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ELECTROMAGNETIC FIELD IN THE SOURCE REGION OF CONTINUOUSLY VARYING CURRENT DENSITY

By

JOHN G. FIKIORIS

Dept. of Electrical and Computer Engineering, National Technical University of Athens, Greece

Abstract. Continuity, analyticity, and the singular points of the vector potential $\mathbf{A}$ and the field vectors $\mathbf{H}, \mathbf{E}$ in a spherical source region ν are investigated thoroughly for, practically, any continuous current density distribution $\mathbf{J}$ in ν. In other words, this is a study of the inhomogeneous Helmholtz equation in ν. Explicit results for $\mathbf{A}, \mathbf{H}, \mathbf{E}$ are obtained by direct integration, extending previous results for constant density in ν to continuously varying ones. The importance of imposing the Hölder condition on $\mathbf{J}$ to insure existence of $\mathbf{E}$ and of certain second derivatives of $\mathbf{A}$ is explicitly demonstrated by a specific continuous $\mathbf{J}$, violating this condition at a point; it is then seen that $\mathbf{E}$ and some second derivatives of $\mathbf{A}$ do not exist, tending to infinity, at that point.

1. Introduction. The problems arising when the electromagnetic field vectors $\mathbf{A}, \mathbf{H}, \mathbf{E}$ are evaluated through volume integrals at points interior to the current density region are well known and have been investigated for many years [1–8]. Additional references are given by Collin in [7] and by Van Bladel in his recent comprehensive book [8]; these references are not specifically cited here, since their approach, the so-called distributional approach, is not followed in this paper. The aforementioned problems are due to the singularities of the kernel of the volume integrals for $\mathbf{A}, \mathbf{H}, \mathbf{E}$ and are coped with by two general approaches, the classical and the distributional. The first deals only with convergent integrals [1–6]; the second, broadly speaking, employs often the concept of generalized functions.

On page 17 of his classical treatise [1], Kellogg explicitly states that the integral formulas used for the evaluation of potential or field functions at points exterior to the source distribution can be used even at interior points, *provided the integrals*, now improper, *converge*. He further states that this assertion, made in connection with Newton's or Coulomb's law, since he deals only with static fields, amounts to a new assumption or an extension of Newton's or Coulomb's law, i.e., to a further hypothesis, whose validity rests solely on its results. In connection with Maxwell's equations the same assumption essentially means that the well-known volume integrals (scalar, vector, or dyadic) for ϕ (the scalar potential), $\mathbf{A}, \mathbf{H}$ of continuous volume current densities can be used at

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interior points. For second derivatives of $\phi, \mathbf{A}$ and for the $\mathbf{E}$ field, however, the kernel contains an R^{-3} singularity at interior points, the integrals are no longer convergent and may not be used in the source region. This is tantamount to saying that, for the $\mathbf{E}$-field integral (and those for second derivatives, in general), interchange of differentiation and integration, or passing the second derivative inside the volume integral, *is not allowed*. The second differentiation should follow, not precede, the explicit evaluation of a convergent volume integral. A related remark is based on the fact that the $\mathbf{E}$ field of a constant current density in an electrically small sphere ν remains constant inside and in the immediate exterior of ν, even as $\nu \to 0$ [4]. Therefore, the $\mathbf{E}$ field cannot be evaluated directly by a convergent volume integral since the integral over the vanishing ν remains constant; it results instead from the derivatives of another convergent integral, whose explicit evaluation must precede the differentiations, as done in [4].

The importance of these remarks will be made clear in the following, based on explicit results for $\mathbf{A}, \mathbf{H}, \mathbf{E}$ arising from, practically, arbitrary continuous current densities $\mathbf{J}(\mathbf{r})$ in a sphere ν of finite radius a. Second derivatives and the $\mathbf{E}$ field will be evaluated both by the correct (integration followed by differentiation) and the wrong (differentiation of the integrand followed by integration) procedures, and the validity of the first will be established on the basis of the inhomogeneous differential equations (Helmholtz's, etc.) that should be satisfied by the resulting field expressions. As in [3, 4] the approach is based on a direct integration of the field equation for $\mathbf{A}$ containing Helmholtz's kernel $\Phi(R) = e^{-jkR}/R$ and provides an extension of previous results by the author, for constant and certain special radial current distributions, to, practically, any continuous $\mathbf{J}(\mathbf{r})$ in spherical regions. Another classical approach by Lee et. al. [5], repeated in [8], is based on splitting the Helmholtz kernel into its static part $\Phi_0(R) = 1/R$ and its nonsingular part $\Phi(R) - \Phi_0(R)$. In the opinion of the present author, working with $\Phi - \Phi_0$ is not more convenient than with the full kernel Φ; this may be better appreciated when explicit results from the integrals containing $\Phi - \Phi_0$ are compared with those obtained herein directly from the singular integrals containing Φ. The latter results, being exact, are not restricted by the value of the derivative of the "excess" density around its zero value, which, if large or infinite, in particular, will prolong the numerical evaluation of certain convergent integrals, like B_{mn} in [5] (denoted by C_{mn} in [8]), for instance.

An additional important restriction that should be imposed on continuous $\mathbf{J}(\mathbf{r})$ to insure existence of second derivatives of ϕ, $\mathbf{A}$ is the Hölder condition. A specific continuous $\mathbf{J}(\mathbf{r})$, not satisfying this condition at a certain point $\mathbf{r}$, will be pointed out as an example; it will be shown explicitly that for this particular $\mathbf{J}(\mathbf{r})$ the $\mathbf{E}$ field as well as some second derivatives of $\mathbf{A}$ do not exist (tend to ∞) at $\mathbf{r}$. In this way a more or less complete study of the inhomogeneous Helmholtz equation is established.

Let us now denote by J_u $(u = x, y, z)$ the Cartesian components of $\mathbf{J}(\mathbf{r})$ and consider them as continuous functions of x, y, z (or the spherical coordinates r, ϑ, ϕ) in a sphere ν of radius a. With assumed time dependence $\exp(j\omega t)$ the Cartesian components of the vector potential at any point $\mathbf{r}$, interior or exterior to ν, are given by

$$A_u(\mathbf{r}) = \frac{\mu}{4\pi} \iiint_\nu J_u(\mathbf{r}') \frac{e^{-jkR}}{R} \, dV', \quad R = |\mathbf{r} - \mathbf{r}'| \quad (u = x, y, z). \tag{1}$$

For constant J_u this integral has been evaluated explicitly in [3, 4], for some special radial distributions $J_u(r)$ in [4] and for even more general ones, as pointed out below, in [6]. In this paper $A_u(\mathbf{r})$ and its first and second derivatives and, therefore, the fields $\mathbf{H}(\mathbf{r}) = \frac{1}{\mu}\nabla \times \mathbf{A}$ and $\mathbf{E}(\mathbf{r}) = -\frac{j}{\omega\varepsilon}(\nabla \times \mathbf{H} - \mathbf{J})$, will be explicitly evaluated for, practically, any continuous $J_u(\mathbf{r})$ in ν providing the full generalization of the subjects investigated in [1, 2, 3, 4]. When $\mathbf{r}$ is interior to ν, the integral in (1) becomes a convergent improper one. The procedure will provide, also, the analytic field exterior to ν due to the sources in ν; the integrals are regular there and derivatives of all orders may be interchanged with the integral operators. In contrast, for $\mathbf{r}$ interior to ν, it will be shown explicitly, and for arbitrary continuous $J_u(\mathbf{r})$ in ν, that only first derivatives may be passed inside the integrals; further differentiation should follow integration. Otherwise, the results so obtained do not satisfy the inhomogeneous Helmholtz equation

$$\nabla^2 \mathbf{A}(\mathbf{r}) + k^2 \mathbf{A}(\mathbf{r}) = -\mu \mathbf{J}(\mathbf{r}) \tag{2}$$

and are, therefore, erroneous. It will be further shown that second derivatives may be interchanged with the integrals only at points $\mathbf{r}$ where $J_u(\mathbf{r}) = 0$ and as long as $J_u(\mathbf{r})$ satisfies a Hölder condition at $\mathbf{r}$. Finally, a specific case will be pointed out, where second derivatives of $A_u(\mathbf{r})$ do not exist at $\mathbf{r}$ (they tend to ∞ as $r \to 0$) because $J_u(\mathbf{r})$ does not satisfy a Hölder condition at $\mathbf{r}$, although this is not necessarily true in general.

All these results are in full agreement with Eqs. (20) and (21) in [3], which indicate the correct way of evaluating the electric field $\mathbf{E}(\mathbf{r})$ at points $\mathbf{r}$ inside ν. In particular, the more general relation (21) of [3], for any point $\mathbf{r}$ in ν, reads as follows:

$$\begin{aligned}\mathbf{E}(\mathbf{r}) = &\iiint_{V-\nu} j\omega\mu \mathbf{J}(\mathbf{r}') \cdot \mathbf{G}(\mathbf{r}'|\mathbf{r})\, dV' + \iiint_{\nu} j\omega\mu [\mathbf{J}(\mathbf{r}') - \mathbf{J}(\mathbf{r})] \cdot \mathbf{G}(\mathbf{r}'|\mathbf{r})\, dV' \\ &+ \frac{1}{j\omega\varepsilon}\left[1 + \frac{1}{k^2}\nabla\nabla\right] \mathbf{J} \left[\frac{\sin(kr)}{kr} e^{-jka}(1 + jka) - 1\right].\end{aligned} \tag{3}$$

In (3) V is the volume of the sources $\mathbf{J}$ and contains ν, a is the radius of ν and r the distance of point $\mathbf{r}$ from the center of ν, $\mathbf{J} = \mathbf{J}(\mathbf{r})$, in the last term, is a constant vector, not operated upon by ∇; again $R = |\mathbf{r} - \mathbf{r}'|$ while

$$\mathbf{G}(\mathbf{r}'|\mathbf{r}) = -\frac{1}{4\pi}\left(\mathbf{I} + \frac{1}{k^2}\nabla'\nabla'\right)\frac{e^{-jkR}}{R} = \frac{1}{4\pi k^2}\nabla' \times \nabla' \times \left(\mathbf{I}\frac{e^{-jkR}}{R}\right) \tag{4}$$

is the Green's dyadic with $\mathbf{I}$ the identity dyadic. The validity of (3) requires, in general, the satisfaction of the Hölder condition by the "excess" density $\mathbf{J}(\mathbf{r}') - \mathbf{J}(\mathbf{r})$ at $\mathbf{r}$. In what follows, the last two terms of (3) are explicitly evaluated for, practically, any continuous $\mathbf{J}(\mathbf{r})$ in ν.

II. The fields of arbitrary continuous current density in ν. As will soon become apparent, it is advantageous to start by considering as source function in ν the "partial density function"

$$J_{un}^m(r, \vartheta, \phi) = f_{un}(r) P_n^m(\cos\vartheta_u) e^{jm\phi_u}, \; u = x, y, z \; (n = 0, 1, 2, \ldots, |m| \leq n) \tag{5}$$

in which $f_{un}(r)$ are arbitrary continuous functions of r in $0 \leq r \leq a$ (continuity of J_u at $r = 0$ requires that $f_{un}(0) = 0$ for $n = 1, 2, \ldots$) and $P_n^m(\cos\vartheta_u)e^{jm\phi_u}$ is the spherical harmonic around the polar axis $\hat{\mathbf{u}}$. With this implied throughout we will, from here on, delete the subscript u from f, ϑ, ϕ. The particular case $f_n(r) = r^p$, with integral $p \geq n-1$, was fully investigated in [6]. The results based on (5) are immediately generalized to any continuous $J_u(\mathbf{r})$, which may be expressed as a finite linear combination of the above partial density functions in the form

$$J_u(\mathbf{r}) = \sum_{n=0}^{N} f_n(r) \sum_{m=-n}^{n} P_n^m(\cos\vartheta)e^{jm\phi}. \tag{6}$$

Even infinite series expansions ($N = \infty$) can be included as long as the convergence of the series is uniform in ν (for instance if $|f_n(r)| < n^{-3}$, etc.). In view of the completeness of the set of spherical harmonics in $0 \leq \vartheta \leq \pi$, $0 \leq \phi \leq 2\pi$ and the continuity of $f_n(r)$, these remarks show that the results of this section can be used for a wide class of practical, continuous, source distributions. Substituting (5) in (1) we get

$$A_{un}^m(r, \vartheta, \phi) = \frac{\mu}{4\pi} \int_0^a \int_0^\pi \int_0^{2\pi} f_n(r')P_n^m(\cos\vartheta')e^{jm\phi'} \frac{e^{-jkR}}{R} r'^2 \sin\vartheta' \, dr' \, d\vartheta' \, d\phi'. \tag{7}$$

We next use for $\Phi(R)$ the well-known expansion in terms of spherical Bessel functions and spherical harmonics [9, pp. 413–414 and 406–408]:

$$\begin{aligned}\Phi(R) = \frac{e^{-jkR}}{R} = -jk \sum_{n'=0}^{\infty} (2n'+1) j_{n'}(kr_<) h_{n'}(kr_>) \sum_{m'=-n'}^{n'} \frac{(n'-m')!}{(n'+m')!} \\ \cdot P_{n'}^{m'}(\cos\vartheta) P_{n'}^{m'}(\cos\vartheta') e^{jm'(\phi-\phi')}\end{aligned} \tag{8}$$

with $h_n = h_n^{(2)}$ throughout and $r_>/r_<$ the greater/smaller of r, r'. This expansion, for any given $\varepsilon > 0$, is uniformly convergent on the domains $[(r', \vartheta', \phi') | 0 \leq r' \leq r - \varepsilon$, $0 \leq \vartheta' \leq \pi$, $0 \leq \phi' \leq 2\pi]$ and $[(r', \vartheta', \phi') | r + \varepsilon \leq r' \leq a$, $0 \leq \vartheta' \leq \pi$, $0 \leq \phi' \leq 2\pi]$ (as well as on domains defined as above with r, ϑ, ϕ and r', ϑ', ϕ' interchanged) [1, 2] and when substituted in (7) allows term-by-term integration as long as the integral is convergent. For then the contribution from an excluded spherical-shell-volume $r - \varepsilon \leq r' \leq r + \varepsilon$ around r tends to 0 with ε. On the contrary, if the singularity of e^{-jkR}/R is raised to R^{-3} by differentiation of the integrand, the integral is no longer convergent and term-by-term integration yields erroneous results. Since (8) changes in the intervals $r' < r$ and $r' > r$ the integration over r' must be separated into two intervals, $0 \leq r' \leq r$ and $r \leq r' \leq a$. Finally, the orthogonality of the spherical harmonics [9, p. 403] leads to

$$\begin{aligned}A_{un}^m(r, \vartheta, \phi) = -j\mu k P_n^m(\cos\vartheta)e^{jm\phi} \bigg(h_n(kr) \int_0^r j_n(kr') f_n(r') r'^2 \, dr' \\ + j_n(kr) \int_r^a h_n(kr') f_n(r') r'^2 \, dr' \bigg).\end{aligned} \tag{9}$$

Unknown to the present author, the same procedure was followed in [6] producing a result similar to (9) for the specific $f_n(r)$ mentioned previously. Calling $kr = t, kr' = t'$,

$$I_n(t) = \int_0^t f_n(t'/k) j_n(t') t'^2 \, dt', \qquad L_n(t) = \int_t^{ka} f_n(t'/k) h_n(t') t'^2 \, dt', \tag{10}$$

we finally obtain

$$A_{un}^m(r, \vartheta, \phi) = -\frac{j\mu}{k^2} P_n^m(\cos\vartheta) e^{jm\phi} F_n(kr), \quad 0 \le r \le a, \tag{11}$$

$$F_n(t) = F_n(kr) = h_n(t) I_n(t) + j_n(t) L_n(t). \tag{12}$$

We first observe that for $n = m = 0$ this general result coincides with the result for $A_u(r)$ obtained in [3, 4], for constant $f_0(r)$ and, in [4], for varying $f_0(r)$, taking into account that $j_0(t) = \frac{\sin t}{t}$ and $h_0(t) = je^{-jt}/t$. We further observe that in this special case (7) is directly integrable and use of expansion (8) can be avoided, as shown in [4]. Explicit evaluation of $I_n(t), L_n(t)$ is possible for a good number of special functions $f_n(r)$, particularly for $n = 0$ [4]. The basic properties of the fields $\mathbf{A}, \mathbf{H}, \mathbf{E}$ can be deduced, however, from the general forms (10) for $I_n(t), L_n(t)$ without explicit evaluation of the integrals. This will become clear in a short while.

The result (11) for A_{un}^m must satisfy the inhomogeneous Helmholtz equation

$$\nabla^2 A_{un}^m + k^2 A_{un}^m = -\mu J_n^m = -\mu f_n(r) P_n^m(\cos\vartheta) e^{jm\phi}. \tag{13}$$

Direct differentiation of (11) yields

$$\begin{aligned} \frac{\partial A_{un}^m}{\partial r} = k\frac{\partial A_{un}^m}{\partial t} &= -\frac{j\mu}{k} P_n^m(\cos\vartheta) e^{jm\phi} F_n'(t), \\ \frac{\partial^2 A_{un}^m}{\partial r^2} &= -j\mu P_n^m(\cos\vartheta) e^{jm\phi} F_n''(t) \end{aligned} \tag{14}$$

where, with the help of the obvious relations

$$I_n'(t) = f_n(t/k) j_n(t) t^2, \quad L_n'(t) = -f_n(t/k) h_n(t) t^2, \quad h_n(t) I_n'(t) + j_n(t) L_n'(t) = 0 \tag{15}$$

we have

$$F_n'(t) = h_n'(t) I_n(t) + j_n'(t) L_n(t), \tag{16}$$

$$F_n''(t) = h_n''(t) I_n(t) + j_n''(t) L_n(t) - j f_n(t/k). \tag{17}$$

The last term on the right of (17), $-jf_n(t/k)$, arises from the Wronskian relation

$$h_n'(t) I_n'(t) + j_n'(t) L_n'(t) = f_n(t/k) t^2 [h_n'(t) j_n(t) - j_n'(t) h_n(t)] = -j f_n(t/k). \tag{18}$$

Substituting in (13), in which the left-hand side assumes the form

$$(\nabla^2 + k^2) A_u = \frac{\partial^2 A_u}{\partial r^2} + \frac{2}{r}\frac{\partial A_u}{\partial r} + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial A_u}{\partial\vartheta}\right) + \frac{1}{r^2 \sin^2\vartheta}\frac{\partial^2 A_u}{\partial\phi^2} + k^2 A_u,$$

and taking into account the partial differential equation satisfied by the spherical harmonic $P_n^m(\cos\vartheta)e^{jm\phi}$ we end up examining whether the following relation is satisfied:

$$h_n'' I_n + j_n'' L_n - jf_n + \frac{2}{t}(h_n' I_n + j_n' L_n) + \left[1 - \frac{n(n+1)}{t^2}\right](h_n I_n + j_n L_n) = -jf_n. \quad (19)$$

Since both $h_n(t)$ and $j_n(t)$ satisfy the spherical Bessel equation

$$z_n''(t) + \frac{2}{t} z_n'(t) + \left[1 - \frac{n(n+1)}{t^2}\right] z_n(t) = 0 \quad (20)$$

the total coefficients multiplying I_n and L_n in (19) are 0 and Eq. (13) is satisfied.

It may be interesting, also, to see what would have happened had the derivatives $\partial A_u/\partial r$ and $\partial^2 A_u/\partial r^2$ been evaluated by reversing the order of differentiation and integration in (7). Based again on the orthogonality of the spherical harmonics the correct result given in (14) would be obtained for $\partial A_u/\partial r$ (an explicit verification of the fact that first derivatives can be passed behind the integral sign [1–4]); for $\partial^2 A_u/\partial r^2$, however, this procedure would lead to the erroneous result

$$\frac{\partial^2 A_{un}^m}{\partial r^2} = -j\mu P_n^m(\cos\vartheta)e^{jm\phi}[h_n''(t)I_n(t) + j_n''(t)L_n(t)]; \quad (21)$$

this differs from the result in (14), (17) by the term $-jf_n(t/k)$. With (21) Eq. (13) is not satisfied, a result hardly surprising since it follows from a not permissible operation. It is included here because of its explicit character and because it further shows that it can still be correct at points, where $f_n(r) = 0$, and as long as $f_n(r')$ (the "excess" density now) satisfies at such points a Hölder condition. So, at points where $f_n(r) = 0$ or $\mathbf{J}(\mathbf{r}) = 0$ it is permissible to pass second derivatives behind the integral, in full agreement with the second and third terms of (3), in which the third is now 0, while in the second the $\mathbf{G}$ operator appears inside the integral. It is important, also, to appreciate the generality and simplicity of the results obtained here for spheres ν, when compared with the formal expressions of Eq. (3).

Having obtained the explicit results (11)–(12) and (14) for $A_{un}^m, \partial A_u/\partial r, \partial^2 A_u/\partial r^2$ it is a simple matter to obtain similar expressions for $\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A}$ and $\mathbf{E} = \frac{-j}{\omega\varepsilon}(\nabla\times\mathbf{H} - \mathbf{J}) = -j\omega\mathbf{A} - j\frac{\omega}{k^2}\nabla(\nabla\cdot\mathbf{A})$ in spherical coordinates around the unit vector $\hat{\mathbf{u}}$. For instance (for $0 \le r \le a$):

$$H_r = \frac{m}{k^2} P_n^m(\cos\vartheta)e^{jm\phi}\frac{F_n(kr)}{r}, \qquad H_\vartheta = \frac{m}{k^2}\cot\vartheta P_n^m(\cos\vartheta)e^{jm\phi}\frac{F_n(kr)}{r},$$

$$H_\phi = \frac{je^{jm\phi}}{k^2 r}\left\{\sin\vartheta P_n^m(\cos\vartheta)[F_n(kr) + krF_n'(kr)] + F_n(kr)\frac{\partial}{\partial\vartheta}[\cos\vartheta P_n^m(\cos\vartheta)]\right\}. \quad (22)$$

The expressions for E_r, E_ϑ, E_ϕ are lengthier, but they follow easily by differentiation; it is easy to verify that the inhomogeneous equations $(\nabla\times\nabla\times -k^2)\mathbf{E} = -j\omega\mu\mathbf{J}$ and $(\nabla\times\nabla\times -k^2)\mathbf{H} = \nabla\times\mathbf{J}$ are, also, satisfied.

Everywhere, the dependence on the radius a appears only in $F_n(kr)$ through $L_n(kr)$. We can further deduce from (12), (16), (17) that $F_n(t), F_n'(t), F_n''(t)$ are all continuous in

$0 \leq t \leq ka$ with the possible exception of the point $t = 0$ (or $r = 0$), owing to the Hankel function $h_n(t)$. We will investigate this point fully in the next section.

Finally, the field exterior to ν, for $r > a$, obtained again from (7) and (8) with $0 \leq r' \leq a < r$, has the simple expression

$$A^m_{un}(r, \vartheta, \phi) = -\frac{j\mu}{k^2} P^m_n(\cos\vartheta) e^{jm\phi} I_n(ka) h_n(kr), \quad a \leq r, \tag{23}$$

$$I_n(ka) = \int_0^{ka} f_n(t') j_n(t') t'^2 \, dt'. \tag{24}$$

The function is obviously analytic in r, ϑ, ϕ. One may, also, easily verify the continuity of $\mathbf{A}(r, \vartheta, \phi), \mathbf{H}, E_\vartheta, E_\phi$ at $r = a$ and the proper step discontinuity of $E_r(a, \vartheta, \phi)$ between interior ($r \leq a$) and exterior ($r \geq a$) values. This was shown in detail in [4] for constant J_u.

III. The Hölder condition. Existence of second derivatives of A. It was observed that with the adoption of (5) or (6) for J_u all possible nonanalyticities of $\mathbf{A}, \mathbf{H}, \mathbf{E}$ in $0 \leq r \leq a$ will arise either from those of $f_n(r)$, which is continuous, but, maybe, nondifferentiable, or from the singularity of $h_n(t)$ at $t = 0$. Those due to $f_n(r)$ may lead to infinite values only in the third (and higher) derivatives of A_u at all points r apart from $r = 0$. This can be seen immediately from (10), (15)–(17), where derivatives of $f_n(t/k)$ will appear for the first time in $F'''_n(t)$. However, for $F''_n(t)$ to exist at $t = 0$ the Hölder condition must, in general, be imposed on $f_n(t/k)$. This is shown by considering a continuous function $f_n(t/k)$ violating this condition at $r = 0$ or $t = 0$:

$$f_n(t/k) = \left[\ln\left(\frac{10ka}{t}\right)\right]^{-1}, \quad 0 \leq t \leq ka. \tag{25}$$

We may call it a "non-Hölder" function at $t = 0$. The particular choice (25) gives $f_n(0) = 0$, making $f_n(t/k)$ equivalent to the "excess" density at $t = 0$, which is of main interest here, and keeps $f_n(t/k)$ in $0 \leq f_n(t/k) \leq 1/\ln(10)$, well below the infinity of the function $1/\ln(1/x)$ at $x = 1$. The function has an infinite derivative at $t = 0$ and rises faster, near this point, than any Hölder function t^q ($q > 0$).

Since we are interested in the behavior of $F''_n(t)$ near $t = 0$, we may keep t in $0 \leq t \leq t_0 << 1$, where the fixed t_0 is chosen small enough to let $j_n(t), h_n(t)$ be well approximated in $0 \leq t \leq t_0$ by the leading terms of their Maclaurin or Laurent expansions, respectively,

$$j_n(t) \cong \frac{2^n n!\, t^n}{(2n+1)!}, \qquad h_n(t) \cong \frac{j(2n)!\, t^{-n-1}}{2^n n!} \quad (n = 0, 1, 2, \ldots).$$

Substituting into the integrals $I_n(t), L_n(t)$ of (10), together with the expression (25) for $f_n(t/k)$, it is readily found through integration by parts that

$$I_n(t) \cong \frac{2^n n!}{(2n+1)!\,(n+3)} \frac{t^{n+3}}{\ln[10ka/t]}, \quad n = 0, 1, 2, \ldots, \tag{26}$$

$$L_n(t) \cong \frac{j(2n)!}{2^n n!\,(n-2)} \frac{t^{-n+2}}{\ln[10ka/t]}, \quad n = 3, 4, \ldots. \tag{27}$$

Combining (26) with the leading terms of $h_n(t), h'_n(t), h''_n(t)$ we obtain

$$h_n(0)I_n(0) = h'_n(0)I_n(0) = h''_n(0)I_n(0) = 0 \tag{28}$$

and, from (17), this part of $F''_n(t)$ goes to 0 at $t = 0$. In addition, the fact that $t^2h_0(t)$ and $t^2h_1(t)$ remain finite, even at $t = 0$, shows from (10) that $L_0(0)$ and $L_1(0)$ remain finite. Finally, for $n = 2$

$$L_2(t) = C(2, t_0, ka) + 3j \int_t^{t_0} \left[\frac{1}{t'} + O(t')\right] \frac{dt'}{\ln(\frac{10ka}{t'})} \tag{29}$$

(with $C(2, t_0, ka) = \int_{t_0}^{ka} f_2(t'/k)h_2(t')t'^2\, dt'$ a constant) and the dominant term is now

$$L_2(t) \cong 3j \ln\ln\left(\frac{10ka}{t}\right). \tag{30}$$

These results for $L_n(t)$ show that $j_n(t)L_n(t), j'_n(t)L_n(t), j''_n(t)L_n(t)$ are all 0 at $t = 0$ for $n = 3, 4, \ldots$; the same holds true for $j'_0(t)L_0(t), j_1(t)L_1(t), j''_1(t)L_1(t)$ while $j_0(t)L_0(t)$, $j''_0(t)L_0(t), j'_1(t)L_1(t)$ remain finite at $t = 0$. For $n = 2$: $j_2(t)L_2(t), j'_2(t)L_2(t)$ are 0 at $t = 0$, but $j''_2(t)L_2(t)$ goes to infinity since the dominant term for it, as well as for $F''_2(t)$, is

$$F''_2(t) \cong j''_2(t)L_2(t) \cong \frac{2j}{5} \ln\ln\left(\frac{10ka}{t}\right). \tag{31}$$

This remarkable result shows that for the continuous, but "non-Hölder" (at $r = 0$) current density

$$J^m_{u2}(r, \vartheta, \phi) = \frac{1}{\ln(\frac{10a}{r})} P^m_2(\cos\vartheta)e^{jm\phi} \quad (m = -2, -1, 0, 1, 2) \tag{32}$$

$\partial^2 A_u/\partial r^2$ and $\mathbf{E}(r, \vartheta, \phi)$ do not exist, but tend to infinity, at $r = 0$. It is worth noting that a) the infinity arises from the term $j''_2(t)L_2(t)$, not from $h''_2(t)I_2(t)$, and only when $n = 2$, b) Müller [2], on pages 118–119, proposes a similar "non-Hölder" function for a static charge density for which the static potential function is not twice differentiable at $r = 0$ (see, also, [5]); again, his result is obtained for $n = 2$, i.e., it is associated with a spherical harmonic of order 2. The results obtained here provide a generalization in the electromagnetic case. On the other hand, a "non-Hölder" current density does not necessarily lead to infinite $\mathbf{E}$, as all previous cases with $n \neq 2$ indicate, or cases where the condition is violated at $r = b < a$ rather than at $r = 0$. Similar infinite results, again only for $n = 2$, were obtained using the "non-Hölder" functions $f_n(r) = (\ln(\frac{10a}{r}))^{-q}$ $(0 < q < 1)$ and $f_n(r) = (\ln\ln(\frac{10a}{r}))^{-1}$.

For $f_n(r) = r^q$ $(0 < q < 1)$, a continuous function satisfying the Hölder condition at $r = 0$, but having no derivative there, it is very easy to establish that $F_n(t), F'_n(t), F''_n(t)$ exist at $t = 0$. For $-1 < q < 0$ we have a density singular at $r = 0$ and, obviously, so are $F''_n(t)$ and $\mathbf{E}$, although A_u and $\mathbf{H}$ exist. For $n = 0$ this case was examined in [4].

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