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DECAY RATES OF SOLUTIONS TO A VON KÁRMÁN SYSTEM FOR VISCOELASTIC PLATES WITH MEMORY

By

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Dedicated to the memory of Prof. Yukiyoshi Ebihara

Abstract. We consider the dynamical von Kármán equations for viscoelastic plates under the presence of a long-range memory. We find uniform rates of decay (in time) of the energy, provided that suitable assumptions on the relaxation functions are given. Namely, if the relaxation decays exponentially, then the first-order energy also decays exponentially. When the relaxation g satisfies

$$-c_1 g^{1+\frac{1}{p}}(t) \leq g'(t) \leq -c_0 g(t)^{1+\frac{1}{p}}, \quad 0 \leq g''(t) \leq c_2 g^{1+\frac{1}{p}}(t), \quad \text{and} \quad g, g^{1+\frac{1}{p}} \in L^1(\mathbb{R}) \quad \text{with} \quad p > 2,$$

then the energy decays as $\frac{1}{(1+t)^p}$. A new Liapunov functional is built for this problem.

1. Introduction. In this paper we study the von Kármán plate model endowed with long-range memory. Our attention will be focused on the asymptotic behaviour of the solution as time goes to infinity. The von Kármán equation is a large-deflection model in the sense that the vertical deflection of the plate may be large in comparison to the thickness of the plate. More precisely, let us denote by Ω an open bounded set of $\mathbb{R}^2$ with smooth boundary Γ. We assume that the boundary is divided into two parts such that

$$\Gamma = \Gamma_0 \cup \Gamma_1 \quad \text{with} \quad \bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \varnothing, \quad \text{and} \quad \Gamma_0 \neq \varnothing. \tag{1.1}$$

Let us denote by $\nu = (\nu_1, \nu_2)$ the external unit normal to Γ, and by $\eta = (-\nu_2, \nu_1)$ the unitary tangent positively oriented on Γ. Finally, by the brackets $[\cdot, \cdot]$ we denote the

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operator given by

$$[u,v] := \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}.$$

The equations that describe small vibrations of a thin homogeneous, isotropic plate of uniform thickness h are given by:

$$u_{tt} - h\Delta u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)\, d\tau = [u,v] \quad \text{in} \quad \Omega \times]0,\infty[, \tag{1.2}$$

$$\Delta^2 v = -[u,u] \quad \text{in} \quad \Omega \times]0,\infty[, \tag{1.3}$$

$$u(x,y,0) = u_0(x,y), \qquad u_t(x,y,0) = u_1(x,y), \qquad (x,y) \in \Omega, \tag{1.4}$$

with the following boundary conditions:

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times]0,\infty[, \tag{1.5}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0 \times]0,\infty[, \tag{1.6}$$

$$\mathcal{B}_1 u - \mathcal{B}_1\left\{\int_0^t g(t-\tau)u(\tau)\, d\tau\right\} = 0 \quad \text{on} \quad \Gamma_1 \times]0,\infty[, \tag{1.7}$$

$$\mathcal{B}_2 u - h\frac{\partial u_{tt}}{\partial \nu} - \mathcal{B}_2\left\{\int_0^t g(t-\tau)u(\tau)\, d\tau\right\} = 0 \quad \text{on} \quad \Gamma_1 \times]0,\infty[, \tag{1.8}$$

where

$$\mathcal{B}_1 u = \Delta u + (1-\mu)B_1 u,$$

$$\mathcal{B}_2 u = \frac{\partial \Delta u}{\partial \nu} + (1-\mu)\frac{\partial B_2 u}{\partial \eta},$$

and B_1 and B_2 are given by

$$B_1 u = 2\nu_1\nu_2 \frac{\partial^2 u}{\partial x \partial y} - \nu_1^2 \frac{\partial^2 u}{\partial y^2} - \nu_2^2 \frac{\partial^2 u}{\partial x^2},$$

$$B_2 u = (\nu_1^2 - \nu_2^2)\frac{\partial^2 u}{\partial x \partial y} + \nu_1\nu_2\left\{\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}\right\}.$$

In (1.2), $u = u(x,y,t)$ denotes the position of the plate and $v = v(x,y,t)$ is the Airy's stress function. We may interpret Eq. (1.2) by saying that the stresses at any instant depend on the complete history of strains that the material has undergone. By $g \in C^2(\mathbb{R},\mathbb{R})$ we are denoting a positive real function and the constant μ is assumed to be in the interval $]0,\frac{1}{2}[$.

Recently, several research papers have been done on the von Kármán model with dissipative effects; we can mention, for example, [1], [5], [6], [16] and the references therein. They consider different types of additional dissipations. Puel and Tucsnak in [16] introduce a damping mechanism acting on the boundary. Horn and Lasiecka in [6] introduce both a frictional damping and a nonlinear dissipation on the boundary. Chueshov in [5] introduces a frictional damping and Bisognin et al study in [1] the von Kármán equations in the presence of thermal effects. In all of the above works, the exponential rate of decay of the solution is shown. But this is, as far as we know, the first time that viscoelastic terms are taken into account.

The main result of this paper (Sections 3 and 4) is the uniform rate of decay to the solution of system (1.2)–(1.8). We prove that the rate of decay of the solution depends on the rate of decay of the relaxation function. That is, when the relaxation decays exponentially, then the vertical deflection as well as the Airy's stress function also decay exponentially. Moreover, if the relaxation decays polynomially, then the vertical deflection and the Airy's stress do decay polynomially. This shows that the memory effect is strong enough to produce the dissipation, which enables us to show that the energy decays uniformly (exponentially or algebraically).

Let us mention some other papers related to the problems we address. Dafermos in [2] proved that the solution to viscoelastic systems goes to zero as time goes to infinity, but without giving an explicit rate of decay. Lagnese in [10] considers the linear viscoelastic equation obtaining uniform rates of decay but introducing additional damping terms acting on the boundary. This is the estabilization problem, which appears in control theory. Uniform rates of decay for the solutions of linear viscoelastic plates with memory were obtained recently by M. Rivera et al [15]. Unfortunately the method used to achieve uniform rates of decay in that work is based on second-order estimates, which are time dependent in our problem. Thus, the methods that have been used for establishing uniform rates of decay fail in the case of viscoelastic von Kármán systems. Therefore, a new asymptotic technique has to be devised.

The method we use here is based on the construction of a functional $\mathcal{L}$ for which an inequality of the form

$$\frac{d}{dt}\mathcal{L}(t) \leq -c\mathcal{L}(t)^{1+\frac{1}{\beta}}$$

holds, with $c, \beta > 0$. To construct such a functional $\mathcal{L}$ we start from the energy of the viscoelastic equation. Then, we look for other functions whose derivatives introduce negative terms such as: $-\int |u_t|^2$, $-\int |\Delta u|^2$, etc. until we are able to construct the whole energy on the right-hand side with negative sign. Finally, we take $\mathcal{L}$ as the summatory of such functions. Unfortunately, the above process also introduces terms without definite sign. To overcome this problem, we introduce a new multiplier (see Lemma 3.1) that allows us to get the appropriate estimates. Finally, we choose carefully the coefficients of each term of $\mathcal{L}$, such that the resulting summatory satisfies the required inequality.

Let us describe briefly all sections in this paper. In Section 2, we prove the existence and uniqueness of weak solutions. Furthermore, we show some regularity results. To do this we assume that g satisfies

$$g, g', g'' \in L^1(0,\infty), \qquad \alpha := 1 - \int_0^\infty g(\tau)\, d\tau > 0, \tag{1.9}$$

$$g(t) \geq 0, \qquad g'(t) \leq 0. \tag{1.10}$$

Since the methods of Section 2 are quite standard we will only give a brief summary of the procedure. In Section 3 we show the exponential decay of the energy assuming that g satisfies additionally

$$-c_0 g(t) \leq g'(t) \leq -c_1 g(t), \qquad 0 \leq g''(t) \leq c_2 g(t). \tag{1.11}$$

Finally, in Section 4, we show that the polynomial rate of decay of the relaxation implies the polynomial rate of decay of the solution. To do this, instead of assumption (1.11) we consider the weaker hypotheses

$$-c_0 g^{1+\frac{1}{p}}(t) \leq g'(t) \leq -c_1 g^{1+\frac{1}{p}}(t), \qquad 0 \leq g''(t) \leq c_2 g^{1+\frac{1}{p}}(t), \quad p > 2, \tag{1.12}$$

$$\beta := \int_0^\infty g^{1-\frac{1}{p}}(\tau)\,d\tau < \infty, \quad p > 2. \tag{1.13}$$

Assumptions (1.12) and (1.13) mean that $g \approx (1+t)^{-p}$ for $p > 2$. All above constants c_i, $i = 0, 1, 2$, are positive.

The notation we use in this paper is standard and can be found in Lions's book [12].

2. Existence and regularity. To study the existence of a solution of system (1.2) we introduce the following spaces:

$$V := \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_0\},$$
$$W := \left\{w \in H^2(\Omega);\ w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_0\right\}.$$

Let $0 < \mu < \frac{1}{2}$. We define the bilinear form $a(\cdot,\cdot)$ as follows:

$$a(u,v) = \int_\Omega \left\{ \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} + \mu\left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}\right) + 2(1-\mu)\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} \right\} dA.$$

Frequently, we shall use the Green's formula for Δ^2:

LEMMA 2.1. Let u and v be functions in $H^4(\Omega) \cap W$. Then, we have

$$\int_\Omega (\Delta^2 u) v\,dA = a(u,v) + \int_{\Gamma_1} \left\{ (\mathcal{B}_2 u)v - (\mathcal{B}_1 u)\frac{\partial v}{\partial \nu} \right\} d\Gamma_1. \tag{2.1}$$

Proof. From Green's formula we get

$$\begin{aligned}\int_\Omega (\Delta^2 u) v\,dA &= \int_{\Gamma_1} \left(\frac{\partial \Delta u}{\partial \nu}\right) v\,d\Gamma_1 - \int_{\Gamma_1} \Delta u \frac{\partial v}{\partial \nu}\,d\Gamma_1 + \int_\Omega \Delta u \Delta v\,dA \\ &= \int_{\Gamma_1} \left(\frac{\partial \Delta u}{\partial \nu}\right) v\,d\Gamma_1 - \int_{\Gamma_1} \Delta u \frac{\partial v}{\partial \nu}\,d\Gamma_1 + a(u,v) \\ &\quad + (1-\mu)\int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2}\right) dA - 2(1-\mu)\int_\Omega \frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}\,dA.\end{aligned}$$

Recalling the definition of B_1 and B_2 and using

$$\int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2}\right) dA - 2\int_\Omega \frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}\,dA = \int_{\Gamma_1} \left\{ \left(\frac{\partial B_2 u}{\partial \eta}\right) v - (B_1 u)\frac{\partial v}{\partial \nu} \right\} d\Gamma_1$$

our result follows. □

In this section we show existence and regularity of solutions of the von Kármán equation. The proof will be organized as follows. First, using the Galerkin method, we prove the existence of weak solutions (see Definition 2.1 below). Then using elliptic regularity and second-order estimates we show the regularity of the solution. To simplify our analysis, we introduce the following binary operators:

$$g \,\square\, \partial^2 u = \int_0^t g(t-\tau)a(u(t)-u(\tau),u(t)-u(\tau))\,d\tau,$$

$$g \,\square\, \nabla u = \int_0^t g(t-\tau)\int_\Omega |\nabla u(x,t)-\nabla u(x,\tau)|^2\,dA\,d\tau,$$

$$g \,\square\, u = \int_0^t g(t-\tau)\int_\Omega |u(x,t)-u(x,\tau)|^2\,dA\,d\tau.$$

With this notation we have

LEMMA 2.2. For any $v \in C^1(0,T;H^2(\Omega))$ we get

$$a\left(\int_0^t g(t-\tau)v(\tau)\,d\tau, v_t\right) = -\frac{1}{2}g(t)a(v,v) + \frac{1}{2}g' \,\square\, \partial^2 v - \frac{1}{2}\frac{d}{dt}\left\{g \,\square\, \partial^2 v - \left(\int_0^t g\,d\tau\right)a(v,v)\right\},$$

$$\int_\Omega \int_0^t g(t-\tau)\nabla v(\tau)\,d\tau \cdot \nabla v_t\,dA = -\frac{1}{2}g(t)\int_\Omega |\nabla v|^2\,dA + \frac{1}{2}g' \,\square\, \nabla v - \frac{1}{2}\frac{d}{dt}\left\{g \,\square\, \nabla v - \left(\int_0^t g\,d\tau\right)\int_\Omega |\nabla v|^2\,dA\right\},$$

$$\int_\Omega \int_0^t g(t-\tau)v(x,\tau)\,d\tau v_t(x,t)\,dA = -\frac{1}{2}g(t)\int_\Omega |v|^2\,dA + \frac{1}{2}g' \,\square\, v - \frac{1}{2}\frac{d}{dt}\left\{g \,\square\, v - \left(\int_0^t g\,d\tau\right)\int_\Omega |v|^2\,dA\right\}.$$

Proof. From the symmetry of $a(\cdot,\cdot)$ we get

$$\begin{aligned}\frac{d}{dt}\{g \,\square\, \partial^2 v\} &= g' \,\square\, \partial^2 v - 2\int_0^t g(t-\tau)a(v(\tau),v_t(t))\,d\tau + 2\int_0^t g(t-\tau)\,d\tau\, a(v,v_t)\\ &= g' \,\square\, \partial^2 v - 2\int_0^t g(t-\tau)a(v(\tau),v_t(t))\,d\tau\\ &\quad + \frac{d}{dt}\left\{\int_0^t g(\tau)\,d\tau\, a(v,v)\right\} - g(t)a(v,v),\end{aligned}$$

which shows the first identity. The proofs of the other two identities are similar. □

Another important Lemma is the following.

LEMMA 2.3. Let u, v, and w be functions in $H^2(\Omega)$ such that $v \in H_0^2(\Omega)$, where Ω is an open bounded and connected subset of $\mathbb{R}^2$ with smooth boundary. Then we have that

$$\int_\Omega w[v,u]\,dA = \int_\Omega v[w,u]\,dA.$$

Proof. Suppose that u and w are $C^4(\Omega)$ and that v belongs to $C_0^4(\Omega)$. Our result will follow using density arguments. It is not difficult to see that

$$\begin{aligned} w[v,u]-v[w,u] &= \left(wv\frac{\partial^2 u}{\partial y^2}\right)_{xx} - 2\left(wv\frac{\partial^2 u}{\partial x\partial y}\right)_{xy} + \left(wv\frac{\partial^2 u}{\partial x^2}\right)_{yy} \\ &\quad + 2\left(\frac{\partial w}{\partial y}v\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial w}{\partial x}v\frac{\partial^2 u}{\partial y^2}\right)_x + 2\left(\frac{\partial w}{\partial x}v\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial w}{\partial y}v\frac{\partial^2 u}{\partial x^2}\right)_y. \end{aligned}$$

A simple integration over Ω gives the required formula. □

The definition of weak solution that we use in this work is given as follows.

DEFINITION 2.1. We say that the couple (u,v) is a weak solution of Eqs. (1.2)–(1.8) if

$$u \in C^1([0,T];V)\cap C^0([0,T];W), \qquad v\in C^0([0,T];H_0^2(\Omega))$$

and satisfies the following identities:

$$\begin{aligned} &\int_0^T\int_\Omega -u_t\theta_t - h\nabla u\cdot\nabla\theta_t\,dA + \int_0^T a(u,\theta)\,dt - \int_0^T a(g*u,\theta)\,dt \\ &\quad = \int_0^T\int_\Omega [u,\theta]v\,dA\,dt + \int_\Omega u_1\theta(\cdot,0)\,dA + h\int_\Omega \nabla u_1\cdot\nabla\theta(\cdot,0)\,dA, \end{aligned} \tag{2.2}$$

$$\int_\Omega \Delta v\Delta\psi\,dA = -\int_\Omega [u,\psi]u\,dA, \tag{2.3}$$

for any function $\theta\in C^1([0,T];W)$ such that $\theta(\cdot,T)=0$, $\theta_t(\cdot,T)=0$ and $\psi\in C_0^2(\Omega)$.

Note that since $\Gamma_0\neq\varnothing$, Korn's Lemma implies that $\sqrt{a(\cdot,\cdot)}$ is a norm equivalent to the usual Sobolev norm $\|\cdot\|_2$ on W. Let us introduce the energy function

$$E(t,v,\psi) := \frac{1}{2}\left\{\int_\Omega(|v_t|^2 + h|\nabla v_t|^2)\,dA + \left(1-\int_0^t g\,d\tau\right)a(v,v) + g\,\Box\,\partial^2 v + \int_\Omega |\Delta\phi|^2\,dA\right\},$$

$$F(t,v) := \frac{1}{2}\left\{\int_\Omega(|v_t|^2 + h|\nabla v_t|^2)\,dA + \left(1-\int_0^t g\,d\tau\right)a(v,v) + g\,\Box\,\partial^2 v\right\},$$

and let us denote by $\{w_i\in W; i\in\mathbb{N}\}$ an orthonormal basis of W. In these conditions we are able to prove the existence of a weak solution for viscoelastic plates.

THEOREM 2.1. *Let us suppose that g satisfies (1.9) and (1.10). Then for any initial data $(u_0,u_1)\in W\times V$, $h>0$ and $T>0$, there exists only one weak solution for Eq. (2.2).*

Proof. Our starting point is to construct the Galerkin approximation u^m of the solution

$$u^m(\cdot,t) = \sum_{i=1}^m h_{i,m}(t)w_i(\cdot), \qquad v^m(\cdot,t) = (\Delta^2)^{-1}(-[u^m,u^m]),$$

which is given by the solution of the approximated equation

$$\int_\Omega u_{tt}^m w_j\,dA + h\int_\Omega \nabla u_{tt}^m \nabla w_j\,dA + a(u^m, w_j) - a(g * u^m, w_j) = \int_\Omega [u^m, w_j]v^m\,dA,$$
$$u^m(\cdot,0) = u_{0,m}, \qquad u_t^m(\cdot,0) = u_{1,m} \tag{2.4}$$

where

$$u_{0,m} = \sum_{i=1}^m \left\{\int_\Omega u_0 w_i\,dA\right\} w_i, \qquad u_{1,m} = \sum_{i=1}^m \left\{\int_\Omega u_1 w_i\,dA\right\} w_i.$$

Let us denote by $A = (a_{ij})$ the matrix given by

$$a_{ij} = \left(\int_\Omega w_i w_j\,dA + h\int_\Omega \nabla w_i\cdot\nabla w_j\,dA\right).$$

We can easily verify that A is a positive definite matrix; so the existence of the approximated solution u^m is guaranteed. Let us multiply Eq. (2.4) by $h'_{j,m}$, sum up the product result in j, and use Lemma 2.2 to conclude that:

$$\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega (|u_t^m|^2 + h|\nabla u_t^m|^2)\,dA + \left(1 - \int_0^t g\,d\tau\right) a(u^m,u^m) + g\,\Box\,\partial^2 u^m\right\}$$
$$= \frac{1}{2}g'\,\Box\,\partial^2 u^m - \frac{1}{2}g(t)a(u^m,u^m) + \int_\Omega [u^m, u_t^m]v^m\,dA.$$

Since

$$\int_\Omega [u^m, u_t^m]v^m\,dA = \frac{1}{2}\int_\Omega \left\{\frac{d}{dt}[u^m,u^m]\right\} v^m\,dA = -\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta v^m|^2\,dA,$$

we have that

$$\frac{d}{dt}E(t,u^m,v^m) = \frac{1}{2}g'\,\Box\,\partial^2 u^m - \frac{1}{2}g(t)a(u^m,u^m).$$

In view of the hypotheses on g we have

$$E(t,u^m,v^m) \le E(0,u^m,v^m);$$

and from our choice of $u_{0,m}$ and $u_{1,m}$ it follows that

$$u^m \text{ is bounded in } C([0,T];W)\cap C^1([0,T];V),$$
$$v^m \text{ is bounded in } C([0,T];H_0^2(\Omega)).$$

Multiplying Eq. (2.4) by $\beta \in C^2([0,T])$ such that $\beta(T) = 0$ and integrating over $[0,T]$ we have

$$\int_0^T\int_\Omega (u^m w_j \beta_{tt} + h\nabla u^m\cdot\nabla w_j\beta_{tt})\,dA\,dt + \int_0^T a(u^m,w_j)\beta\,dt - \int_0^T a(g*u^m,w_j)\beta\,dt$$
$$= \int_0^T\int_\Omega [u^m,w_j]v^m\beta\,dA - \int_\Omega u_{0,m}w_j\beta_t(0)\,dA + \int_\Omega u_{1,m}w_j\beta(0)\,dA$$
$$- h\int_\Omega \nabla u_{0,m}\nabla w_j\beta_t(0)\,dA + h\int_\Omega \nabla u_{1,m}\nabla w_j\beta(0)\,dA.$$

Since $[u^m, w_j]$ converges weakly and v^m strongly, letting $m \to \infty$ and using the density of the terms $\{w_j\beta;\ j \in \mathbb{N},\ \beta \in C([0,T])\}$, we get that u is a weak solution of (2.2). The uniqueness follows by standard methods for hyperbolic equations.

To prove the regularity of the solution we introduce the following definition:

DEFINITION 2.2. We say that (u_0, u_1, f) is k-regular if

$$u_j \in H^{2+k-j}(\Omega) \cap W, \quad j = 0, \dots, k, \quad u_{k+1} \in V,$$

where u_j is obtained by the following recursive formula:

$$u_{j+2} - h\Delta u_{j+2} = \Delta^2 u_j + g_j - f_k(0),$$
$$h\frac{\partial u_{j+2}}{\partial \nu} = \mathcal{B}_2 u_j \quad \text{on} \quad \Gamma_1$$

and also:

$$u_j = \frac{\partial u_j}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0, \qquad \mathcal{B}_1 u_l = 0 \quad \text{on} \quad \Gamma_1, \quad \forall j = 1, \dots, k$$

where

$$f_0 := f,$$
$$f_{j+1} = \dot{f}_j(t) - g(t)\Delta^2 u_j, \qquad f_j \in L^1(0,T; L^2(\Omega), \quad j = 1, \dots, k,$$
$$g_j = \partial_t^j \{[u(x,t), v(x,t)]\}_{t=0}.$$

To show the regularity result we will use the following Lemma:

LEMMA 2.4. Suppose that $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma_1)$, and $h \in H^{3/2}(\Gamma_1)$. Then, any solution of

$$a(v, w) = \int_\Omega fw\, dA + \int_{\Gamma_1} gw\, d\Gamma_1 + \int_{\Gamma_1} h\frac{\partial w}{\partial \nu}\, d\Gamma_1 \quad \forall w \in W$$

satisfies:

$$v \in H^4(\Omega)$$

and also:

$$\Delta^2 v = f,$$
$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0,$$
$$\mathcal{B}_1 v = h, \quad \mathcal{B}_2 v = g \quad \text{on} \quad \Gamma_1.$$

Moreover, if $(f, g, h) \in H^k(\Omega) \times H^{k+1/2}(\Gamma_1) \times H^{k+3/2}(\Gamma_1)$, then $v \in H^{k+4}(\Omega)$.

Proof. See [13].

Before proving the regularity result we establish a Lemma that will be useful in what follows.

LEMMA 2.5. Let us suppose that $u, v \in H^2(\Omega)$ and $w \in H_0^2(\Omega)$. Then, we have that the following inequality

$$\left| \int_\Omega [u, w] v\, dA \right| \le c(\|u\|_1 \|w\|_1)^{\frac{1}{2}} (\|u\|_2 \|w\|_2)^{\frac{1}{2}} \|v\|_2$$

holds. Here $\|\cdot\|_j$ denotes the norm of the Sobolev space $H^j(\Omega)$.

Proof. $[u,v]$ is a continuous operator such that

$$\|[u,v]\|_{-2} \leq C\|u\|_{2-\varepsilon}\|v\|_{1+\varepsilon}.$$

Here $\|\cdot\|_{-j}$ denotes the norm of the dual space $H^{-j}(\Omega)$. Using interpolation we have, for $\varepsilon = \frac{1}{2}$,

$$\begin{aligned}\left|\int_\Omega [u,w]v\,dA\right| &\leq c\|[u,v]\|_{-2}\|v\|_2\\ &\leq c\|u\|_{2-\varepsilon}\|w\|_{1+\varepsilon}\|v\|_2\\ &\leq c\|u\|_{\frac{3}{2}}\|w\|_{\frac{3}{2}}\|v\|_2\\ &\leq c(\|u\|_1\|w\|_1)^{\frac{1}{2}}(\|u\|_2\|w\|_2)^{\frac{1}{2}}\|v\|_2,\end{aligned}$$

from which our conclusion follows. □

The regularity of the solution is established in the next Theorem.

THEOREM 2.2. Let us suppose that the initial data (u_0, u_1, f) is 2-regular and that hypothesis (1.1) holds. Then the weak solution of Eq. (1.2) satisfies:

$$u \in C^1([0,T]; V\cap H^3(\Omega)) \cap C^0([0,T]; W\cap H^4(\Omega))$$

and also:

$$u \in C^j([0,T]; V\cap H^{4-j}(\Omega)); \quad j = 1,\ldots,4.$$

Proof. Differentiating Eq. (2.4) with respect to the time we get

$$\begin{aligned}\int_\Omega u^m_{ttt}w_j\,dA + h\int_\Omega \nabla u^m_{ttt}\nabla w_j\,dA + a(u^m_t, w_j) - a(g*u^m_t, w_j) = \int_\Omega [u^m_t, w_j]v^m\,dA\\ + \int_\Omega [u^m, w_j]v^m_t\,dA + g(t)a(u_{0,m}, w_j). \quad (2.5)\end{aligned}$$

Multiplying Eq. (2.5) by $h''_{j,m}$, summing up the product result in j, and using Lemma 2.2 we conclude that:

$$\begin{aligned}\frac{d}{dt}F(t,u^m_t) = \frac{1}{2}g'\,\Box\,\partial^2u^m - \frac{1}{2}g(t)a(u^m_t, u^m_t) + g(t)a(u_{0,m}, u^m_{tt})\\ + \int_\Omega [u^m_t, u^m_{tt}]v^m\,dA + \int_\Omega [u^m, u^m_{tt}]v^m_t\,dA + g(t)a(u_{0,m}, u^m_{tt}).\end{aligned} \quad (2.6)$$

Notice that

$$\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega [u^m_t, u^m_t]v^m\,dA\right\} = \frac{1}{2}\int_\Omega [u^m_t, u^m_t]v^m_t\,dA + \int_\Omega [u^m_t, u^m_{tt}]v^m\,dA.$$

Inserting the above expression into identity (2.6) we get

$$\begin{aligned}\frac{d}{dt}\left\{F(t,u_t) - \frac{1}{2}\int_\Omega [u^m_t, u^m_t]v^m\,dA\right\} = \frac{1}{2}g'\,\Box\,\partial^2u^m\\ - \frac{1}{2}g(t)a(u^m_t, u^m_t) + g(t)a(u_{0,m}, u^m_{tt})\underbrace{-\frac{1}{2}\int_\Omega [u^m_t, u^m_t]v^m_t\,dA + \int_\Omega [u^m, u^m_{tt}]v^m_t\,dA}_{:=R}. \quad (2.7)\end{aligned}$$

Note that R satisfies

$$\begin{aligned} R &= -\frac{3}{2}\int_\Omega [u_t^m, u_t^m] v_t^m \, dA + \int_\Omega \frac{d}{dt}\{[u^m, u_t^m]\} v_t^m \, dA \\ &= -\frac{3}{2}\int_\Omega [u_t^m, u_t^m] v_t^m \, dA + \int_\Omega [u^m, u_{tt}^m] v_t^m \, dA \\ &= -\frac{3}{2}\int_\Omega [u_t^m, u_t^m] v_t^m \, dA + \frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta v_t^m|^2 \, dA. \end{aligned}$$

Substitution of the above identity into (2.7) yields

$$\begin{aligned} \frac{d}{dt}\left\{ E(t, u_t^m, v_t^m) - \frac{1}{2}\int_\Omega [u_t^m, u_t^m] v^m \, dA \right\} &= \frac{1}{2} g' \,\Box\, \partial^2 u^m - \frac{1}{2} g(t) a(u^m, u^m) \\ &\quad - \frac{3}{2}\int_\Omega [u_t^m, u_t^m] v_t^m \, dA + g(t) a(u_{0,m}, u_{tt}^m). \end{aligned} \tag{2.8}$$

Integrating Eq. (2.8) we get

$$\begin{aligned} E(t, u_t^m, v_t^m) \le E(0, u_t^m, v_t^m) &+ \frac{1}{2}\int_\Omega [u_t^m, u_t^m] v^m \, dA \\ &- \frac{3}{2}\int_0^t \int_\Omega [u_t^m, u_t^m] v_t^m \, dA \, dt + \int_0^t g(\tau) a(u_{0,m}, u_{tt}^m) \, d\tau. \end{aligned}$$

Note that after an integration by parts we get that

$$\int_0^t g(\tau) a(u_{0,m}, u_{tt}^m) \, d\tau = g(\tau) a(u_{0,m}, u_t^m)\Big|_{\tau=0}^{\tau=t} - \int_0^t g'(\tau) a(u_{0,m}, u_t^m) \, d\tau,$$

from which it follows that

$$\int_0^t g(\tau) a(u_{0,m}, u_{tt}^m) \, d\tau \le C_\varepsilon E(0, u_t^m, v_t^m) + \varepsilon \|u_t^m\|_2^2 + c\int_0^t \|u_t^m\|_2^2 \, d\tau.$$

Using Lemma 2.5, the first-order estimate, and the inequality

$$\begin{aligned} \int_\Omega [u_t^m, u_t^m] v_t^m \, dA &\le \|u_t^m\|_1 \|u_t^m\|_2 \|v_t^m\|_2 \\ &\le C\|u_t^m\|_2 \|v_t^m\|_2 \end{aligned}$$

we get

$$E(t, u_t, v_t) \le CE(0, u_t, v_t) + C\int_0^t E(\tau, u_t, v_t) \, d\tau.$$

Using Gronwall's inequality we find that

$$u_{tt}^m \text{ is bounded in } L^\infty(0, T; H^1(\Omega)), \tag{2.9}$$

$$u_t^m \text{ is bounded in } L^\infty(0, T; H^2(\Omega)), \tag{2.10}$$

$$v_t^m \text{ is bounded in } L^\infty(0, T; H_0^2(\Omega)). \tag{2.11}$$

Differentiating Eq. (2.5) with respect to time we get

$$\int_\Omega u^m_{tttt} w_j \, dA + h\int_\Omega \nabla u^m_{tttt} w_j \, dA + a(u^m_{tt}, w_j) - a(g * u^m_{tt}, w_j) = \int_\Omega [u^m_{tt}, w_j] v^m \, dA$$
$$+ 2\int_\Omega [u^m_t, w_j] v^m_t \, dA + \int_\Omega [u^m, w_j] v^m_{tt} \, dA + \{g(t) + g'(t)\} a(u_{0,m}, w_j). \tag{2.12}$$

Multiplying Eq. (2.12) by $h'''_{j,m}$, summing up the product result in j, and using Lemma 2.2 we conclude that:

$$\frac{d}{dt} F(t, u_{tt}) = \frac{1}{2} g' \square \partial^2 u^m_{tt} - \frac{1}{2} g(t) a(u^m_{tt}, u^m_{tt}) + g(t) a(u_{0,m}, u^m_{tt})$$
$$+ \underbrace{\int_\Omega [u^m_{tt}, u^m_{ttt}] v^m \, dA + 2\int_\Omega [u^m_t, u^m_{ttt}] v^m_t \, dA + \int_\Omega [u^m, u^m_{ttt}] v^m_{tt} \, dA.}_{:= R_2} \tag{2.13}$$

Note that R_2 can be written as follows:

$$R_2 = \frac{1}{2}\frac{d}{dt}\int_\Omega [u^m_{tt}, u^m_{tt}] v^m \, dA - \frac{1}{2}\int_\Omega [u^m_{tt}, u^m_{tt}] v^m_t \, dA$$
$$+ 2\int_\Omega [u^m_t, u^m_{ttt}] v^m_t \, dA + \int_\Omega [u^m, u^m_{ttt}] v^m_{tt} \, dA.$$

On the other hand, we have that

$$\int_\Omega [u^m_t, u^m_{ttt}] v^m_t \, dA = \frac{d}{dt}\left\{\int_\Omega [u^m_t, u^m_{tt}] v^m_t \, dA\right\} - \int_\Omega [u^m_{tt}, u^m_{tt}] v^m_t \, dA - \int_\Omega [u^m_t, u^m_{tt}] v^m_{tt} \, dA.$$

Differentiating Eq. (2.3) three times and multiplying by v^m_{tt} we get

$$\frac{d}{dt}\int_\Omega |\Delta v^m_{tt}|^2 \, dA = -6\int_\Omega [u^m_t, u^m_{tt}] v^m_{tt} \, dA - 2\int_\Omega [u^m, u^m_{ttt}] v^m_{tt} \, dA,$$

from which it follows that

$$R_2 = \frac{d}{dt}\left\{\int_\Omega [u^m_{tt}, u^m_{tt}] v^m \, dA - 2\int_\Omega [u^m_t, u^m_{tt}] v^m_t \, dA - \frac{1}{2}\int_\Omega |\Delta v^m_{tt}|^2 \, dA\right\}$$
$$- \frac{7}{2}\int_\Omega [u^m_{tt}, u^m_{tt}] v^m_t \, dA - 5\int_\Omega [u^m_t, u^m_{tt}] v^m_{tt} \, dA.$$

Inserting the above expression into identity (2.13) we get

$$\frac{d}{dt}\left\{E(t, u^m_{tt}, v^m_{tt}) - \int_\Omega [u^m_{tt}, u^m_{tt}] v^m \, dA + 2\int_\Omega [u^m_t, u^m_{tt}] v^m_t \, dA\right\} = \frac{1}{2} g' \square \partial^2 u^m_{tt}$$
$$-\frac{1}{2} g(t) a(u^m_{tt}, u^m_{tt}) + g(t) a(u_{0,m}, u^m_{tt}) - \frac{7}{2}\int_\Omega [u^m_{tt}, u^m_{tt}] v^m_t \, dA - 5\int_\Omega [u^m_t, u^m_{tt}] v^m_{tt} \, dA. \tag{2.14}$$

Integrating inequality (2.14) with respect to the time we get that

$$E(t, u^m_{tt}, v^m_{tt}) \le E(0, u^m_{tt}, v^m_{tt}) \int_\Omega [u^m_{tt}, u^m_{tt}] v^m \, dA - 2\int_\Omega [u^m_t, u^m_{tt}] v^m_t \, dA$$
$$+ \int_0^t g(\tau) a(u_{0,m}, u^m_{tt}) \, d\tau - \frac{7}{2}\int_0^t \int_\Omega [u^m_{tt}, u^m_{tt}] v^m_t \, dA \, d\tau - 5\int_0^t \int_\Omega [u^m_t, u^m_{tt}] v^m_{tt} \, dA \, d\tau. \tag{2.15}$$

Using Lemma 2.5 and the estimates (2.9)–(2.10) we conclude that there exists a positive constant C such that

$$\int_\Omega [u_{tt}^m, u_{tt}^m]v_t^m\,dA \le C\|u_{tt}^m\|_1\|u_{tt}^m\|_2\|v_t^m\|_2 \le C\|u_{tt}^m\|_2\|v_t^m\|_2.$$

Similarly we have

$$\begin{aligned}\int_\Omega [u_t^m, u_{tt}^m]v_{tt}^m\,dA &\le C\|u_t^m\|_1^{\frac12}\|u_{tt}^m\|_1^{\frac12}\|u_t^m\|_2^{\frac12}\|u_{tt}^m\|_2^{\frac12}\|v_{tt}^m\|_2\\ &\le C\|u_{tt}^m\|_2^{\frac12}\|u_t^m\|_2^{\frac12}\|v_{tt}^m\|_2\\ &\le C\{\|u_{tt}^m\|_2^2+\|u_t^m\|_2^2+\|v_{tt}^m\|_2^2\},\end{aligned}$$

$$\int_\Omega [u_{tt}^m, u_{tt}^m]v^m\,dA \le CE(0,u_t,v_t)\|u_{tt}^m\|_2,$$

$$\int_\Omega [u_t^m, u_{tt}^m]v_t^m\,dA \le CE(0,u_t,v_t)\|u_{tt}^m\|_2.$$

Substituting the above inequalities into (2.15) and applying Gronwall's inequality we get

$$u_{ttt}^m \text{ is bounded in } L^\infty(0,T;H^1(\Omega)), \tag{2.16}$$

$$u_{tt}^m \text{ is bounded in } L^\infty(0,T;H^2(\Omega)), \tag{2.17}$$

$$v_{tt}^m \text{ is bounded in } L^\infty(0,T;H_0^2(\Omega)). \tag{2.18}$$

This implies that u and v have the following regularity:

$$u_{ttt} \in L^\infty(0,T;H^1(\Omega)), \tag{2.19}$$

$$u_{tt} \in L^\infty(0,T;H^2(\Omega)), \tag{2.20}$$

$$v_{tt} \in L^\infty(0,T;H_0^2(\Omega)). \tag{2.21}$$

Integrating by parts Eq. (2.2) with respect to time and denoting by $U = u - g*u$ we get that

$$a(U,w) = -\int_\Omega (\{u_{tt} - h\Delta u_{tt}\}w - [u,v]w)\,dA - h\int_{\Gamma_1}\frac{\partial u_{tt}}{\partial\nu}w\,d\Gamma_1,$$

for any $w \in W$. From Lemma 2.4 we get that

$$U \in L^\infty(0,T;H^4(\Omega)).$$

Using Volterra's resolvent equation, we get that $u \in L^\infty(0,T;H^4(\Omega))$. Moreover, we have that u, v verify system (1.2)–(1.8) in the strong sense. The proof is now complete. □

3. Asymptotic behaviour: Exponential decay. In this section we study the asymptotic behaviour of the solution of (1.2)–(1.8). Note that as deduced in Section 2, the energy function satisfies

$$\frac{d}{dt}E(t,v,\psi) = \frac12 g'\square\partial^2 v - \frac12 g(t)a(v,v). \tag{3.1}$$

To prove the exponential decay we will denote by $w = u - g * u$. It follows that w satisfies:

$$w_{tt} - h\Delta w_{tt} + \Delta^2 w + g'(0)\{u - h\Delta u\} + g(0)\{u_t - h\Delta u_t\} + g'' * \{u - h\Delta u\} = [u, v] \quad \text{in } \Omega \times]0,\infty[, \tag{3.2}$$

$$\Delta^2 v = -[u, u] \quad \text{in } \Omega \times]0,\infty[, \tag{3.3}$$

$$w(x,y,0) = u_0(x,y), \qquad w_t(x,y,0) = u_1(x,y) - g(0)u_0(x,y) \tag{3.4}$$

with the following boundary conditions:

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times]0,\infty[, \tag{3.5}$$

$$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Gamma_0 \times]0,\infty[, \tag{3.6}$$

$$\mathcal{B}_1 w = 0 \quad \text{on } \Gamma_1 \times]0,\infty[, \tag{3.7}$$

$$\mathcal{B}_2 w - h\frac{\partial w_{tt}}{\partial \nu} + g(0)h\frac{\partial u_t}{\partial \nu} + g'(0)h\frac{\partial u}{\partial \nu} + h\frac{\partial g'' * u}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \times]0,\infty[. \tag{3.8}$$

Next we introduce the new energy function associated to system (3.2)–(3.8),

$$\mathcal{E}(t) = \frac{1}{2}\left\{\int_\Omega (|w_t|^2 + h|\nabla w_t|^2 + |\Delta v|^2)\, dA\right\} + \sum_{i=1}^{6} S_i(t)$$

where $S_i(t)$ is given by

$$S_1(t) = -\frac{g'(0)}{2}\int_\Omega \left\{|u|^2 + \left(\int_0^t g\, d\tau\right)|u|^2 - 2(g * u)u\right\} dA - \frac{1}{2} g\,\Box\, u,$$

$$S_2(t) = -\frac{g'(0)}{2} h\left\{\int_\Omega \left[|\nabla u|^2 + \left(\int_0^t g\, d\tau\right)|\nabla u|^2 - 2(g * \nabla u)\nabla u\right] dA - hg\,\Box\,\nabla u\right\},$$

$$S_3(t) = \frac{g(0)}{2}\left\{\int_\Omega \left[g(0)|u|^2 + \left(\int_0^t g'\, d\tau\right)|u|^2\right] dA - g'\,\Box\, u\right\},$$

$$S_4(t) = \frac{g(0)}{2} h\left\{\int_\Omega g(0)\left[|\nabla u|^2 + \left(\int_0^t g'\, d\tau\right)|\nabla u|^2\right] dA - g'\,\Box\,\nabla u\right\},$$

$$S_5(t) = -\frac{1}{2}\left\{g''\,\Box\, u + \int_\Omega \left[|g' * u|^2 - \left(\int_0^t g''\, d\tau\right)|u|^2\right] dA\right\},$$

$$S_6(t) = -\frac{1}{2} h\left\{g''\,\Box\,\nabla u + \int_\Omega \left[|g' * \nabla u|^2 - \left(\int_0^t g''\, d\tau\right)|\nabla u|^2\right] dA\right\}.$$

The point of departure of this study is to establish an energy identity given in the next Lemma.

LEMMA 3.1. Under the above conditions, the solution of system (3.2)–(3.8) satisfies the following inequality:

$$\frac{d}{dt}\mathcal{E}(t) \le C_\varepsilon\left\{\int_\Omega g(t)|\nabla u|^2\, dA + g\,\Box\,\nabla u\right\} + c\{\|v\|_\infty g(t)a(u,u) + g\,\Box\,\partial^2 u\} + \varepsilon\int_\Omega |\nabla u|^2\, dA - g(0)\int_\Omega |u_t|^2 + h|\nabla h_t|^2\, dA.$$

Proof. Let us multiply Eq. (3.2) by w_t. Applying Green's formula and using the boundary conditions we get

$$\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega |w_t|^2\,dA + h\int_\Omega |\nabla w_t|^2\,dA + a(w,w)\right\} = \underbrace{\int_\Omega [u,v]w_t\,dA}_{=I_0(t)}$$

$$\underbrace{-\,g'(0)\int_\Omega (uw_t + h\nabla u\nabla w_t)\,dA}_{:=I_1+I_2} \underbrace{-\,g(0)\int_\Omega (u_tw_t + h\nabla u_t\nabla w_t)\,dA}_{:=I_3+I_4}$$

$$\underbrace{-\int_\Omega \{g''*u + hg''*\nabla u\}w_t\,dA}_{:=I_4+I_5}$$

$$\underbrace{-\,h\int_{\Gamma_1}\left\{\frac{\partial w_{tt}}{\partial\nu} - g'(0)\frac{\partial w}{\partial\nu}w_t - g(0)\frac{\partial w_t}{\partial\nu}\right\}w_t\,d\Gamma - \int_{\Gamma_1}\left[(\mathcal{B}_2w)w_t - (\mathcal{B}_1w)\frac{\partial w_t}{\partial\nu}\right]d\Gamma}_{=0}.$$

Now we will consider each term I_i. Using Lemma 2.2 we get

$$\begin{aligned} I_1(t) &= -\frac{g'(0)}{2}\frac{d}{dt}\int_\Omega |u|^2\,dA - g'(0)\frac{d}{dt}\int_\Omega (g*u)u\,dA + g'(0)\int_\Omega (g*u)u_t\,dA \\ &= -g'(0)\frac{d}{dt}\underbrace{\left\{\frac{1}{2}\int_\Omega\left[|u|^2 + g*uu + \frac{1}{2}\left(\int_0^t g\,d\tau\right)|u|^2\right]dA - \frac{1}{2}g\,\Box\, u\right\}}_{=S_1(t)} \\ &\quad + \frac{g'(0)}{2}g'\,\Box\, u - \frac{g'(0)g(t)}{2}\int_\Omega |u|^2\,dA. \end{aligned}$$

Similarly we get that

$$I_2(t) = \frac{d}{dt}S_2(t) + h\frac{g'(0)}{2}g'\,\Box\,\nabla u - \frac{g'(0)g(t)}{2}h\int_\Omega |\nabla u|^2\,dA.$$

Now we will consider I_3:

$$\begin{aligned} I_3(t) &= -g(0)\int_\Omega |u_t|^2 + g(0)\int_\Omega (g(0)uu_t + (g'*u)u_t)\,dA \\ &= -g(0)\int_\Omega |u_t|^2\,dA - \frac{g(0)}{2}\frac{d}{dt}\underbrace{\left\{g'\,\Box\, u - \int_0^t g'\,d\tau\int_\Omega |u|^2\,dA - g(0)\int_\Omega |u|^2\,dA\right\}}_{=S_2(t)} \\ &\quad + \frac{g(0)}{2}g''\,\Box\, u - \frac{g(0)g'(t)}{2}\int_\Omega |u|^2\,dA. \end{aligned}$$

By symmetry we get

$$I_4(t) = \frac{d}{dt}S_4(t) - g(0)h\int_\Omega |\nabla u_t|^2 + \frac{g(0)}{2}hg''\,\Box\,\nabla u - \frac{g(0)g'(t)}{2}h\int_\Omega |\nabla u|^2\,dA.$$

Finally we consider I_5:

$$\begin{aligned}
I_5(t) &= -\int_\Omega [g'' * uu_t - \left(\frac{d}{dt}g' * u\right)(g' * u) + g(0)u(g' * u) - g(0)u(g'' * u)]\,dA \\
&= \frac{1}{2}\frac{d}{dt}\left\{g'' \Box u - \int_\Omega \left[|g' * u|^2 - \int_0^t g''\,d\tau |u|^2\right] dA\right\} \\
&\quad - \int_\Omega u \int_0^t \{g(0)g''(t-\tau) - g'(0)g'(t-\tau)\}\{u(\tau) - u(t)\}\,d\tau\,dA \\
&\quad - \{g(0)g'(t) - g'(0)g(t)\}\int_\Omega |u|^2\,dA.
\end{aligned}$$

Analogously

$$\begin{aligned}
I_6(t) &= \frac{d}{dt}S_6(t) - h\{g(0)g''(t) - g'(0)g'(t)\}\int_\Omega |\nabla u|^2\,dA \\
&\quad - h\int_\Omega \nabla u \cdot \left(\int_0^t \{g(0)g'(t-\tau) - g'(0)g(t-\tau)\}\{\nabla u(\tau) - \nabla u(t)\}\,d\tau\right) dA.
\end{aligned}$$

From the above inequalities, Poincaré's inequality, and the hypotheses on g we obtain

$$I_i(t) \le \frac{d}{dt}S_i(t) + C_\varepsilon\left\{\int_\Omega g(t)|\nabla u|^2\,dA + g\Box\nabla u\right\} + \varepsilon\int_\Omega |\nabla u|^2\,dA,$$

for $i \neq 3, 4$, and

$$\begin{aligned}
I_3(t) &= \frac{d}{dt}S_3(t) - g(0)\int_\Omega |u_t|^2 + c\left\{g\Box u + g(t)\int_\Omega |u|^2\,dA\right\}, \\
I_4(t) &= \frac{d}{dt}S_4(t) - g(0)h\int_\Omega |\nabla u_t|^2 + c\left\{g\Box\nabla u + g(t)\int_\Omega |\nabla u|^2\,dA\right\}.
\end{aligned}$$

Finally, we will estimate the term I_0:

$$\begin{aligned}
I_0(t) &= \int_\Omega [u,v]u_t - [u,v]g' * u - g(0)[u,v]u\,dA \\
&= -\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta v|^2\,dA - \int_\Omega [u,v]\underbrace{\int_0^t g'(t-\tau)\{u(\cdot,\tau) - u(\cdot,t)\}\,d\tau}_{:=F}\,dA \\
&\qquad\qquad - g(t)\int_\Omega [u,v]u\,dA \\
&= -\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta v|^2\,dA - \int_\Omega [u,F]v\,dA - g(t)\int_\Omega [u,u]v\,dA \\
&\le -\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta v|^2\,dA + c_\varepsilon\|v\|_\infty g\Box\partial^2 u + \varepsilon a(u,u) + c\|v\|_\infty g(t)a(u,u).
\end{aligned}$$

Since the energy $E(t)$ is bounded, then we have that $\|v\|_\infty^2 \le \int_\Omega |\Delta v|^2\,dA$ is also bounded. So, from the last inequality our conclusion follows. □

REMARK 3.1. It is not difficult to see that there exists a positive constant C such that

$$\mathcal{E}(t) \le CE(t).$$

To prove it we will show that there exists a positive constant c such that

$$\int_\Omega (|w_t|^2 + h|\nabla w_t|^2)\,dA + a(w,w) \le cE(t).$$

We only prove the inequality

$$\int_\Omega |w_t|^2\,dA \le cE(t),$$

the others being similar. Note that

$$w_t = u_t - g(t)u - \int_0^t g'(t-\tau)\{u(\cdot,\tau) - u(\cdot,t)\}\,d\tau,$$

from which it follows that

$$\int_\Omega |w_t|^2\,dA \le c\left\{\int_\Omega |u_t|^2\,dA + g(t)\int_\Omega |u|^2\,dA + g\,\Box\, u\right\}.$$

Using Körn's inequality our conclusion follows. □

Let us introduce the functional

$$J(t,w) := \int_\Omega (w_t w + h\nabla w_t \nabla w)\,dA.$$

Now we are in a condition to prove the exponential decay of the solutions.

THEOREM 3.1. Suppose that the initial data (u_0, u_1) is 2-regular and the kernal g satisfies conditions (1.11) and (1.9). Then there exist positive constants κ_0 and κ_1 such that

$$E(t,u,v) \le \kappa_0 E(0,u,v)e^{-\kappa_1 t}.$$

Proof. Multiplying Eq. (3.2) by w, integrating over Ω, and using the boundary conditions we get

$$\frac{d}{dt}J(t) = \int_\Omega (|u_t|^2 + h|\nabla u_t|^2)\,dA - a(u,u) + a\left(\int_0^t g(t-\tau)u(\tau)\,d\tau, u\right) \tag{3.9}$$

$$- \int_\Omega |\Delta v|^2\,dA \tag{3.10}$$

$$\le \int_\Omega (|u_t|^2 + h|\nabla u_t|^2)\,dA - (1-\varepsilon)a(w,w)$$

$$+ \frac{1}{4\varepsilon}\left(\int_0^t g\,d\tau\right) g\,\Box\,\partial^2 u - \int_\Omega |\Delta v|^2\,dA. \tag{3.11}$$

Using Lemma 3.1 we find that

$$\underbrace{\frac{d}{dt}\left\{NE(t) + \mathcal{E}(t) + \frac{g(0)}{2}J(t)\right\}}_{=\mathcal{L}(t)}$$

$$\le -\kappa_0 \underbrace{\left\{\int_\Omega (|u_t|^2 + h|\nabla u_t|^2 + |\Delta v|^2)\,dA + a(u,u) + g\,\Box\,\partial^2 u\right\}}_{:=\mathcal{N}(t)}.$$

It is not difficult to see that

$$\frac{N}{2}\mathcal{N}(t) \leq \mathcal{L}(t) \leq N\mathcal{N}(t),$$

for N large enough. From the last two inequalities our conclusion follows. □

4. Polynomial decay. In this section we will show that when the kernel g decays polynomially then the first-order energy also decays at the same rate. To do so, we consider hypotheses (1.9), (1.12), (1.13). To prove the main result of this section we will use the following Lemma.

LEMMA 4.1. Suppose that g and h are continuous functions satisfying $g \in L^{1+\frac{1}{q}}(0,\infty) \cap L^1(0,\infty)$ for some $q \geq 1$ and that $g^r \in L^1(0,\infty)$ for some $0 < r < 1$. Then we have that

$$\int_0^t |g(t-\tau)h(\tau)|\,d\tau \leq \left\{\int_0^t |g(t-\tau)|^{1+\frac{1-r}{q}}|h(\tau)|\,d\tau\right\}^{\frac{q}{q+1}} \left\{\int_0^t |g(t-\tau)|^r|h(\tau)|\,d\tau\right\}^{\frac{1}{q+1}}.$$

Proof. For any fixed t we have

$$\int_0^t |g(t-\tau)h(\tau)|\,d\tau = \int_0^t \underbrace{|g(t-\tau)|^{\frac{r}{q+1}}|h(\tau)|^{\frac{1}{q+1}}}_{:=u}\,\underbrace{|g(t-\tau)|^{1-\frac{r}{q+1}}|h(\tau)|^{\frac{q}{q+1}}}_{:=\nu}\cdot d\tau.$$

Note that $u \in L^p(0,\infty)$, $v \in L^{p'}(0,\infty)$, where $p = q+1$ and $p' = \frac{q+1}{q}$. Using Hölder's inequality, we get

$$\int_0^t |g(t-\tau)h(\tau)|\,d\tau \leq \left\{\int_0^t |g(t-\tau)|^r|h(\tau)|\,d\tau\right\}^{\frac{1}{q+1}} \left\{\int_0^t |g(t-\tau)|^{1+\frac{1-r}{q}}|h(\tau)|\,d\tau\right\}^{\frac{q}{q+1}}.$$

This completes the proof.

LEMMA 4.2. Let us suppose that $z \in C(0,T;H^2(\Omega))$ and that g is a continuous function satisfying hypotheses (1.12)–(1.13). Then for $0 < r < 1$ we have

$$g\,\square\,\partial^2 z \leq 2\left\{\int_0^t g^r\,d\tau \|z\|^2_{C(0,T;H^2)}\right\}^{\frac{1}{1+(1-r)p}} \{g^{1+\frac{1}{p}}\,\square\,\partial^2 z\}^{\frac{(1-r)p}{1+(1-r)p}},$$

while for $r = 0$ we get

$$g\,\square\,\partial^2 z \leq 2\left\{\int_0^t \|z(\tau)\|_2^2\,d\tau + t\|z(t)\|_2^2\right\}^{\frac{1}{p+1}} \{g^{1+\frac{1}{p}}\,\square\,\partial^2 z\}^{\frac{p}{1+p}}.$$

Proof. From the hypotheses of z and Lemma 4.1 we get

$$\begin{aligned} g\,\square\,\partial^2 z &= \int_0^t g(t-\tau)\,\underbrace{a(z(t)-z(\tau), z(t)-z(\tau))}_{=h(\tau)}\,d\tau \\ &\leq \left\{\int_0^t g^r(t-\tau)h(\tau)\,d\tau\right\}^{\frac{1}{(1-r)p+1}} \left\{\int_0^t g^{1+\frac{1}{p}}(t-\tau)h(\tau)\,d\tau\right\}^{\frac{(1-r)p}{(1-r)p+1}} \qquad (4.1) \\ &\leq \{g^r\,\square\,\partial^2 z\}^{\frac{1}{(1-r)p+1}} \{g^{1+\frac{1}{p}}\,\square\,\partial^2 z\}^{\frac{(1-r)p}{(1-r)p+1}}. \end{aligned}$$

For $0 < r < 1$ we have

$$g^r \,\square\, \partial^2 z = \int_0^t g^r(t-\tau)a(z(t)-z(\tau), z(t)-z(\tau))\,d\tau$$
$$\leq 4\int_0^t g^r(\tau)\,d\tau \|z\|^2_{C(0,T;H^2)},$$

from which the first inequality of Lemma 4.2 follows. To prove the last part, let us take $r = 0$ to obtain

$$1 \,\square\, \partial^2 z = \int_0^t a(z(t)-z(\tau), z(t)-z(\tau))\,d\tau$$
$$\leq 2t\|z(t)\|_2^2 + 2\int_0^t \|z(\tau)\|_2^2\,d\tau.$$

Substitution of the above inequality into (4.1) yields the second inequality. The proof is now complete. □

From the above Lemma we get

$$g \,\square\, \partial^2 u \leq c_0 (g^{1+\frac{1}{p}} \,\square\, \partial^2 u)^{\frac{(1-r)p}{1+(1-r)p}}, \tag{4.2}$$

for $0 < r < 1$.

LEMMA 4.3. Let us suppose that f is a nonnegative C^1 function satisfying

$$f'(t) \leq -k_0[f(t)]^{1+\frac{1}{p}} + \frac{k_1}{(1+t)^{p+1}},$$

for some $p \geq 1$ and positive constants k_0 and k_1. Under these conditions, there exists a positive constant c_1 such that

$$f(t) \leq \frac{f(0) + 2k_1}{\left(1 + \frac{c_1}{p}\sqrt[p]{f(0)}t\right)^p}.$$

Proof. Let us denote by $h(t)$ and $F(t)$ the functions defined by

$$h(t) := \frac{2\kappa_1}{p(1+t)^p}, \qquad F(t) := f(t) + h(t).$$

So we have

$$F'(t) = f'(t) - \frac{2\kappa_1}{(1+t)^{p+1}}$$
$$\leq -\kappa_0[f(t)]^{1+\frac{1}{p}} - \frac{\kappa_1}{(1+t)^{p+1}}$$
$$\leq -\kappa_0\left\{[f(t)]^{1+\frac{1}{p}} + \frac{p^{1+\frac{1}{p}}}{2\kappa_0\kappa_1^{\frac{1}{p}}}[h(t)]^{1+\frac{1}{p}}\right\}.$$

From this it follows that there exists a positive constant c for which we have

$$F'(t) \leq -c\{[f(t)]^{1+\frac{1}{p}} + [h(t)]^{1+\frac{1}{p}}\} \leq -c[F(t)]^{1+\frac{1}{p}},$$

which gives the required inequality. □

LEMMA 4.4. Under the above conditions, the solution of system (3.2)–(3.8) satisfies the following inequality:

$$\frac{d}{dt}\mathcal{E}(t) \leq C_\varepsilon \left\{ \int_\Omega g(t)|\nabla u|^2\, dA + g^{1+\frac{1}{p}} \square \nabla u \right\}$$
$$+ c\{\|v\|_\infty g(t) a(u,u) + g^{1+\frac{1}{p}} \square \partial^2 u\} + \varepsilon \int_\Omega |\nabla u|^2\, dA - g(0) \int_\Omega (|u_t|^2 + h|\nabla u_t|^2)\, dA$$

Proof. The only difference with respect to the proof of Lemma 3.1 is to estimate the following term:

$$-\int_\Omega u \int_0^t \{g(0)g''(t-\tau) - g'(0)g'(t-\tau)\}\{u(\tau) - u(t)\}\, d\tau\, dA.$$

All other estimates follow using the same argument and the hypotheses of g. Denoting by h

$$h(t-\tau) = g(0)g''(t-\tau) - g'(0)g'(t-\tau)$$

we have that

$$|h(t)| \leq cg^{1+\frac{1}{p}},$$

from which our conclusion follows.

We are now in a condition to prove the main result of this section.

THEOREM 4.1. Let us suppose that the initial data (u_0, u_1) is 2-regular, and that (1.9), (1.12), (1.13) hold. Then any solution of system (1.2)–(1.8) satisfies:

$$E(t,u) \leq CE(0,u)(1+t)^{-p},$$

for $p > 2$.

Proof. As in Theorem 3.1 we arrive at the following inequality:

$$\frac{d}{dt}\underbrace{\left\{ NE(t) + \mathcal{E}(t) + \frac{g(0)}{2}J(t) \right\}}_{=\mathcal{L}(t)}$$
$$\leq -\kappa_0 \left\{ \underbrace{\int_\Omega (|u_t|^2 + h|\nabla u_t|^2 + |\Delta v|^2)\, dA + a(u,u)}_{:=\mathcal{N}(t)} + g^{1+\frac{1}{p}} \square \partial^2 u \right\}.$$

Since the energy is bounded, Lemma 4.2 implies

$$\mathcal{N}(t) \geq c\mathcal{N}(t)^{\frac{1+(1-r)p}{(1-r)p}}, \qquad g^{1+\frac{1}{p}} \square \partial^2 u \geq c\{g \square \partial^2 u\}^{\frac{1+(1-r)p}{(1-r)p}}.$$

It is not difficult to see that we can take N large enough such that $\mathcal{L}$ satisfies

$$c\{E(t,u)\} \leq \mathcal{L}(t,u) \leq c_1\{\mathcal{N}(t) + g^{1+\frac{1}{p}} \square \partial^2 u\}^{\frac{(1-r)p}{1+(1-r)p}}, \tag{4.3}$$

from which it follows that

$$\frac{d}{dt}\mathcal{L}(t,u) \leq -c_2 \mathcal{L}(t,u)^{\frac{1+(1-r)p}{(1-r)p}},$$

which implies that

$$\mathcal{L}(t,u) \le C\mathcal{L}(0,u)\frac{1}{(1+t)^{(1-r)p}},$$

from which it follows that the energies decay to zero uniformly. Using Lemma 4.2 with $r = 0$ we get that

$$\mathcal{N}(t) \ge c\mathcal{N}(t)^{\frac{1+p}{p}}, \qquad g\Box\partial^2 u \ge c\{g\Box\partial^2 u\}^{\frac{1+p}{p}}.$$

Repeating the same reasoning as above we get

$$\mathcal{L}(t,u) \le C\mathcal{L}(0,u)\frac{1}{(1+t)^{p}},$$

from which our result follows. The proof is now complete. □

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