(p)|\le c|p|^3$, $|\lambda_{\mathrm{im}}(p)-\tilde{\lambda}_{\mathrm{im}}(p)|\le c|p|^3$ and $(\lambda_{\mathrm{r}}+\tilde{\lambda}_{\mathrm{r}})t/2\ge[c_0(\alpha)+b]p^2t/2$, $\lambda_{\mathrm{r}}\ge bp^2/2$ as $|p|\le p_0$, we deduce that

$$|J_3|\le ct\left[\int\exp(-3bp^2t/4)|p|^3\,dp+\int\exp(-bp^2t/2)|p|^3\,dp\right]\le c(\alpha)t^{-1},\quad t\to+\infty,$$

where $c(\alpha)\to\infty$ as $\alpha\to 1$.

Noting that $|A_k(p)-A_k(0)|\le c|p|$, $k=1,2$, as $|p|\le p_0$, and taking into account (3.11) we obtain

$$|J_4|\le c\int_{|p|\le p_0}\exp(-bp^2t)|p|\,d|p|=O(t^{-1})\quad\text{as } t\to+\infty.$$

Finally, using (3.14) and (3.15) we find that for $t>0$

$$|J_5|\le c\int_{|p|\ge p_0}\exp(-\lambda_{\mathrm{r}}(p)t)|p|^{-5}\,dp\le c\exp(-bp_0^2t/2).$$

Thus, we have singled out the major term in $u^{(0)}(x,t)$ and estimated the remainder as $O(t^{-1})$, $t\to+\infty$ uniformly in $x\in\mathbf{R}^1$. It remains to obtain a long-time estimate of the series $\sum_{N=1}^{\infty}\varepsilon^{N+1}u^{(N)}(x,t)$, which corresponds to the nonlinear contribution in (3.23).

(b) *Estimate of the nonlinear part of the solution.* We set

$$\hat{v}^{(N)}(p,t)=\widehat{V}^{(N)}(p,t)+\widehat{W}^{(N)}(p,t),\quad N\ge 1,$$

where $\widehat{V}^{(N)}(p,t)$ and $\widehat{W}^{(N)}(p,t)$ are solutions of the equations

$$\begin{aligned} \widehat{V}^{(N)}(p,t)&=-\beta p^2\int_0^t K_1(p,t,\tau)\sum_{j=1}^{N}\hat{v}^{(j-1)}(p,\tau)*\hat{v}^{(N-j)}(p,\tau)\,d\tau, \\ \widehat{W}^{(N)}(p,t)&=-\beta p^2\int_0^t K_2(p,t,\tau)\sum_{j=1}^{N}\hat{v}^{(j-1)}(p,\tau)*\hat{v}^{(N-1)}(p,\tau)\,d\tau, \end{aligned}\tag{5.1}$$

and the integral kernels $K_{1,2}$ are defined by (3.14). It is easy to show that the functions $\widehat{V}^{(N)}(p,t)$ have the same bounds (3.18) as $\hat{v}^{(N)}(p,t)$. Using these estimates we can find that

$$\begin{aligned} |\widehat{W}^{(N)}|&\le cm^2(p)\int_0^t d\tau\exp[-2\kappa(p)(t-\tau)]\sum_{j=1}^{N}c^{j-1}c^{N-j}j^{-7}(N+1-j)^{-7} \\ &\times\int M^{-5}(p-q)M^{-5}(q)\exp[-\kappa((p-q)/j)\tau]\exp[-\kappa(q/(N+1-j))\tau]\,dq. \end{aligned}$$

By virtue of the same procedure as for deducing (3.18) we obtain for $t > 0$, $N \geq 1$, $p \in \mathbf{R}^1$,

$$|\widehat{W}^{(N)}(p,t)| \leq c^N (N+1)^{-7} m(p) M^{-5} \exp[-\kappa(p/(N+1))]. \tag{5.2}$$

Consequently, the contribution of the series containing $\widehat{W}^{(N)}(p,t)$ to the asymptotics in question can be estimated as $O(t^{-1/2})$ as $t \to +\infty$.

Next we rewrite the first equation in (4.1) as

$$\widehat{V}^{(N)}(p,t) = -\beta p^2 \int_0^t K_1(p,t,\tau) \sum_{j=1}^{N} [\widehat{V}^{(j-1)} * \widehat{V}^{(N-j)} + 2\widehat{W}^{(j-1)} * \widehat{V}^{(N-j)} + \widehat{W}^{(j-1)} * \widehat{W}^{(N-j)}] \, d\tau$$

and note that only the term containing $\sum_{j=1}^{N} \widehat{V}^{(j-1)} * \widehat{V}^{(N-1)}$ must be left in the integrand for obtaining the major term of the asymptotic formula of the solution of (2.1). Conducting the same arguments as in [20] we can prove that the contribution of the other terms in the integrand to the asymptotics as $t \to +\infty$ is $O(t^{-1+r})$ with some small $\gamma > 0$.

Thus, for finding the major term of the asymptotics we can consider the solutions of the integral equation

$$\widehat{V}^{(N)}(p,t) = -\beta p^2 \int_0^t K_1(p,t,\tau) \sum_{j=1}^{N} \widehat{V}^{(j-1)} * \widehat{V}^{(N-1)}(p,\tau) \, d\tau, \quad N \geq 1.$$

Next we shall expand the kernel K_1 as $p \to 0$. We can represent it as follows:

$$K_1(p,t,\tau) = \widetilde{K}(p,t,\tau) + \sum_{j=3}^{6} K_j(p,t,\tau),$$

where

$$\begin{aligned}
\widetilde{K} &= \exp[-\tilde{\lambda}_{\mathrm{r}}(t-\tau)] \sin[\tilde{\sigma}(t-\tau)]/\tilde{\sigma}, \\
K_3 &= \left(\frac{\lambda_1 - \lambda_{\mathrm{r}}}{\sigma} - \frac{1}{\tilde{\sigma}}\right) \exp[-\lambda_{\mathrm{r}}(t-\tau)] \frac{\sin[\sigma(t-\tau)]}{R(p)}, \\
K_4 &= \frac{\exp[-\lambda_{\mathrm{r}}(t-\tau)]}{R(p)\tilde{\sigma}} \{\sin[\sigma(t-\tau)] - \sin[\tilde{\sigma}(t-\tau)]\}, \\
K_5 &= \{\exp[-\lambda_{\mathrm{r}}(t-\tau)] - \exp[-\tilde{\lambda}_{\mathrm{r}}(t-\tau)]\} \frac{\sin[\sigma(t-\tau)]}{R(p)\tilde{\sigma}}, \\
K_6 &= \left[\frac{1}{R(p)} - 1\right] \exp[-\tilde{\lambda}_{\mathrm{r}}(t-\tau)] \frac{\sin[\tilde{\sigma}(t-\tau)]}{\tilde{\sigma}}.
\end{aligned}$$

By virtue of (3.9) and (3.11), the following estimates can be established:

$$\begin{aligned}
|K_3| &\leq c m(p) M^{-3}(p) \exp[-\lambda_{\mathrm{r}}(t-\tau)], \\
|K_{4,5}| &\leq c M^{-3}(p) \exp[-2\kappa(p)(t-\tau)] \begin{cases} |p|^3 t, & |p| \leq p_0, \\ 1, & |p| > p_0, \end{cases} \\
|K_6| &\leq c m(p) M^{-1}(p) \exp[-2\kappa(p)(t-\tau)].
\end{aligned} \tag{5.3}$$

Taking K_5 as an example we shall show that eliminating $K_{4,5}$ in the above written integral equation will cause an $O(t^{-1})$ as $t \to +\infty$ perturbation of the solution of (2.1). Set $\widehat{V}^{(N)}(p,t) = \hat{y}^{(N)}(p,t) + \widehat{Y}^{(N)}(p,t)$, $N \geq 1$, where $\hat{y}^{(N)}$ and $\widehat{Y}^{(N)}$ are the solutions of the equations

$$\hat{y}^{(N)}(p,t) = -\beta p^2 \int_0^t K_M(p,t,\tau) \sum_{j=1}^{N} \widehat{V}^{(j-1)} * \widehat{V}^{(N-j)}\, d\tau,$$

$$\widehat{Y}^{(N)}(p,t) = -\beta p^2 \int_0^t K_5(p,t,\tau) \sum_{j=1}^{N} \widehat{V}^{(j-1)} * \widehat{V}^{(N-j)}\, d\tau, \quad K_1 = K_M + K_5. \tag{5.4}$$

Next, using the upper line in (4.3) and the (3.15)-type estimates for $\widehat{V}^{(N)}$, we deduce that for $|p| \leq p_0$, $N \geq 1$, $t > 0$,

$$\begin{aligned} |\widehat{Y}^{(N)}(p,t)| &\leq c|p|^5 t \int_0^t d\tau \exp[-2\kappa(p)(t-\tau)] \sum_{j=1}^{N} c^{j-1} c^{N-1} j^{-7} (N+1-j)^{-7} \\ &\quad \times \int dq \exp\left[-\kappa\left(\frac{p-q}{j}\right)\tau - \kappa\left(\frac{q}{N+1-j}\right)\tau\right] M^{-5}(p-q) M^{-5}(q) \\ &\leq c^N |p|^4 t (N+1)^{-7} M^{-5}(p) \exp[-\kappa(p/(N+1))t] \\ &\leq c^N (N+1)^{-5} |p|^2 \exp[-\kappa(p/(N+1))t/2], \end{aligned}$$

where we have used the inequality $\exp[-\kappa(\frac{p}{N+1})t] \leq c[\kappa(\frac{p}{N+1})t]^{-1} \exp[-\kappa(\frac{p}{N+1})t/2]$. The lower line in (4.3) together with (3.15) for $\widehat{V}^{(N)}$ allows us to establish the (3.15)-type estimates for $\widehat{Y}^{(N)}$ as $|p| \geq p_0$. Thus, for all $p \in \mathbf{R}^1$, $N \geq 1$, $t > 0$ we have

$$|\widehat{Y}^{(N)}(p,t)| \leq c^N (N+1)^{-5} m^2(p) M^{-5}(p) \exp\left[-\kappa\left(\frac{p}{N+1}\right) t/2\right]. \tag{5.5}$$

Rewriting the sum in the integrand of the first equation in (4.4) as

$$\sum_{j=1}^{N} [\hat{y}^{(j-1)} * \hat{y}^{(N-j)} + 2\hat{y}^{(j-1)} * \widehat{Y}^{(N-j)} + \widehat{Y}^{(j-1)} * \widehat{Y}^{(N-j)}]$$

and estimating the terms containing the convolutions with $\widehat{Y}^{(N-j)}$ by means of (4.5), we can conclude that the contribution of the corresponding series to the asymptotic formula will be $O(t^{-1})$ as $t \to +\infty$.

If we eliminate the kernels K_3 and K_6 in the integral equation, we shall make the error $O(t^{-1+\gamma})$ with some small $\gamma > 0$ in the asymptotics as $t \to +\infty$ of the solution.

It remains to study the equation

$$\hat{y}^{(N)}(p,t) = -\beta p^2 \int_0^t \widetilde{K}(p,t,\tau) \sum_{j=1}^{N} \hat{y}^{(j-1)}(p,\tau) * \hat{y}^{(N-j)}(p,\tau)\, d\tau, \quad N \geq 1,$$

where

$$\begin{aligned} \widetilde{K}(p,t,\tau) = \{&\exp[-bp^2(t-\tau) + i\sqrt{\alpha}p(t-\tau)] \\ &- \exp[-bp^2(t-\tau) - i\sqrt{\alpha}p(t-\tau)]\}/[1/(2i\sqrt{\alpha}p)]. \end{aligned}$$

The analysis of its solution has been done in [20]. Its major part has been singled out, and it has been shown that the residual term gives a contribution $O(t^{-(1+\delta)/2})$, $0<\delta<1/2$, as $t\to+\infty$ to the solution of (2.1). Now everything is ready to present the main results.

(c) *Combining the long-time estimates.* Recalling the integral J_2 representing the major term of the linear part of the solution and the estimates of its nonlinear part, we can summarize our results. The solution of the problem (2.1) has the following asymptotics as $t\to+\infty$:

$$\begin{aligned}
u(x,t) &= F^{-1}(\hat{y}(p,t)) + O(t^{-(1+\delta)/2}), \quad 0<\delta<1/2,\\
\hat{y}(p,t) &= \sum_{N=0}^{\infty} \varepsilon^{N+1}\hat{y}^{(N)}(p,t);\\
\hat{y}^{(N)}(p,t) &= \hat{y}_1^{(N)}(p,t) + \hat{y}_2^{(N)}(p,t), \quad N\ge 0,\\
\hat{y}_1^{(0)}(p,t) &= A_2(0)\exp(-\tilde{\lambda}_2(p)t),\\
\tilde{y}_2^{(0)}(p,t) &= A_3(0)\exp(-\tilde{\lambda}_3(p)t),\\
\hat{y}_1^{(N)}(p,t) &= \frac{i\beta p}{2\sqrt{\alpha}}\int_0^t \exp[-\tilde{\lambda}_2(p)(t-\tau)]\sum_{j=1}^{N}\hat{y}_1^{(j-1)}(p,\tau)*\hat{y}_1^{(N-j)}(p,\tau)\,d\tau,\\
\hat{y}_2^{(N)}(p,t) &= -\frac{i\beta p}{2\sqrt{\alpha}}\int_0^t \exp[-\tilde{\lambda}_3(p)(t-\tau)]\sum_{j=1}^{N}\hat{y}_2^{(j-1)}(p,\tau)*\hat{y}_2^{(N-j)}(p,\tau)\,d\tau.
\end{aligned}$$

We can represent the Fourier image of the major term of the asymptotics in another way by setting $\hat{y}(p,t)=\hat{y}_1(p,t)+\hat{y}_2(p,t)$, where $\hat{y}_1$ and $\hat{y}_2$ are the solutions of the Cauchy problems

$$\begin{aligned}
\partial_\tau\hat{y}_1 + \tilde{\lambda}_2(p)\hat{y}_1 &= \frac{i\beta p}{2\sqrt{\alpha}}\hat{y}_1*\hat{y}_1, &\qquad \partial_t\hat{y}_2 + \tilde{\lambda}_3(p)\hat{y}_2 &= -\frac{i\beta p}{2\sqrt{\alpha}}\hat{y}_2*\hat{y}_2,\\
\hat{y}_1(p,0) &= \varepsilon A_2(0), & \hat{y}_2(p,0) &= \varepsilon A_3(0).
\end{aligned}$$

Introducing the new variable $t_1=\sqrt{\alpha}t$ and setting $b_1=b/\sqrt{\alpha}$, $\beta_1=\beta/\sqrt{\alpha}$, we obtain the following problems for the pre-images $y_{1.2}(x,t)=F^{-1}(\hat{y}_{1.2}(p,t))$:

$$\begin{aligned}
\partial_{t_1}y_1 - \partial_x y_1 - b\partial_x^2 y_1 &= -(\beta_1/2)\partial_x(y_1^2), &\qquad \partial_{t_1}y_2 + \partial_x y_2 - b_1\partial_x^2 y_2 &= (\beta_1/2)\partial_x(y_2^2),\\
y_1(x,0) &= \varepsilon A_2(0)\delta(x), & y_2(x,0) &= \varepsilon A_3(0)\delta(x),
\end{aligned}$$

where $\delta(x)$ is the usual Dirac function.

These Cauchy problems for Burgers' equation with a transfer can be solved by the well-known procedure. For example, to find y_1 we put $\xi=x+t_1$, keeping the second variable unchanged and applying the Hopf-Cole transformation [23] $y_1(\xi,t_1) = -2(b_1/\beta_1)\partial_\xi \ln f(\xi,t_1)$. Solving the reduced problem we find that

$$y_1(\xi,t_1) = \frac{\int(\xi-\eta)\exp[-(\xi-\eta)^2/(4b_1t_1)]f(\eta,0)\,d\eta}{\beta_1 t_1\int\exp[-(\xi-\eta)^2/(4b_1t_1)]f(\eta,0)\,d\eta},$$

where $f(\eta,0)=\exp[-\frac{\beta_1}{2b_1}\int_{-\infty}^{\eta}y_1(\zeta,0)\,d\zeta]$. Analogously, we can set $\rho=x-t_1$ and deduce the expression for $y_2(\rho,t_1)$. Returning to the original variables and renaming the functions $u_-(x,t) = y_1(\xi, t_1), u_+(x,t) = y_2(\rho, t_1)$, we obtain the major term of the asymptotic formula (2.3). The proof of the theorem is complete.

6. Discussion. Now we shall give an intuitive explanation of the method applied and will clarify the reason for the similarity of the long-time behavior of solutions of (1.4) and the damped version of (1.3) examined in [20]. We shall consider both equations in parallel:

$$u_{ttt} - u_{xxt} + u_{tt} - \alpha u_{xx} = \beta(u^2)_{xx}, \quad 1/3 \le \alpha < 1, \tag{6.1}$$

$$u_{xxxx} - u_{xxt} + u_{tt} - \alpha u_{xx} = \beta(u^2)_{xx}, \quad \alpha > 0, \tag{6.2}$$

with the initial conditions (2.1) for (6.1) and $u(x,0) = \varepsilon\varphi(x)$, $u_t(x,0) = \varepsilon\psi(x)$ for (6.2).

Applying the Fourier transform in x and factoring the differential operators in t, we get

$$(\partial_t + \lambda_1(p))(\partial_t + \lambda_2(p))(\partial_t + \lambda_3(p))\hat{u} = -\beta p^2 \hat{u} * \hat{u}, \tag{6.3}$$

$$(\partial_t + \Lambda_2(p))(\partial_t + \Lambda_3(p))\hat{u} = -\beta p^2 \hat{u} * \hat{u}, \tag{6.4}$$

where $\lambda_i(p), \Lambda_i(p)$ are the roots of the cubic equation (3.3) and the quadratic equation $\Lambda^2 - p^2\Lambda + \alpha p^2 + p^4 = 0$, respectively. It is easy to see that $\Lambda_{2,3}(p) = \frac{p^2}{2} \mp ip\sqrt{\frac{3}{4}p^2 + \alpha}$.

Since the limit as $p \to 0$ is responsible for the long-time behavior of solutions we shall use the following asymptotic formulas:

$$\lambda_1(p) = 1 - (1-\alpha)p^2 + O(p^3),$$

$$\lambda_{2,3}(p) = \mp i\sqrt{\alpha}p + \tfrac{1}{2}(1-\alpha)p^2 + O(p^3), \tag{6.5}$$

$$\Lambda_{2,3}(p) = \mp i\sqrt{\alpha}p + \tfrac{1}{2}p^2 + O(p^3). \tag{6.6}$$

Solving (6.3) we obtain

$$\hat{u}(p,t) = \sum_{j=1}^{3} \{\varepsilon A_j(p)\exp(-\lambda_j(p)t) - \beta p^2 \int_0^t f_j(p)\exp[-\lambda_j(p)(t-\tau)](\hat{u} * \hat{u})(p,\tau)\, d\tau\} \tag{6.7}$$

where $A_j(p), f_j(p)$ are defined by (3.9). An analogous formula (only with the sum of two terms corresponding to $\Lambda_{2,3}(p)$) is valid for the solution of (6.4). The main idea consists in extracting the long-time asymptotics of $\hat{u}(p,t)$ uniformly in p for $|p| \le ct^{-1/2}$. For this purpose we must obtain the asymptotics as $p \to 0$ of the functions $A_j(p), f_j(p)$.

Assuming that the initial data have good enough decay for (3.16) to be valid we have:

$$\Delta(p) = 2i\sqrt{\alpha}p + O(p^3),$$

$$f_1(p) = 1 + O(p^2), \qquad f_2(p) = \frac{-i}{2\sqrt{\alpha}p} + O(1), \qquad f_3(p) = \frac{i}{2\sqrt{\alpha}p} + O(1),$$

$$A_1(p) = \hat{\chi}(0) + O(p),$$

$$A_2(p) = \frac{-i}{2\sqrt{\alpha p}}[\hat{\psi}(0) + \hat{\chi}(0)] + \frac{\hat{\varphi}(0) + \hat{\chi}(0)}{2} - \frac{\hat{\psi}_1 + \hat{\chi}_1}{2\sqrt{\alpha}} + O(p),$$

$$A_3(p) = \frac{i}{2\sqrt{\alpha p}}[\hat{\psi}(0) + \hat{\chi}(0)] + \frac{\hat{\varphi}(0) + \hat{\chi}(0)}{2} + \frac{\hat{\psi}_1 + \hat{\chi}_1}{2\sqrt{\alpha}} + O(p).$$

Define U_j, $j = 1, 2$, to satisfy

$$\widehat{U}_j(p,t) = \varepsilon\left[\frac{\hat{\varphi}(0) + \hat{\chi}(0)}{2} \mp \frac{\hat{\psi}_1 + \hat{\chi}_1}{2\sqrt{\alpha}}\right] \exp\left[\pm i\sqrt{\alpha}pt - \frac{1}{2}(1-\alpha)p^2 t\right]$$
$$- \beta p^2 \int_0^t \frac{\mp i}{2\sqrt{\alpha p}} \exp\left[\pm i\sqrt{\alpha}p(t-\tau) - \frac{1}{2}(1-\alpha)p^2(t-\tau)\right] (\widehat{U} * \widehat{U})(p,\tau)\, d\tau,$$

where we have used the upper sign when $j = 2$ and the lower sign when $j = 3$. We obtained these equations by substituting the leading-order expressions for $f_j(p)$ and $A_j(p)$ into (6.7). It is not hard to see that the U_j satisfy Burgers' equations with a transfer. U_2 is a solitary wave pulse, and U_3 is another one traveling in the opposite direction. The leading-order terms of $A_j(p)$ lead us to consider

$$\hat{h}(p,t) = \frac{-i\varepsilon}{2\sqrt{\alpha p}}[\hat{\psi}(0) + \hat{\chi}(0)] \exp\left[-\frac{1}{2}(1-\alpha)p^2 t\right] \{\exp[i\sqrt{\alpha}pt] - \exp[-i\sqrt{\alpha}pt]\}.$$

We note that $h(x,t)$ is the antiderivative of the difference of two solutions of two heat equations with opposite drift. The support of h spreads linearly with time, and therefore overlaps sufficiently with the supports of both U_2 and U_3. If present, this h term will couple with U_j through the nonlinearity $\hat{u} * \hat{u}$, at least through the cross terms $\hat{h} * \widehat{U}_j$. For this reason the simplifying assumption $\hat{\psi}(0) + \hat{\chi}(0) = 0$ is made, so that we do not have to contend with this problem. We can reduce (2.1) to the following system:

$$\begin{aligned} u_t + u &= v, & u(x,0) &= \varepsilon\varphi(x), \\ v_t &= w_x, & v(x,0) &= \varepsilon[\varphi(x) + \psi(x)], \\ w_t - \alpha v_x - (1-\alpha)w_{xx} + (1-\alpha)u_{xxt} &= \beta(u^2)_x, & w(x,0) &= \varepsilon\int_{-\infty}^{x} [\psi(y) + \chi(y)]\, dy. \end{aligned}$$

This system also makes the assumption made above seem natural.

We have already shown that for all p, $\alpha \le \lambda_1(p) < 1$; hence, we expect the term with $j = 1$ in the sum in (6.7) to be small for large t (the factor of p^2 tames the integral part). Therefore, it follows from (6.7) that

$$\hat{u}(p,t) = \widehat{U}_2(p,t) + \widehat{U}_3(p,t) + o(1), \quad |p| \le ct^{-1/2}.$$

The cross term $\widehat{U}_2 * \widehat{U}_3$ is quite small because of the positioning of the supports of the solitary wave pulses and the fact that they have opposite drifts. On these intuitive grounds one should expect a result like

$$u(x,t) = U_2(x,t) + U_3(x,t) + O(t^{-1/2}) \quad \text{as } t \to +\infty.$$

A very similar situation arises in the study of the long-time behavior of solutions of (6.2). There one is lead to the simplifying assumption $\hat{\psi}(0) = 0$. This assumption is appropriate for the following system form of (6.2) considered in [20]:

$$u_t = v_x, \qquad u(x,0) = \varepsilon\varphi(x),$$
$$v_t - \alpha u_x - v_{xx} + u_{xxx} = \beta(u^2)_x, \quad v(x,0) = \varepsilon \int_{-\infty}^{x} \psi(y)\, dy,$$

where as before $u_t(x,0) = \varepsilon\psi(x)$. Thus, the new analytical challenge of Eq. (6.1) over (6.2) is to deal with the terms arising from $\lambda_1(p), f_1(p)$, and $A_1(p)$ and to develop the method of studying the higher-order semilinear dissipative evolution equations in the process (without reducing them to systems).

Although the estimate of the residual term in the long-time asymptotics (2.3) is uniform in space, due to the presence of the exponential multipliers, the asymptotic formula is effective only in the regions $|x \pm \sqrt{\alpha}t| \leq c\sqrt{t}\ln t$ surrounding the wave fronts $x = \pm\sqrt{\alpha}t$. Outside these zones, (2.3) gives a rough estimate of the long-time behavior of the solution in question, that is, $u = O(t^{-(1+\delta)/2})$ as $t \to +\infty$.

The parameter α accounts for both dispersion and dissipation. As α increases and approaches one, the wave pulses become steeper on the front side (with respect to the direction of the movement) and smoother on the rear one. With the decrease of α the pulses turn smoother and more symmetric. Such a behavior is typical of the weak shock waves governed by Burgers' equation (see [23]). Taking the limit $\alpha \to 1$ in (2.3) is impossible since the constant in the estimate of the remainder tends to infinity in this case. Consequently, (2.3) does not describe the discontinuity jumps. The expressions $\frac{\varepsilon\beta}{4b\sqrt{\alpha}}[\hat{\varphi}(0) + \hat{\psi}(0) \mp \frac{\hat{\psi}_1 + \hat{\chi}_1}{\sqrt{\alpha}}]$ play the role of the Reynolds number in the theory of viscous fluids showing the dependence on dispersion, diffusion, nonlinearity, and the initial data.

Passing to the limit $\beta \to 0$ is quite possible and yields

$$u_-(x,t) \to \frac{\varepsilon[\hat{\varphi}(0) + \hat{\psi}(0) - (\hat{\psi}_1 + \hat{\chi}_1)/\sqrt{\alpha}]}{2\sqrt{\pi b t}} \exp\left[-\frac{(x+\sqrt{\alpha}t)^2}{4bt}\right],$$
$$u_+(x,t) \to \frac{\varepsilon[\hat{\varphi}(0) + \hat{\psi}(0) + (\hat{\psi}_1 + \hat{\chi}_1)/\sqrt{\alpha}]}{2\sqrt{\pi b t}} \exp\left[-\frac{(x-\sqrt{\alpha}t)^2}{2\sqrt{\pi b t}}\right]$$

while the estimate of the remainder improves and becomes $O(t^{-1})$ as $t \to +\infty$. Here the fundamental solutions of the heat equation can be recognized easily.

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