QUARTERLY OF APPLIED MATHEMATICS
VOLUME LVIII, NUMBER 1
MARCH 2000, PAGES 171–176

A NOTE ON THE UNIQUENESS OF THE CLOSED ORBIT OF THE FITZHUGH-NAGUMO SYSTEM

By

MAKOTO HAYASHI

Department of Mathematics, College of Science and Technology, Nihon University, 7-24-1, Narashinodai, Funabashi, 274-8501, Japan

Abstract. A parameter range for which the ordinary differential equations governing the FitzHugh nerve system have a unique nontrivial closed orbit is given. It is wider than those already known.

1. Introduction. We consider the following two-dimensional autonomous system of ordinary differential equations:

$$\begin{cases} \dot{x} = y - \{\frac{1}{3}x^3 + \eta x^2 + (\eta^2 - \eta_0^2)x\}, \\ \dot{y} = -\frac{\rho b}{3}\{x^3 + 3\eta x^2 + 3(\eta^2 + \frac{1}{b} - 1)x\}, \end{cases} \tag{1}$$

where the dot $(\dot{})$ denotes differentiation, b, ρ and η_0 are constants such that $0 < b < 1$, $0 < \rho < 1/b$, and $\eta_0 = \sqrt{1 - \rho b}$, and η is some real parameter. The system (1) is called the FitzHugh nerve system and is obtained by transforming the well-known FitzHugh-Nagumo system (see [Fi] and [Na]). It was studied in such papers as [H1], [K-S], [Su], and [T-V]. We shall prove the following

THEOREM. The system (1) has a unique nontrivial closed orbit if $\eta^2 < \eta_0^2$.

This result improves those given in [H1] and [K-S]. In fact, the result that "If either $\eta^2 \leq 4^{-1}\eta_0^2$ or $\{\rho b^2 - 7b + 6 < 0$ and $\eta^2 < \eta_0^2\}$, then the system (1) has a unique nontrivial closed orbit" was given in [K-S]. In [H1] the result that "There is a positive constant $\eta_1 \leq \eta_0$ such that the system (1) has a unique nontrivial closed orbit for $|\eta| \leq \eta_1$" was given. Therefore, the result of the above theorem is clearly stronger than those in [K-S] and [H1] (it was shown in [H1] that the result in [H1] is not included in that of [K-S]). In Sec. 4 it will be shown that there is a concrete case that can be treated by the above theorem, but not by the results in [K-S] and [H1].

Received April 20, 1998.
2000 *Mathematics Subject Classification.* Primary 34C07, 34C25, 34C26, 34D20.
E-mail address: `mhayashi@penta.ge.cst.nihon-u.ac.jp`

2. Lemmas. In this section we prepare some lemmas to be used in the next section to prove the Theorem. We consider the Liénard system of the following form:

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x), \end{cases} \tag{2}$$

where F is continuously differentiable and g is continuous. We assume the following conditions for the system (2):

[C1] $xg(x) > 0$ if $x \neq 0$;

[C2] there exist $a_1 < 0$ and $a_2 > 0$ such that $F(a_1) = F(a_2) = 0$, $xF(x) \leq 0$ for $a_1 < xa_2$, $F(x)$ is nondecreasing for $x < a_1$ and $x > a_2$;

[C3] $\lim_{x\to\pm\infty} \int_0^x \{F'(\xi) + |g(\xi)|\} d\xi = \pm\infty$.

To prove the Theorem we shall use the following three lemmas.

LEMMA 1. Assume that the system (2) satisfies the conditions [C1], [C2], [C3] and besides

[C4] $G(a_1) > G(a_2)$ and there exists a constant $\alpha \geq 0$ such that $\frac{F(x)}{G^\alpha(x)}$ is nondecreasing for $x \in (a_1, x_1) \cup (a_2, +\infty)$; moreover, there exists a constant $\delta > 0$ such that $\frac{F(x)}{G^\alpha(x)}$ is strictly increasing in x with $0 < |x| < \delta$,

where $G(x) = \int_0^x g(\xi)\, d\xi$ and $x_1 < 0$ is a number satisfying the equation $G(a_2) = G(x_1)$. Then the system (2) has a unique nontrivial closed orbit.

Proof of Lemma 1. Under the conditions [C1], [C2], and [C3] the system (2) has at least one nontrivial closed orbit. See [H1] or [H2]. Moreover, by [Ze-1] or [Ze-2], under the conditions [C1], [C2], and [C4] the system (2) has at most one nontrivial closed orbit. □

Suppose that the condition $\eta^2 < \eta_0^2$ in the Theorem holds with $\eta \geq 0$ (the proof for the case $\eta < 0$ is essentially the same).

LEMMA 2. Let

$$\Gamma(x) = \left\{2\eta_0^2 + 3\left(\frac{1}{b} - 1\right)\right\} x^2 + \eta\left\{2(\eta_0^2 - \eta^2) + 3\left(\eta_0^2 + \frac{1}{b} - 1\right)\right\} x \\ - 3(\eta^2 - \eta_0^2)\left(\eta^2 + \frac{1}{b} - 1\right).$$

Then $\Gamma(\varepsilon(\eta - \eta_0)) > 0$ if $\eta^2 < \eta_0^2$, where

$$\varepsilon = \frac{3(\eta + \eta_0)\left(\eta^2 + \frac{1}{b} - 1\right)}{2\eta_0\left\{2(\eta_0^2 - \eta^2) + 3\left(\eta_0^2 + \frac{1}{b} - 1\right)\right\}}.$$

Proof of Lemma 2. If $\eta^2 < \eta_0^2$, we have

$$\begin{aligned}\Gamma(\varepsilon(\eta-\eta_0)) &= \varepsilon^2(\eta-\eta_0)^2\left\{2\eta_0^2+3\left(\frac{1}{b}-1\right)\right\}\\ &\quad+\varepsilon\eta\left\{2(\eta_0^2-\eta^2)+3\left(\eta_0^2+\frac{1}{b}-1\right)\right\}(\eta-\eta_0)-3(\eta^2-\eta_0^2)\left(\eta^2+\frac{1}{b}-1\right)\\ &= \varepsilon^2(\eta-\eta_0)^2\left\{2\eta_0^2+3\left(\frac{1}{b}-1\right)\right\}\\ &\quad+3\left(\frac{\eta}{2\eta_0}-1\right)(\eta^2-\eta_0^2)\left(\eta^2+\frac{1}{b}-1\right)>0. \quad \square\end{aligned}$$

LEMMA 3. Let

$$g(x)=\frac{\rho b}{3}\left\{x^3+3\eta x^2+3\left(\eta^2+\frac{1}{b}-1\right)x\right\} \quad \text{and} \quad G(x)=\int_0^x g(\xi)\,d\xi.$$

Then $G(a)-G(-a)\geq 0$ for every $a>0$.

Proof of Lemma 3. Since $G(a)-G(-a)=\frac{2}{3}\rho b\eta a^3\geq 0$, the proof is completed. $\square$

3. Proof of Theorem. We shall prove the Theorem by using the above three lemmas. We set $F(x)=(1/3)x^3+\eta x^2+(\eta^2-\eta_0^2)x$. Then, if $\eta^2<\eta_0^2$, we see easily that the system (2) satisfies the conditions [C1], [C2], and [C3]. We shall check the condition [C4] in Lemma 1. We have for $\eta^2<\eta_0^2$,

$$a_1=\frac{-3\eta-\sqrt{12\eta_0^2-3\eta^2}}{2}<0 \quad \text{and} \quad a_2=\frac{-3\eta+\sqrt{12\eta_0^2-3\eta^2}}{2}>0.$$

Then we get

$$G(a_1)-G(a_2)=\frac{\rho b}{4}\left\{\eta^3+2\eta_0^2+6\left(\frac{1}{b}-1\right)\right\}\sqrt{12\eta_0^2-3\eta^2}>0.$$

If $\eta^2<\eta_0^2$, since $0<\varepsilon<1$, we have $a_2>\frac{3}{2}(\eta_0-\eta)>\varepsilon(\eta_0-\eta)$. Let $x_1<0$ be a number satisfying the equation $G(a_2)=G(x_1)$. From the above fact, the monotonicity of G and Lemma 3, it follows that

$$G(x_1)=G(a_2)>G(\varepsilon(\eta_0-\eta))\geq G(\varepsilon(\eta-\eta_0)).$$

Using the fact that $a_1<x_1<\varepsilon(\eta-\eta_0)<0$, we shall show that $F(x)/G^\alpha(x)$ is nondecreasing for $x\in(a_1,\varepsilon(\eta-\eta_0))\cup(a_2,+\infty)$. This means that $F'(x)G(x)-\alpha F(x)g(x)\geq 0$ for $x\in(a_1,\varepsilon(\eta-\eta_0))\cup(a_2,+\infty)$.

From the calculation in [H1] (see [H1], p. 333), we see that the above claim means that

$$\begin{aligned}\Phi(x,\alpha) = {} & (3-4\alpha)x^4 + 6\eta(3-4\alpha)x^3 \\ & + 3\left\{5(3-4\alpha)\eta^2 - (1-4\alpha)\eta_0^2 + 2(3-2\alpha)\left(\frac{1}{b}-1\right)\right\}x^2 \\ & + 12\eta\left\{2(2-3\alpha)\eta^2 - (1-3\alpha)\eta_0^2 + 3(1-\alpha)\left(\frac{1}{b}-1\right)\right\}x \\ & + 18(\eta^2-\eta_0^2)\left(\eta^2+\frac{1}{b}-1\right)(1-2\alpha) \\ \geq {} & 0\end{aligned}$$

for $x \in (a_1, \varepsilon(\eta-\eta_0)) \cup (a_2, +\infty)$.

Let $\alpha = \frac{3}{4}$. Thus we get the following expression, which is of degree 2 in x:

$$\begin{aligned}& \Phi\left(x, \tfrac{3}{4}\right) \\ & \quad = 3\left[\left\{2\eta_0^2 + 3\left(\frac{1}{b}-1\right)\right\}x^2 + \eta\left\{2(\eta_0^2-\eta^2) + 3\left(\eta_0^2+\frac{1}{b}-1\right)\right\}x \right. \\ & \qquad \left. -3(\eta^2-\eta_0^2)\left(\eta^2+\frac{1}{b}-1\right)\right] \\ & \quad = 3\Gamma(x).\end{aligned}$$

If $\eta^2 < \eta_0^2$, from the fact that Γ is a function of degree 2, the inequality $\Gamma(0) > 0$ and Lemma 2, we conclude that $\Phi(x, \frac{3}{4}) \geq 0$ for $x \in (a_1, \varepsilon(\eta-\eta_0)) \cup (a_2, +\infty)$. Therefore, the condition [C4] in Lemma 1 is satisfied. □

4. A numerical example. We shall present the phase portrait of the following system as an example illustrating the application of the Theorem. We consider the system (1) with $b = 1/2, \rho = 1$, and $\eta^2 = 3/8$:

$$\begin{cases} \dot{x} = y - \left(\dfrac{1}{3}x^3 + \dfrac{\sqrt{6}}{4}x^2 - \dfrac{1}{8}x\right), \\ \dot{y} = -\dfrac{1}{6}\left(x^3 + \dfrac{3\sqrt{6}}{4}x^2 + \dfrac{33}{8}x\right). \end{cases} \tag{3}$$

In this case, since $\eta_0^2 = 1 - \rho b = 1/2 > \eta^2$, the system (3) satisfies the condition in the Theorem. Thus we see that the system (3) has a unique nontrivial closed orbit; see Figure 1. We note that this system does not satisfy the condition in [H1] nor that of [K-S], either.

REMARK. In [Su] it was shown that the system (1) has no nontrivial closed orbits if it satisfies the conditions

$$\eta^2 \geq \eta_0^2 \text{ and } \eta^4 - 4\eta^2\eta_0^2 + \eta_0^4 + 2(\tfrac{1}{2}-1)\eta^2 - 4(\tfrac{1}{b}-1)\eta_0^2 + 4(\tfrac{1}{b}-1)^2 \geq 0$$

or

$$2\{\eta_0^2 + (\tfrac{1}{b}-1)\}^3 < \eta^2\{\eta^2 + 3(\tfrac{1}{b}-1)\}^2.$$

After that, Treskov and Volokitin improved the result of [Su] by applying bifurcation theory. It is difficult to give the condition by the explicit formulas (see [T-V]).

We do not know yet what happens in the region in the (η, η_0)-plane in which $\eta^2 \geq \eta_0^2$, but the condition of [Su] or [T-V] is not satisfied. But some numerical experiments tell us that the system may have two (or more) nontrivial closed orbits if (η, η_0) is in the above-mentioned region. For instance, consider the system (1) with $b = \frac{5}{6}, \rho = \frac{93}{250}$, and $\eta^2 = \frac{7}{10}$. It satisfies neither the condition [Su] nor [T-V]. From the phase portrait of the system it seems that it has several nontrivial closed orbits. We hope to show this mathematically in the future.

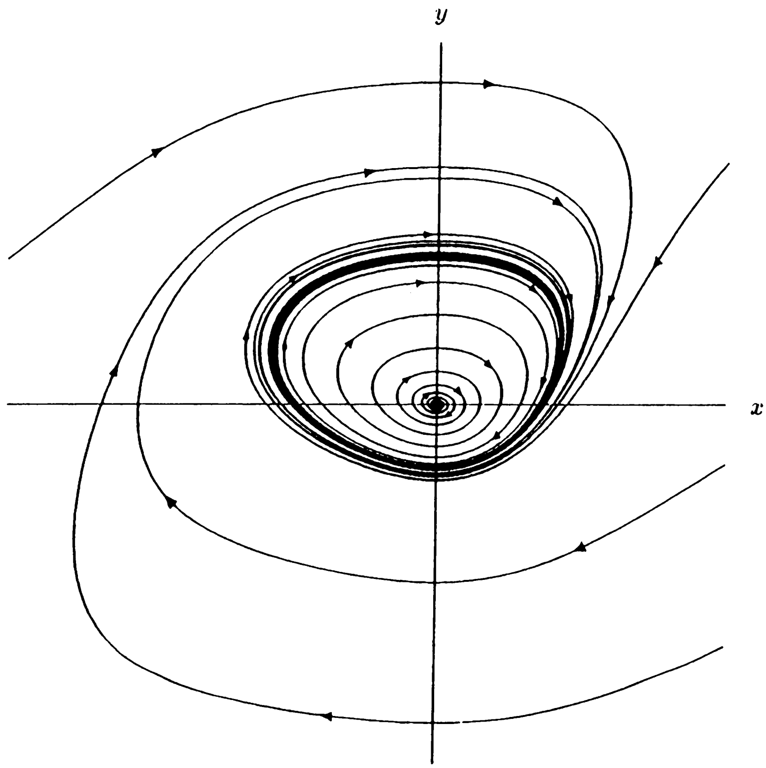

FIG. 1

Acknowledgment. The author would like to express his sincere gratitude to Professor Takesi Yamanaka at Nihon University for having given him continuous encouragement.

REFERENCES

[Fi] R. FitzHugh, *Impulses and physiological states in theoretical models of nerve membrane*, Biophys. J. **1**, 445–466 (1961)

[H1] M. Hayashi, *On the uniqueness of the closed orbit of FitzHugh-Nagumo system*, Math. Japon. **46** (2), 331–336 (1997)

[H2] M. Hayashi, *On the uniqueness of the closed orbit of the Liénard system*, Math. Japon. **46** (3), 371–376 (1997)

[K-S] E. Kaumann and U. Staude, *Uniqueness and nonexistence of limit cycles for the FitzHugh equation*, Lecture Notes in Math. **1017**, Springer-Verlag, New York, 1983, pp. 313–321

[Na] J. Nagumo, S. Arimoto, and S. Yoshizawa, *An active pulse transmission line simulating nerve axon*, Proc. IRE **50**, 2061–2070 (1962)

[Su] J. Sugie, *Nonexistence of periodic solutions for the FitzHugh nerve system*, Quart. Appl. Math. **49**, 543–554 (1991)

[T-V] S. A. Treskov and E. P. Volokitin, *On existence of periodic orbits for the FitzHugh nerve system*, Quart. Appl. Math. **54**, 601–607 (1996)

[Ze-1] X. Zeng, *On the uniqueness of limit cycle of Liénard's equation*, Scientia Sinica (Series A) **25**, 583–592 (1982)

[Ze-2] X. Zeng, Z. Zhang, and S. Gao, *On the uniqueness of the limit cycle of the generalized Liénard equation*, Bull. London Math. Soc. **26**, 213–247 (1994)