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ON THE LOCATION OF DEFECTS IN STATIONARY SOLUTIONS OF THE GINZBURG-LANDAU EQUATION IN $\mathbf{R}^2$

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1. Introduction. In this paper we study boundary-value problems for the stationary Ginzburg-Landau equation in two space dimensions in the limit of fast reaction and slow diffusion:

$$\Delta u + \frac{1}{\varepsilon^2} u(1 - |u|^2) = 0 \quad \text{in } \Omega. \tag{1.1}$$

Here Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$, $u \in \mathbf{R}^2$ and ε a positive constant such that $\varepsilon \ll 1$. We are interested in solutions u_ε with singular behavior as $\varepsilon \to 0$ near a zero of u_ε in Ω. Such a zero represents a regularized topological singularity, also referred to as a *defect* or a *vortex*.

Equation (1.1) describes the stationary states of evolution equations such as the *Ginzburg-Landau equation* [GL]

$$iu_t = \Delta u + u(1 - |u|^2)$$

or its dissipative version

$$u_t = \Delta u + u(1 - |u|^2).$$

These equations arise in many areas of physics. We mention amplitude equations for general classes of pattern-forming systems of partial differential equations [CNR,K] and the theory of superconductivity [D, KT]. In these and other contexts, the most important and best studied phenomena are associated with the dynamics and interaction of defects [N, PRo, PRu1, PRu2].

For definiteness we impose at the boundary of Ω the Dirichlet condition

$$u(x) = g(x) \quad \text{for } x \in \partial\Omega; \tag{1.2}$$

analogous results for the Neumann problem will be given later. Here $g\colon \partial\Omega \to \mathbf{R}^2$ is a given smooth function about which we assume

Hg1: $|g(x)| = 1$ for all $x \in \partial\Omega$;

Hg2: $\arg g(x)$ increases by 2π when x traverses $\partial\Omega$ once in the positive direction (see extension below).

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It is known that this problem has a variational solution u_ε which for ε sufficiently small has precisely *one* zero a_ε,

$$u_\varepsilon(a_\varepsilon) = 0, \tag{1.3}$$

when Ω is star shaped [BBH1,2]. Our objective will be to find the possible locations a_0 of this zero a_ε in the limit as $\varepsilon \to 0$ and to give an asymptotic description of u_ε.

The criterion that we develop for the location a_0 has an immediate analog, valid when (1.2) is replaced by the zero Neumann condition on $\partial\Omega$. In this case the existence of singular solutions comparable to u_ε is not guaranteed.

Returning to the Dirichlet problem, we see by compactness that a_ε tends to a limit a_0 as $\varepsilon \to 0$ along a subsequence, and it was shown in [BBH1,2] that $a_0 \in \Omega$. We shall see by a formal analysis that

$$u_\varepsilon(x) \to e^{i\theta_0(x)} \quad \text{as } \varepsilon \to 0,$$

uniformly on compact subsets that do not include a_0. The phase θ_0 is a real-valued function that is *harmonic* in $\Omega\backslash\{a_0\}$ and satisfies the boundary condition

$$e^{i\theta_0(x)} = g(x) \quad \text{for all } x \in \partial\Omega. \tag{1.4}$$

Here we have identified $\mathbf{R}^2$ with $\mathbf{C}$ and expressed u in complex notation. This practice will be continued throughout most of this paper. By assumption Hg1 we can write

$$g(x) = e^{i\chi(x)}, \tag{1.5}$$

where $\chi(x)$ is a real-valued function. Thus

$$\theta_0(x) = \chi(x) \quad \text{for all } x \in \partial\Omega. \tag{1.6}$$

Let $\varphi_a(x) = \arg(x - a)$. Then we can define for any $a \in \Omega$ the function

$$\psi_0(x\,;a) = \theta_0(x) - \varphi_a(x), \tag{1.7}$$

which for $a = a_0$ will be shown to satisfy

$$(\mathrm{P})\begin{cases} \Delta\psi_0 = 0 & \text{in } \Omega\backslash\{a\}, \\ \psi_0 = \chi - \varphi_a & \text{on } \partial\Omega \end{cases}$$

and to have a removable singularity at $x = a_0$.

Our main observation can now be formulated as follows.

Suppose (a_ε) *is a sequence of zeros of solutions* u_ε *of Problem* (1.1), (1.2), *which converges to a point* $a_0 \in \Omega$ *as* $\varepsilon \to 0$. *Then*

$$\nabla\psi_0(x\,;a_0)|_{x=a_0} = 0. \tag{1.8}$$

From a physical perspective the condition (1.8) can be elucidated by viewing ψ_0 as a field generated by the defect, in combination with the vector field prescribed on the boundary of the domain. Condition (1.8) then states that the limit point a_0

must have the property that

> *if a defect is located at* a_0*, then its own field* $\psi_0(\cdot\,;a_0)$ *has a stationary point at* a_0.

After this result was obtained, Bethuel, Brezis, and Hélein in a remarkable paper [BBH2,3] proved the validity of this *vanishing gradient* condition for the location of a_0, as well as its extension to the case when there is more than one defect by means of variational methods. Our analysis is formal, but provides additional insight into the nature of u_ϵ.

As a simple example, we consider Problem (1.1), (1.2) on the unit disc, i.e., $\Omega = B_1$ and choose for $\chi = \arg g$, in polar coordinates (r, φ),

$$\chi(1, \varphi; 0) = \varphi + A\sin(2\varphi), \qquad \varphi = \arg x, \quad A \in \mathbf{R}.$$

By symmetry we anticipate that the defect will be located at the origin. Thus, setting $a_0 = 0$, the boundary condition in Problem (P) becomes

$$\psi_0(1, \varphi; 0) = A\sin(2\varphi) \quad \text{on } \partial B_1.$$

For the solution of Problem (P) we now find

$$\psi_0(r, \varphi; 0) = Ar^2\sin(2\varphi),$$

so that $\nabla\psi_0$ does indeed vanish at the origin, as expected.

The analysis given in this paper for a single zero of u_ε can easily be extended to solutions with N zeros of degree 1. Suppose they are located at the points $a_\varepsilon^1, \dots, a_\varepsilon^N$. Then, as before, we find that

$$u_\varepsilon(x) \to e^{i\theta_0(x)} \quad \text{as } \varepsilon \to 0$$

uniformly on compact sets in $\Omega\backslash\mathbf{a}_0$, where $\mathbf{a}_0$ denotes the set of limit points a_0^i of a_ε^i. The phase function θ_0 can now be written as

$$\theta_0(x) = \sum_{i=1}^{N} \varphi_{a_0^i}(x) + \psi_0(x; \mathbf{a}_0),$$

in which $\varphi_{a_0^i}(x) = \arg(x - a_0^i)$, so that ψ_0 is single valued on $\partial\Omega$. We now find that ψ_0 is the solution of the problem

$$(\mathrm{P}')\begin{cases} \Delta\psi_0 = 0 & \text{in } \Omega, \\ \psi_0 = \chi - \displaystyle\sum_{i=1}^{N}\varphi_{a_0^i} & \text{on } \partial\Omega. \end{cases}$$

For the possible locations of the zeros we obtain the set of conditions

$$\left.\left(\nabla\psi_0(x; \mathbf{a}_0) + \sum_{i=1,\ i\neq j}^{N} \varphi_{a_0^i}(x)\right)\right|_{x=a_0^j} = 0 \quad \text{for } j = 1, 2, \dots, N. \tag{1.8a}$$

In the inner expansion near a zero a_0^i, the first term is unaffected by the other zeros and is still the ground state defined by Problem (II). The second term, however, is affected, because it depends on the second derivatives of ψ_0 and the angle functions $\varphi_{a_0'}$ of the other zeros.

For the Neumann problem, when the boundary condition

$$\frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

is imposed, the vanishing gradient condition (1.8) can also be shown to hold for a suitably modified function ψ_0. The boundary condition in Problem (P) then needs to be replaced by

$$\frac{\partial \psi_0}{\partial \nu} = -\frac{\partial \varphi_{a_0}}{\partial \nu} \quad \text{on } \partial\Omega.$$

Since $\int_{\partial\Omega} \partial\varphi_{a_0}/\partial\nu = 0$, the existence of ψ_0 is ensured and it is unique, except for an additional constant. Our analysis below can easily be modified to apply to this case. For example, there is now no need to introduce the phase shift α_0 and the boundary condition on ψ_1 becomes $\partial\psi_1/\partial\nu = 0$, so that ψ_1 will be constant.

It should be pointed out, however, that there is no guarantee that a point a_0 for which (1.8) holds indeed exists in this case.

The plan of the paper is the following. In Sec. 2 we give an outline of the method leading to condition (1.8). Since it is a formal one, based on matched asymptotic expansions, we shall discuss here the underlying assumptions and limitations. In Sec. 3 we derive the outer expansion and in Sec. 4 we derive the inner expansion and show how the matching conditions lead to condition (1.8). This requires a technical result, which we leave to Sec. 5.

2. Outline. In this section we introduce some notation, formulate our basic assumptions and sketch the ideas that lead to condition (1.8).

It is well known that Problem (1.1), (1.2) has a variational solution u_ε which has a zero a_ε. We shall assume that for ε small enough, a_ε is unique, and that there exists an infinite sequence (ε_n) tending to zero such that

$$u_{\varepsilon_n} \to u_0 \quad \text{and} \quad a_{\varepsilon_n} \to a_0 \in \Omega \quad \text{as } n \to \infty, \tag{2.1}$$

the convergence being uniform in C^2 on compact subsets of $\Omega\backslash\{a_0\}$. It has recently been shown that this is indeed the case when Ω is star shaped [BBH1,2]. Throughout, when we write $\varepsilon \to 0$, it is understood that the limit is taken along this subsequence.

For the analysis it is convenient to shift the origin to the zero a_ε of u_ε and introduce a phase factor $e^{i\alpha_\varepsilon}$, to be chosen later. We write

$$y = x - a_\varepsilon \quad \text{and} \quad \tilde{u}_\varepsilon(y) = u_\varepsilon(x)e^{i\alpha_\varepsilon}, \tag{2.2}$$

although we shall drop the tilde again. We can then write

$$u_\varepsilon(y) = \rho_\varepsilon(y)e^{i\theta_\varepsilon(y)}, \tag{2.3}$$

which yields upon substitution into (1.1) the following system for ρ_ε and θ_ε:

$$\begin{cases} \varepsilon^2\Delta\rho + \rho(1 - \varepsilon^2|\nabla\theta|^2 - \rho^2) = 0, & \text{(2.4a)} \\ \Delta\theta + \dfrac{2}{\rho}\nabla\rho\cdot\nabla\theta = 0, & \text{(2.4b)} \end{cases}$$

defined in the translated domain $\Omega - a_\varepsilon = \{y\colon y + a_\varepsilon \in \Omega\}$. From (2.1) we conclude that outside any arbitrary small ball around the origin,

$$\rho_\varepsilon(y) \to 1 \quad \text{and} \quad \theta_\varepsilon(y) \to \theta_0(y) \quad \text{as } \varepsilon \to 0, \tag{2.5}$$

uniformly in compact subsets of $(\Omega - a_\varepsilon)\backslash\{0\}$, and that

$$\theta_0(y) = \varphi_0(y) + \psi_0(y),$$

where $\psi_0(x-a)$ is a solution of Problem (P).

Our basic assumptions about the family u_ε are the following. Here we use polar coordinates $y = re^{i\varphi}$ and suppress the dependence on φ for the time being.

1. There exist smooth functions $u_i(r)$, $i = 0, 1, 2, \ldots$ defined in $\Omega\backslash\{a_0\}$ such that for any fixed $r_0 > 0$,

$$u_\varepsilon(r) = u_0(r) + \varepsilon u_1(r) + \varepsilon^2 u_2(r) + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0 \tag{2.6}$$

uniformly for $r \geq r_0$.

2. There exist smooth functions $U_i(s)$, $i = 0, 1, 2, \ldots$ defined in all of $\mathbf{R}^2$ and real-valued positive gauge functions $\gamma_i(\varepsilon)$ with $\gamma_{i+1}(\varepsilon) = o(\gamma_i(\varepsilon))$ as $\varepsilon \to 0$, such that for any fixed $s_0 > 0$,

$$U_\varepsilon(s) \stackrel{\text{def}}{=} u_\varepsilon(\varepsilon s) = \gamma_0(\varepsilon)U_0(s) + \gamma_1(\varepsilon)U_1(s) + \gamma_2(\varepsilon)U_2(s) + o(\gamma_2(\varepsilon)) \quad \text{as } \varepsilon \to 0 \tag{2.7}$$

uniformly for $s \leq s_0$.

3. These order relations may be differentiated twice with respect to (r, φ) and (s, φ).

4. There exist functions $\nu(\varepsilon)$, $\sigma_1(\varepsilon)$, and $\sigma_2(\varepsilon)$ with the properties

$$\nu(\varepsilon) \to 0, \quad \sigma_1(\varepsilon) \to 0, \quad \sigma_2(\varepsilon) \to \infty \quad \text{as } \varepsilon \to 0 \tag{2.8}$$

and $\varepsilon\sigma_2(\varepsilon) \geq 2\sigma_1(\varepsilon)$, such that for some constants $C > 0$, $r_1 > 0$, and $s_1 > 0$,

$$\varepsilon^2|u_2(r)| \leq C\nu(\varepsilon) \quad \text{uniformly for } \sigma_1(\varepsilon) \leq r \leq r_1, \tag{2.9a}$$

$$\gamma_2(\varepsilon)|U_2(s)| \leq C\nu(\varepsilon) \quad \text{uniformly for } s_1 \leq s \leq \sigma_2(\varepsilon), \tag{2.9b}$$

and (2.6) and (2.7) both hold with remainder term $o(\nu(\varepsilon))$ uniformly for $r > \sigma_1(\varepsilon)$ and $s < \sigma_2(\varepsilon)$, respectively.

As a consequence of these assumptions, we have the following matching conditions:

$$\gamma_0(\varepsilon)U_0(s) + \gamma_1(\varepsilon)U_1(s) + \gamma_2(\varepsilon)U_2(s) = u_0(\varepsilon s) + \varepsilon u_1(\varepsilon s) + \varepsilon^2 u_2(\varepsilon s) + o(\nu(\varepsilon)) \quad \text{as } \varepsilon \to 0 \tag{2.10}$$

uniformly for $\frac{1}{2}\sigma_2(\varepsilon) \leq s \leq \sigma_2(\varepsilon)$. For a suitable choice of ν, σ_1, and σ_2 it is typically true that (2.10) suffices to relate the large s behavior of U_i to the small r behavior of u_i in such a way that both are determined uniquely.

We shall operate on the basis of these assumptions in the following way.

A. We substitute the assumed representation (2.6) into (1.1) and (1.2) to obtain the *outer* functions u_i.

B. We rewrite (2.6) in terms of the stretched variable $s = r/\varepsilon$ and substitute into (2.10) to obtain the gauge functions γ_i.

C. We substitute (2.7) into (1.1), rewritten in terms of s and φ, and apply (2.10) again to obtain the *inner* functions U_i.

Observe that if we set $s = \tau\sigma_2(\varepsilon)$, where $\frac{1}{2} \le \tau \le 1$, then (2.10) holds uniformly in ε and uniformly with respect to τ in that interval. Letting $\varepsilon \to 0$, we obtain various necessary relations connecting the asymptotic behavior of the functions $u_i(r)$ for small r with that of the functions $U_i(s)$ for large s. In this way we find a necessary condition on the location a_0 of the defect.

In our case it will turn out that if (1.8) is not satisfied, then

$$\gamma_0(\varepsilon) = 1, \qquad \gamma_1(\varepsilon) = \varepsilon \log \varepsilon, \qquad \gamma_2(\varepsilon) = \varepsilon,$$

and we find that for any $r_1 > 0$, $s_1 > 0$ there exists a constant $C > 0$ such that

$$|u_2(r)| \le \frac{C}{r^2} \quad \text{for } r < r_1 \quad \text{and} \quad |U_2(s)| \le Cs \quad \text{for } s > s_1.$$

This allows us to choose

$$\nu(\varepsilon) = \varepsilon^{2/3}, \qquad \sigma_1(\varepsilon) = \varepsilon^{2/3}, \qquad \sigma_2(\varepsilon) = 2\varepsilon^{-1/3}. \tag{2.11}$$

Once we have shown that (1.8) must be satisfied, we obtain

$$\gamma_0(\varepsilon) = 1 \quad \text{and} \quad \gamma_1(\varepsilon) = \varepsilon^2$$

and proceed as above.

The basic inner approximation $U_0(s, \varphi)$ will be found to satisfy (1.1) in all of $\mathbf{R}^2$ with $\varepsilon = 1$, to vanish at the origin, and to represent a mapping of degree 1 from each circle centered at the origin into $\mathbf{R}^2$. It is well known that there exists a solution with this property of the form $U(s, \varphi) = \rho(s)e^{i\varphi}$, in which ρ is real. In this paper we assume that U_0 is of this form. A similar assumption (see (4.13)) is made about the possible form of solutions of the linearization (4.12) of (1.1) (with $\varepsilon = 1$) about U_0 with certain prescribed behavior of infinity.

3. The outer expansion. For convenience we restate the problem

$$(\mathrm{I})\begin{cases} \varepsilon^2\Delta u + u(1 - |u|^2) = 0 & \text{in } \Omega_\varepsilon, & (3.1\text{a}) \\ u = ge^{i\alpha_\varepsilon} = e^{i(\chi+\alpha_\varepsilon)} & \text{on } \partial\Omega_\varepsilon, & (3.1\text{b}) \end{cases}$$

where $\Omega_\varepsilon = \Omega - a_\varepsilon$, and we seek to express u_ε, a_ε, and α_ε in the form

$$u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \tag{3.2a}$$

$$a_\varepsilon = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots, \tag{3.2b}$$

$$\alpha_\varepsilon = \alpha_0 + \varepsilon \alpha_1 + \varepsilon^2 \alpha_2 + \cdots. \tag{3.2c}$$

It will often be convenient to write u_ε in polar coordinates

$$u_\varepsilon(y) = \rho_\varepsilon(y)e^{i\theta_\varepsilon(y)}, \quad y \in \Omega_\varepsilon \tag{3.3}$$

and to expand ρ_ε and θ_ε as

$$\rho_\varepsilon = \rho_0 + \varepsilon\rho_1 + \varepsilon^2\rho_2 + \cdots, \tag{3.4a}$$

$$\theta_\varepsilon = \theta_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \cdots, \tag{3.4b}$$

where

$$\theta_0 = \varphi + \psi_0. \tag{3.5}$$

Then u_0, u_1, and u_2 can be written as

$$u_0 = \rho_0 e^{i\theta_0}, \tag{3.6a}$$

$$u_1 = \rho_1 e^{i\theta_0} + \rho_0 \psi_1 i e^{i\theta_0}, \tag{3.6b}$$

$$u_2 = (\rho_2 - \tfrac{1}{2}\rho_0\psi_1^2)e^{i\theta_0} + (\rho_0\psi_2 + \rho_1\psi_1)ie^{i\theta_0}. \tag{3.6c}$$

We now substitute (3.2) into (3.1a) and equate the coefficients of equal powers of ε. The terms of $O(1)$ yield

$$|u_0| = 1$$

so that by (3.6a), $\rho_0 = 1$ and we obtain

$$u_0 = e^{i\theta_0}. \tag{3.7}$$

The terms of $O(\varepsilon)$ yield

$$u_0 \cdot u_1 = 0$$

and hence, by (3.6b), $\rho_1 = 0$ and thus

$$u_1 = \psi_1 i e^{i\theta_0}. \tag{3.8}$$

The terms of $O(\varepsilon^2)$ yield the equation

$$\Delta u_0 - u_0(2u_0 \cdot u_2 + |u_1|^2) = 0 \tag{3.9}$$

because $u_0 \cdot u_1 = 0$. Since

$$\Delta u_0 = e^{i\theta_0}(i\Delta\theta_0 - |\nabla\theta_0|^2),$$

we can write (3.9) as

$$e^{i\theta_0}(i\Delta\theta_0 - |\nabla\theta_0|^2) = e^{i\theta_0}(2u_0 \cdot u_2 + \psi_1^2),$$

or, in components,
the $e^{i\theta_0}$*-component*:

$$u_0 \cdot u_2 = -\tfrac{1}{2}|\nabla\theta_0|^2 - \tfrac{1}{2}\psi_1^2,$$

the $ie^{i\theta_0}$*-component*:

$$\Delta\theta_0 = 0. \tag{3.10}$$

From (3.6c) we conclude that

$$\rho_2 = -\tfrac{1}{2}|\nabla\theta_0|^2. \tag{3.11}$$

Thus, we find for u_2,

$$u_2 = -\tfrac{1}{2}(|\nabla\theta_0|^2 + \psi_1^2)e^{i\theta_0} + \psi_2 i e^{i\theta_0}. \tag{3.12}$$

In (3.7), (3.8), and (3.12) we have expressed u_0, u_1, and u_2 in terms of ψ_0, ψ_1, and ψ_2. In what follows we derive some properties of these functions.

From (3.10) we conclude after shifting back over a_0 that

$$\Delta\psi_0(x) = 0 \quad \text{for all } x = y + a_0 \in \Omega\backslash\{a_0\} \tag{3.13a}$$

and that

$$\psi_0 = \chi + \alpha_0 - \varphi \quad \text{on } \partial\Omega. \tag{3.13b}$$

Because $\chi + \alpha_0 - \varphi$ is single valued on $\partial\Omega$, (3.13) may be uniquely solved by a smooth function ψ_0. The constant α_0 will now be chosen so that this smooth harmonic function satisfies

$$\psi_0(y) = 0 \quad \text{when } y = 0. \tag{3.14}$$

However, since (3.13a) is only required to hold for $x \neq a_0$, the regular function ψ_0 may be supplemented by a function p_0 that is harmonic, except for a singularity at a_0, and that vanishes on $\partial\Omega$. The singularity of p_0 must be of the form of a linear combination of $\log|x-a_0|$ and its derivatives. We therefore effect a notational change, replacing ψ_0 in (3.5) by $\psi_0 + p_0$. In due course we shall show that $p_0 = 0$.

We determine ψ_1 and ψ_2 from the equation (2.4b) for θ_ε,

$$\Delta\theta_\varepsilon + \frac{2}{\rho_\varepsilon}\nabla\rho_\varepsilon \cdot \nabla\theta_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon\backslash\{0\}. \tag{3.15}$$

Because we have found that

$$\rho_\varepsilon = 1 + \varepsilon^2\rho_2 + \cdots,$$

we conclude that

$$\Delta\psi_1 = 0 \qquad \text{in } \Omega_\varepsilon\backslash\{0\}, \tag{3.16}$$

$$\Delta\psi_2 = \nabla(|\nabla\theta_0|^2)\cdot\nabla\theta_0 \quad \text{in } \Omega_\varepsilon\backslash\{0\}. \tag{3.17}$$

As to the boundary condition for ψ_1, we write

$$\begin{aligned} u_\varepsilon(y) &= u_0(x - a_\varepsilon) + \varepsilon u_1(x - a_\varepsilon) + \cdots \\ &= u_0(x - a_0) + \varepsilon\{u_1(x - a_0) - a_1\cdot\nabla u_0(x - a_0)\} + \cdots, \end{aligned}$$

where a_1 was introduced in (3.2b). This yields the condition

$$u_1 = a_1\cdot\nabla u_0 + i\alpha_1 g e^{i\alpha_0} \quad \text{on } \partial(\Omega - a_0).$$

Thus, shifting back to the original domain Ω, and remembering (3.7), (3.8), and (3.13b), we find that ψ_1 satisfies

$$\begin{cases} \Delta\psi_1 = 0 & \text{in } \Omega\backslash\{a_0\}, \quad (3.18a) \\ \psi_1 = a_1\cdot\nabla\theta_0 + \alpha_1 & \text{on } \partial\Omega. \quad (3.18b) \end{cases}$$

As before we use the symbol ψ_1 to denote the solution of (3.18) that is regular in Ω so that the general solution of (3.18) is $\psi_1 + p_1$, where p_1 is harmonic except at $x = a_0$, where it will have a singularity in the form of a linear combination of $\log|x - a_0|$ and its derivatives. We therefore replace ψ_1 in (3.4b) by $\psi_1 + p_1$. As with ψ_0 we select the constant α_1 so that

$$\psi_1(y) = 0 \quad \text{when } y = 0. \tag{3.19}$$

Finally, we need to compute the behavior of ψ_2 near the origin in Ω_ε when $p_0 = 0$. We write $r = |y|$ and

$$e_r = (\cos\varphi, \sin\varphi) = e^{i\varphi} \quad \text{and} \quad e_\varphi = (-\sin\varphi, \cos\varphi) = ie^{i\varphi}.$$

We first compute the asymptotic behavior of the right-hand side of (3.17) as $r \to 0$.

LEMMA 3.1. Suppose $p_0 = 0$. Then

$$\nabla(|\nabla\theta_0|^2)\cdot\nabla\theta_0 = \frac{A}{r^3} + \frac{B}{r^2} + O\left(\frac{1}{r}\right) \quad \text{as } r \to 0,$$

in which

$$A = -4(\psi_{0x}\cos\varphi + \psi_{0y}\sin\varphi) = -4(\nabla\psi_0, e_r),$$

$$B = -\left\{4\psi_{0x}\psi_{0y} + (\psi_{0xx} - \psi_{0yy})\right\}\cos 2\varphi + 2\left\{(\psi_{0x}^2 - \psi_{0y}^2) - \psi_{0xy}\right\}\sin 2\varphi,$$

and the partial derivatives of ψ_0 are evaluated at the origin.

Proof. Recall that

$$\theta_0 = \varphi + \psi_0$$

and therefore

$$\nabla\theta_0 = \nabla\varphi + \nabla\psi_0 = \frac{1}{r}e_\varphi + \nabla\psi_0.$$

Hence

$$|\nabla\theta_0|^2 = \frac{1}{r^2} + \frac{2}{r}(\nabla\psi_0, e_\varphi) + |\nabla\psi_0|^2. \tag{3.20}$$

Thus

$$\nabla(|\nabla\theta_0|^2) = -\frac{2}{r^3}e_r - \frac{2}{r^2}(\nabla\psi_0, e_\varphi)e_r + \frac{2}{r}\nabla(\nabla\psi_0, e_\varphi) + \nabla(|\nabla\psi_0|^2) \tag{3.21}$$

and so

$$\begin{aligned}\nabla(|\nabla\theta_0|^2)\cdot\nabla\theta_0 = &-\frac{2}{r^3}(\nabla\psi_0, e_r) - \frac{2}{r^2}(\nabla\psi_0, e_\varphi)(\nabla\psi_0, e_r)\\ &+\frac{2}{r}\left(\nabla(\nabla\psi_0, e_\varphi), \nabla\psi_0\right) + \nabla(|\nabla\psi_0|^2)\cdot\nabla\psi_0\\ &+\frac{2}{r^2}\left(\nabla(\nabla\psi_0, e_\varphi), e_\varphi\right) + \frac{1}{r}\nabla(|\nabla\psi_0|^2)\cdot e_\varphi.\end{aligned} \tag{3.22}$$

But

$$\nabla(\nabla\psi_0, e_\varphi) = -\frac{1}{r}(\nabla\psi_0, e_r)e_\varphi + (-\nabla\psi_{0x}\sin\varphi + \nabla\psi_{0y}\cos\varphi),$$

so that we can write (3.22) as

$$\begin{aligned}\nabla(|\nabla\theta_0|^2)\cdot\nabla\theta_0 = &-\frac{4}{r^3}(\nabla\psi_0, e_r) - \frac{4}{r^2}(\nabla\psi_0, e_\varphi)(\nabla\psi_0, e_r)\\ &+\frac{2}{r^2}\left((-\nabla\psi_{0x}\sin\varphi + \nabla\psi_{0y}\cos\varphi), e_\varphi\right) + O\left(\frac{1}{r}\right).\end{aligned} \tag{3.23}$$

Since we can write

$$(\nabla\psi_0, e_\varphi)(\nabla\psi_0, e_r) = \psi_{0x}\psi_{0y}\cos 2\varphi - \tfrac{1}{2}(\psi_{0x}^2 - \psi_{0y}^2)\sin 2\varphi$$

and because ψ_0 is harmonic, we have

$$(-\nabla\psi_{0x}\sin\varphi + \nabla\psi_{0y}\cos\varphi)\cdot e_\varphi = -\tfrac{1}{2}(\psi_{0xx} - \psi_{0yy})\cos 2\varphi - \psi_{0xy}\sin 2\varphi,$$

and the desired expression readily follows in the case $p_0 = 0$.

It is interesting to note that the $O(r^{-3})$ term involves functions with period 2π in φ and the $O(r^{-2})$ term involves functions with period π in φ. In the following lemma we analyze the consequences of this behavior for solutions of (3.17) in a reduced neighborhood of the origin.

LEMMA 3.2. Let the functions u and v satisfy the equations

$$\Delta u = \frac{\sin\varphi}{r^3} \quad \text{and} \quad \Delta v = \frac{\sin 2\varphi}{r^2}$$

in a reduced neighborhood Σ of the origin. Then

$$u(r, \varphi) = -\frac{1}{2}\frac{\log r}{r}\sin\varphi + h_1(r, \varphi),$$

$$v(r, \varphi) = -\frac{1}{4}\sin 2\varphi + h_2(r, \varphi),$$

where h_1 and h_2 are harmonic functions in Σ.

The proof is given by direct computation.

From Lemmas 3.1 and 3.2 we deduce that ψ_2 has the following asymptotic behavior near the origin.

LEMMA 3.3. Suppose $p_0 = 0$. Then

$$\psi_2(r, \varphi) = 2\frac{\log r}{r}(\nabla\psi_0, e_r) + p_2(r, \varphi) + O(1) \quad \text{as } r \to 0,$$

where $\nabla\psi_0$ is evaluated at the origin and p_2 is harmonic in a reduced neighborhood of the origin.

Proof. Since the equation for ψ_2 is linear we can think of ψ_2 as composed of three terms, each one corresponding to one term on the right-hand side of (3.17) and a harmonic function p_2 in a reduced neighborhood of the origin. The contribution of the first two terms is given in Lemma 3.2. The third term on the right lies in L^p for any $p < 2$ and so by Sobolev's imbedding, its contribution is continuous in a neighborhood of the origin.

Thus, having determined u_0 in (3.7), u_1 in (3.8) and (3.18), and u_2 in (3.12), (3.18), and Lemma 3.3, we arrive at the following expansion when $p_0 = 0$, $p_1 = 0$, and $p_2 = 0$:

Outer expansion. We have

$$\begin{aligned} u_\varepsilon(r, \varphi) = e^{i\theta_0}&\left\{1 - \frac{1}{2}\varepsilon^2\left(\frac{1}{r^2} + \frac{2}{r}(\nabla\psi_0, e_\varphi) + |\nabla\psi_0|^2 + \psi_1^2\right) + O(\varepsilon^3)\right\} \\ &+ ie^{i\theta_0}\left\{\varepsilon\psi_1 + 2\varepsilon^2\frac{\log r}{r}(D\psi_0, e_r) + O(\varepsilon^2)\right\}, \end{aligned} \tag{3.24}$$

where $D\psi_0$ denotes the value of $\nabla\psi_0$ at the origin, and $O(\varepsilon^k)$, $k = 2, 3$, is understood to be uniform in a neighborhood of the origin.

When $p_1 \neq 0$ or $p_2 \neq 0$, the coefficients of ε and ε^2 in (3.24) also contain terms that are more singular as $r \to 0$ than those indicated.

4. The inner expansion. To obtain an expansion for the solution u_ε in a neighborhood of its zero, which we have shifted to the origin, we scale the spatial variable so that the factor $1/\varepsilon^2$ disappears from Eq. (1.1). We thus introduce the new variables

$$s = \frac{r}{\varepsilon} \quad \text{and} \quad U_\varepsilon(s, \varphi) = u_\varepsilon(r, \varphi). \tag{4.1}$$

Then U_ε will be a solution of the equation

$$\Delta U + U(1 - |U|^2) = 0 \quad \text{for } x/\varepsilon \in \Omega/\varepsilon \tag{4.2}$$

which has a zero at the origin

$$U(0) = 0 \tag{4.3}$$

with index 1.

As indicated in Sec. 2, we shall look for a solution in the form of the expansion

$$U_\varepsilon(s, \varphi) = \sum_{i=0}^{\infty} \gamma_i(\varepsilon) U_i(s, \varphi), \quad \gamma_0 = 1, \tag{4.4}$$

in which the real-valued coefficients $\gamma_1, \gamma_2, \dots$ have the property that

$$\gamma_1(\varepsilon) = o(1) \quad \text{and} \quad \gamma_{i+1}(\varepsilon) = o(\gamma_i(\varepsilon)) \quad \text{as } \varepsilon \to 0.$$

They will be determined when we match this *inner* expansion to the *outer* expansion obtained in the previous section.

When we substitute (4.4) into Eq. (4.2) and expand the domain to all of $\mathbf{R}^2$ according to (4.2), we find at once that U_0 must be a solution of the problem

$$\begin{cases} \Delta U + U(1 - |U|^2) = 0 \quad \text{in } \mathbf{R}^2, & \text{(4.5a)} \\ U(0) = 0, & \text{(4.5b)} \end{cases}$$

which has index 1. We assume that U_0 is of the form

$$U_0(s, \varphi) = \rho_0(s) e^{i\varphi}. \tag{4.6}$$

In that case, ρ_0 must solve the problem

$$\text{(II)} \begin{cases} \rho'' + \dfrac{1}{s}\rho' + \rho\left(1 - \dfrac{1}{s^2} - \rho^2\right) = 0, \quad \rho > 0 \quad \text{for } 0 < s < \infty, \\ \rho(0) = 0, \qquad \rho(\infty) = 1. \end{cases}$$

Problem (II) is known from the theory of spiral waves. We refer to [G, H], where the following results have been formulated.

LEMMA 4.1. Problem (II) has a unique solution ρ_0. The solution has the following properties:

(a) $$\rho_0'(s) > 0 \quad \text{and} \quad \rho_0(s) < 1 \quad \text{for all } s > 0;$$

(b) $$\rho_0(s) = As\left(1 - \tfrac{1}{8}s^2 + O(s^4)\right) \quad \text{as } s \to 0$$

for some positive constant A;

(c) $$\rho_0(s) = 1 - \frac{1}{2s^2} + O(s^{-4}) \quad \text{as } s \to \infty.$$

Since neither [G] nor [H] contains a proof of this Lemma we sketch one in the Appendix.

To obtain the coefficients $\gamma_1, \gamma_2, \dots$ we go back to the outer expansion, set $r = \varepsilon s$, and regroup the terms in ascending order of growth in ε.

LEMMA 4.2. We have

$$U_\varepsilon(s, \varphi) = \widehat{U}_0(s, \varphi) + \varepsilon \log \varepsilon \widehat{U}_1(s, \varphi) + \varepsilon \widehat{U}_2(s, \varphi) + O(\varepsilon^2 \log \varepsilon), \tag{4.7}$$

in which

$$\widehat{U}_0(s, \varphi) = \left(1 - \frac{1}{2s^2}\right) e_r, \tag{4.8a}$$

$$\widehat{U}_1(s, \varphi) = \frac{2}{s}(D\psi_0, e_r)e_\varphi, . \tag{4.8b}$$

$$\widehat{U}_2(s, \varphi) = \left(2\frac{\log s}{s} - \frac{1}{2s} + s\right)(D\psi_0, e_r)e_\varphi - \frac{1}{s}(D\psi_0, e_\varphi)e_r, \tag{4.8c}$$

where $D\psi_0$ is the gradient of ψ_0 evaluated at the origin.

Thus, from Lemma 4.2 we conclude that

$$\gamma_0 = 1, \quad \gamma_1 = \varepsilon \log \varepsilon, \quad \gamma_2 = \varepsilon, \quad \gamma_3 = \varepsilon^2 \log \varepsilon. \tag{4.9}$$

Proof. To begin with we assume that $p_0 = 0$, $p_1 = 0$, and $p_2 = 0$. When we then set $r = \varepsilon s$ in (3.24) we obtain

$$\begin{aligned} U_\varepsilon(s, \varphi) = e^{i\theta_0} \Big\{ 1 - \frac{1}{2s^2} &- \varepsilon(\nabla\psi_0(\varepsilon s, \varphi), e_\varphi)\frac{1}{s} \\ &- \frac{\varepsilon^2}{2}\left(|\nabla\psi_0(\varepsilon s, \varphi)|^2 + \psi_1^2(\varepsilon s, \varphi)\right) + O(\varepsilon^3)\Big\} \\ + ie^{i\theta_0}\Big\{ 2\varepsilon \log \varepsilon(D\psi_0, e_r)\frac{1}{s} &+ \varepsilon\psi_1(\varepsilon s, \varphi) \\ &+ 2\varepsilon(D\psi_0, e_r)\frac{\log s}{s} + O(\varepsilon^2)\Big\}. \end{aligned} \tag{4.10}$$

Next we expand ψ_0, ψ_1, and $\nabla\psi_0$ near the origin. Since by assumption $\psi_0(0) = 0$ and $\psi_1(0) = 0$ we have

$$\psi_i(x) = (D\psi_i, x) + \tfrac{1}{2}(x, D^2\psi_i x) + O(x^3) \quad \text{as } x \to 0 \quad \text{for } i = 0, 1,$$

where $D^2\psi_i$ is the Hessian of ψ_i evaluated at the origin. Thus, with $x = \varepsilon s e_r$ we obtain

$$\psi_i(x) = \varepsilon s(D\psi_i, e_r) + \tfrac{1}{2}\varepsilon^2 s^2(e_r, D^2\psi_i e_r) + O(\varepsilon^3 s^3), \quad i = 0, 1.$$

Similarly we obtain

$$(\nabla\psi_0(x), e_r) = (D\psi_0, e_r) + \varepsilon s(e_r, D^2\psi_0 e_r) + O(\varepsilon^2 s^2).$$

We now substitute the expansions for ψ_0, ψ_1, and $\nabla\psi_0$ into (4.10) and collect terms with the same growth in ε to obtain the desired expansion (4.7) with terms given by (4.8).

Lemma 4.2 yields the following matching conditions for U_0, U_1, and U_2 when $p_0 = 0$, $p_1 = 0$, and $p_2 = 0$:

$$U_0(s, \varphi) \sim \left(1 - \frac{1}{2s^2}\right) e_r \quad \text{as } s \to \infty, \tag{4.11a}$$

$$U_1(s, \varphi) \sim \frac{2}{s}(D\psi_0, e_r)e_\varphi \quad \text{as } s \to \infty, \tag{4.11b}$$

$$U_2(s, \varphi) \sim -\frac{1}{s}(D\psi_0, e_\varphi)e_r + \left(s + 2\frac{\log s}{s} - \frac{1}{2s}\right)(D\psi_0, e_r)e_\varphi \quad \text{as } s \to \infty. \tag{4.11c}$$

By Lemma 4.1b the condition (4.11a) is indeed satisfied by the function U_0 defined in (4.6).

We now turn to a discussion of the functions p_0, p_1, and p_2.

LEMMA 4.3. We have

$$p_0 = 0, \quad |p_1(r)| \le O(\log r) \quad \text{and} \quad |p_2(r)| \le O(r^{-1}) \quad \text{as } r \to 0.$$

Proof. We first show that $p_0 = 0$. Suppose, for example, that $p_0 = \log r$ (a similar argument holds for the other cases). Then in (3.24) the exponent $i\theta_0$ would have an extra term $i \log r = i(\log \varepsilon + \log s)$. This would introduce a factor $e^{i \log \varepsilon}$ in the $O(1)$ term of (2.7), that is, in the γ_0 term, which would now no longer be real valued. This contradicts our assumption that the coefficients γ_i be real.

Next we turn to p_1 and suppose for definiteness that $p_1 = r^{-1} \sin \varphi$. Then in (3.24) we obtain an extra term $\varepsilon r^{-1} \sin \varphi$ in the coefficient of $ie^{i\theta_0}$ and so in (4.8a) and in (4.11a) we obtain an extra term $s^{-1} \sin \varphi$. This term cannot be matched with U_0. The case that $p_1 = O(r^{-k})$, $k \ge 2$ can be handled similarly, although in this case the basic hypothesis that the coefficients γ_i be real is also violated.

When $p_1 = A \log r$, where A is a constant, we obtain two extra terms, $A\varepsilon \log \varepsilon$ and $A\varepsilon \log s$, in the coefficient of $e^{i\theta_0}$ in (4.10). They yield the following terms in (4.8):

$$Ae_\varphi \quad \text{in (4.8b)} \quad \text{and} \quad A \log s e_\varphi \quad \text{in (4.8c)},$$

leading to corresponding terms in the matching conditions (4.11b) and (4.11c).

The assertion about p_2 is established by similar arguments.

To obtain U_1 and U_2 we substitute (4.4) into Eq. (4.2) and equate the coefficient of $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$ to zero. We then find that both of them satisfy the equation

$$\Delta U - 2U_0(U_0 \cdot U) + U(1 - \rho_0^2) = 0. \tag{4.12}$$

From (4.11b) and (4.11c) it is clear that as $s \to \infty$, U_1 and U_2 have the asymptotic form

$$U(s, \varphi) = f(s)(D\psi_0, e_\varphi)e^{i\varphi} + g(s)(D\psi_0, e_r)ie^{i\varphi}. \tag{4.13}$$

We conjecture that every solution of (4.12), which solves (4.13) in that asymptotic sense, in fact solves it exactly for some real functions f and g. We show that solutions of this form exist.

LEMMA 4.4. Suppose U is given by (4.13) and $D\psi_0 \neq 0$. Then $f(s)$ and $g(s)$ are solutions of the system

$$\begin{cases} f'' + \frac{1}{s}f' - \frac{2}{s^2}f - \frac{2}{s^2}g + (1 - 3\rho_0^2)f = 0, & (4.14a) \\ g'' + \frac{1}{s}g' - \frac{2}{s^2}g - \frac{2}{s^2}f + (1 - \rho_0^2)g = 0, & (4.14b) \end{cases}$$

and vanish at the origin:

$$f(0) = 0 \quad \text{and} \quad g(0) = 0. \tag{4.15}$$

Proof. Since $U_0 = \rho_0 e^{i\varphi}$, we have

$$U_0 \cdot U = \rho_0(s)f(s)(D\psi_0, e_\varphi)$$

so that the equation for U becomes

$$\Delta U - 2\rho_0^2(s)f(s)(D\psi_0, e_\varphi)e^{i\varphi} + U(1 - \rho_0^2) = 0. \tag{4.16}$$

We next compute ΔU which we write as

$$\Delta U = U_{ss} + \frac{1}{s}U_s + \frac{1}{s^2}U_{\varphi\varphi}.$$

Since

$$(D\psi_0, e_\varphi)_\varphi = -(D\psi_0, e_r) \quad \text{and} \quad (D\psi_0, e_r)_\varphi = (D\psi_0, e_\varphi),$$

we obtain

$$U_\varphi = iU - f(s)(D\psi_0, e_r)e^{i\varphi} + g(s)(D\psi_0, e_\varphi)ie^{i\varphi},$$

and, when we differentiate once more,

$$U_{\varphi\varphi} = -2U - 2\{f(s)(D\psi_0, e_r)ie^{i\varphi} + g(s)(D\psi_0, e_\varphi)e^{i\varphi}\}.$$

Thus, we obtain

$$\begin{aligned} \Delta U = &\left(f'' + \frac{1}{s}f' - \frac{2}{s^2}f - \frac{2}{s^2}g\right)(D\psi_0, e_\varphi)e^{i\varphi} \\ &+ \left(g'' + \frac{1}{s}g' - \frac{2}{s^2}g - \frac{2}{s^2}f\right)(D\psi_0, e_r)ie^{i\varphi}. \end{aligned} \tag{4.17}$$

If we substitute (4.17) into (4.16) we obtain for the components, after dividing the first one by $(D\psi_0, e_\varphi)$ and the second one by $(D\psi_0, e_r)$, the required two differential equations for f and g.

The initial conditions for f and g follow at once from the fact that $U_1(0) = 0$ and $U_2(0) = 0$.

In Sec. 5 we shall show that the initial conditions (4.15) determine the solution pair (f, g) of (4.14) *uniquely* up to a multiplicative constant:

LEMMA 4.5. For every $\kappa \in \mathbf{R}$ there exists a unique solution (f_κ, g_κ) of (4.14) and (4.15) such that

$$f_\kappa(s) \sim \kappa s^2 \quad \text{and} \quad g_\kappa(s) \sim \kappa s^2 \quad \text{as } s \to 0.$$

It follows from Lemma 4.5 that the asymptotic behaviour of the functions $U_1(s, \varphi)$ and $U_2(s, \varphi)$ we have constructed must be similar. This is incompatible with the matching requirements set forth in (4.11b) and (4.11c), unless we have

$$D\psi_0 = 0, \tag{4.18}$$

and $p_1 = 0$ as well.

Having found that the zero a_ε of u_ε converges to a point where the limiting function ψ_0 has a vanishing gradient, it follows from Lemma 4.2 that the terms involving $\varepsilon \log \varepsilon$ and ε disappear. We thus need to start the process of finding an appropriate inner expansion anew.

As before we set $\gamma_0 = 1$, but we leave $\gamma_1 = \gamma(\varepsilon)$ still to be chosen so that we write

$$U(s, \varphi; \varepsilon) = U_0(s, \varphi) + \gamma(\varepsilon)U_1(s, \varphi) + \cdots$$

and again we take

$$U_0(s, \varphi) = \rho_0(s)e^{i\varphi}.$$

Then

$$\Delta U_1 - 2\rho_0^2(e^{i\varphi} \cdot U_1)e^{i\varphi} + U_1(1 - \rho_0^2) = 0. \tag{4.19}$$

We may write

$$U_1(s, \varphi) = F(s, \varphi)e^{i\varphi} + iG(s, \varphi)e^{i\varphi}, \tag{4.20}$$

in which F and G are real-valued functions. Then

$$\begin{aligned}\Delta U_1 = &\left\{F_{ss} + \frac{1}{s}F_s - \frac{1}{s^2}F + \frac{1}{s^2}F_{\varphi\varphi} - \frac{2}{s^2}G_\varphi\right\}e^{i\varphi} \\ &+ \left\{G_{ss} + \frac{1}{s}G_s - \frac{1}{s^2}G + \frac{1}{s^2}G_{\varphi\varphi} + \frac{2}{s^2}F_\varphi\right\}ie^{i\varphi},\end{aligned} \tag{4.21}$$

and we can write the $e^{i\varphi}$-component of (4.19) as

$$F_{ss} + \frac{1}{s}F_s - \frac{1}{s^2}F + \frac{1}{s^2}F_{\varphi\varphi} - \frac{2}{s^2}G_\varphi + (1 - 3\rho_0^2)F = 0 \tag{4.22a}$$

and the $ie^{i\varphi}$-components as

$$G_{ss} + \frac{1}{s}G_s - \frac{1}{s^2}G + \frac{1}{s^2}G_{\varphi\varphi} + \frac{2}{s^2}F_\varphi + (1 - \rho_0^2)G = 0. \tag{4.22b}$$

The system (4.22) can be reduced to a pair of ordinary differential equations by taking a Fourier series expansion in φ. The various modes decouple; so it suffices to consider F and G of the form

$$F(s, \varphi) = f(s)e^{in\varphi}, \qquad G(s, \varphi) = ig(s)e^{in\varphi}, \tag{4.23}$$

or rather the real parts of these expressions. We get

$$f'' + \frac{1}{s}f' - \frac{1+n^2}{s^2}f + \frac{2n}{s^2}g + (1 - 3\rho_0^2)f = 0, \tag{4.24a}$$

$$g'' + \frac{1}{s}g' - \frac{1+n^2}{s^2}g + \frac{2n}{s^2}f + (1 - \rho_0^2)g = 0. \tag{4.24b}$$

We can find the possible algebraic behavior of the solutions of (4.24) for large s by setting

$$f(s) = s^{\alpha}[1 + o(1)] \quad \text{and} \quad g(s) = ks^{\alpha+2}[1 + o(1)] \quad \text{as } s \to \infty. \tag{4.25}$$

When we substitute these expressions into (4.24) and use the asymptotic behavior of $\rho_0(s)$, we find that α and k must satisfy

$$2(kn - 1) = 0 \quad \text{and} \quad k[(\alpha + 2)^2 - n^2] = 0,$$

so that

$$\alpha = \pm n - 2 \quad \text{and} \quad k = \frac{1}{n}. \tag{4.26}$$

For example, when $n = 2$ we must have $\alpha = 0$ or $\alpha = -4$, and $k = \frac{1}{2}$.

There are other exponential-type solutions as well.

The behavior of all solutions as $s \to 0$ can be found in a similar way by setting

$$f(s) = s^{\alpha}[1 + o(1)] \quad \text{and} \quad g(s) = ks^{\alpha}[1 + o(1)] \quad \text{as } s \to 0.$$

We then find

$$\alpha^2 = 1 + n^2 - 2nk \quad \text{and} \quad k = \pm 1 \tag{4.27}$$

so that

$$\alpha = \pm(1 + n) \quad \text{and} \quad k = -1 \quad \text{or} \quad \alpha = \pm(1 - n) \quad \text{and} \quad k = 1. \tag{4.28}$$

The only solutions $f(s)$ and $g(s)$ that vanish at least like s^n at the origin are therefore

$$f = As^{n+1} + \cdots \quad \text{and} \quad g = -As^{n+1} + \cdots.$$

We are now ready to match the inner solution to the outer solution given by (3.24), in which we have put $D\psi_0 = 0$. Setting $r = \varepsilon s$ in (3.24) we find that we must take

$$\gamma_1(\varepsilon) = \varepsilon^2$$

and that the matching conditions become

$$U_0(s, \varphi) \sim \left(1 - \frac{1}{2s^2}\right) e_r \quad \text{as } s \to \infty, \tag{4.29}$$

$$U_1(s, \varphi) \sim -(e_r, D^2\psi_0 e_\varphi)e_r + \tfrac{1}{2}s^2(e_r, D^2\psi_0 e_r)e_\varphi \quad \text{as } s \to \infty. \tag{4.30}$$

This is consistent with our assumption that

$$U_0(s, \varphi) = \rho_0(s)e^{i\varphi}.$$

As regards U_1 we note that

$$(e_r, D^2\psi_0 e_\varphi) = \lambda e^{2i\varphi} \quad \text{and} \quad (e_r, D^2\psi_0 e_r) = -\lambda i e^{2i\varphi},$$

where $\lambda = (\psi_{0xy}(0), -\psi_{0xx}(0))$. Thus we must set $n = 2$ in (4.23) and require that

$$f(s) \to -\lambda \quad \text{and} \quad g(s) \to -\tfrac{1}{2}\lambda s^2 \quad \text{as } s \to \infty.$$

This means that we must choose $\alpha = 0$ and $k = \frac{1}{2}$, which is entirely consistent with the relations between α, n, and k given in (4.26).

5. Proof of Lemma 4.5. In this section we shall establish some properties of the solution (f, g) of the system

$$\begin{cases} f'' + \frac{1}{s}f' - \frac{2}{s^2}f - \frac{2}{s^2}g + (1 - 3\rho_0^2)f = 0, & \text{(4.14a)} \\ g'' + \frac{1}{s}g' - \frac{2}{s^2}g - \frac{2}{s^2}f + (1 - \rho_0^2)g = 0. & \text{(4.14b)} \end{cases}$$

Specifically, we shall show that apart from an arbitrary constant factor, there exists precisely one solution (f, g) of (4.14) such that

$$f(0) = 0 \quad \text{and} \quad g(0) = 0. \tag{4.15}$$

We begin with a few preliminary estimates.

LEMMA 5.1. Suppose (f, g) is a solution of (4.14) that satisfies (4.15). Then

$$f(s), g(s) = O(s^2) \quad \text{and} \quad f(s) - g(s) = O(s^6) \quad \text{as } s \to 0.$$

Proof. We write

$$u = f + g, \quad v = f - g, \quad \text{and } t = -\log s. \tag{5.1}$$

Then (4.14) becomes

$$u'' - 4u = -h_1(t) \quad \text{and} \quad v'' = -h_2(t), \tag{5.2}$$

where

$$h_1(t) = e^{-2t}\{u - \rho_0^2(3f + g)\} \quad \text{and} \quad h_2(t) = e^{-2t}\{v - \rho_0^2(3f - g)\}. \tag{5.3}$$

In view of condition (4.15), we have $h_i(t) = O(e^{-2t})$ $(i = 1, 2)$ as $t \to \infty$. This implies by an elementary argument that

$$u(t) = O(te^{-2t}) \quad \text{and} \quad v(t) = O(e^{-2t}) \quad \text{as } t \to \infty.$$

Using these bounds to sharpen the asymptotic estimates of $h_1(t)$ and $h_2(t)$, we eventually find that

$$u(t) = O(e^{-2t}) \quad \text{and} \quad v(t) = O(e^{-6t}) \quad \text{as } t \to \infty,$$

which yields the desired behavior of $f(s)$ and $g(s)$ as $s \to 0$.

The estimates derived in Lemma 5.1 allow us to use the variation of constants formula to write (4.14) as an integral equation.

To simplify the notation we write $w = \text{col}(f, g)$. Then (4.14) becomes

$$w'' + \frac{1}{s}w' - \frac{1}{s^2}Aw = -B(s)w, \tag{5.4}$$

where

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad B(s) = \begin{pmatrix} 1 - 3\rho_0^2(s) & 0 \\ 0 & 1 - \rho_0^2(s) \end{pmatrix}. \tag{5.5}$$

One readily finds for the eigenfunctions $\varphi_i(s)\eta_i$ $(i = 1, 2, 3, 4)$ of the left-hand side of (5.4):

$$\varphi_1(s) = s^2, \quad \varphi_2(s) = s^{-2}, \quad \varphi_3(s) = \log s, \quad \varphi_4(s) = 1 \tag{5.6a}$$

and

$$\eta_1 = \eta_2 = \mathrm{col}(1\,,\,1)\,, \qquad \eta_3 = \eta_4 = \mathrm{col}(1\,,\,-1). \tag{5.6b}$$

The variation of constant formula now becomes

$$w(s) = \left\{\lambda s^2 + s^2 J_1(s) + \frac{1}{s^2} J_2(s)\right\}\eta_1 + \{\log s\, J_3(s) + J_4(s)\}\eta_3\,, \tag{5.7}$$

where

$$J_1(s) = -\frac{1}{8}\int_0^s \eta_1 B(\sigma) w(\sigma)\,\frac{d\sigma}{\sigma}\,, \tag{5.8a}$$

$$J_2(s) = +\frac{1}{8}\int_0^s \eta_1 B(\sigma) w(\sigma)\sigma^3\,d\sigma\,, \tag{5.8b}$$

$$J_3(s) = -\frac{1}{2}\int_0^s \eta_3 B(\sigma) w(\sigma)\sigma\,d\sigma\,, \tag{5.8c}$$

$$J_4(s) = +\frac{1}{2}\int_0^s \eta_3 B(\sigma) w(\sigma)\sigma\log\sigma\,d\sigma. \tag{5.8d}$$

This integral equation allows us to obtain the following asymptotic estimate.

Lemma 5.2. We have

$$w(s) = \left\{\lambda s^2 + O(s^4)\right\}\eta_1 + O(s^6)\eta_3 \quad \text{as } s \to 0. \tag{5.9}$$

Regarding existence and uniqueness we obtain

Theorem 5.3. For every $\kappa \in \mathbf{R}$ there exists a unique solution $(f_\kappa\,,\, g_\kappa)$ of (4.14) such that

$$f_\kappa(s) \sim \kappa s^2 \quad \text{and} \quad g_\kappa(s) \sim \kappa s^2 \quad \text{as } s \to 0. \tag{5.10}$$

Proof. The origin is a regular singular point for (4.14) and the proof follows by standard construction of a convergent power series.

Appendix. In this Appendix we establish the existence and uniqueness of symmetric solutions of Problem (4.5) in $\mathbf{R}^2$ of the form

$$U(s\,,\,\varphi) = \rho(s)e^{iN\varphi}\,,$$

in which N is a positive integer and represents the index of the solution, which vanishes at the origin only. In Sec. 4, we introduced such a solution U_0 with $N = 1$. Substitution yields for ρ the two-point boundary-value problem

$$\text{(II)}\begin{cases} \rho'' + \dfrac{1}{s}\rho' + \rho\left(1 - \dfrac{N^2}{s^2} - \rho^2\right) = 0\,, \quad \rho > 0 \quad \text{for } 0 < s < \infty\,, & \text{(A.1a)} \\ \rho(0) = 0\,, \quad \rho(\infty) = 1. & \text{(A.1b)} \end{cases}$$

The existence and uniqueness of this solution ρ_0 is well known and can be established by methods used in [G] for a closely related, but different equation. For completeness, we give here the proof for Eq. (A.1a).

THEOREM A.1. Problem (II) has a unique solution ρ_0. This solution has the following properties:

(a) $$\rho_0'(s) > 0 \quad \text{and} \quad \rho_0(s) < 1 \quad \text{for all } s > 0;$$

(b) $$\rho_0(s) = As^N \left(1 - \frac{1}{4(N+1)}s^2 + O(s^4)\right) \quad \text{as } s \to 0$$

for some positive constant A;

(c) $$\rho_0(s) = 1 - \frac{N^2}{2s^2} + O(s^{-4}) \quad \text{as } s \to \infty.$$

If ρ_0 is a solution of Problem (II), then the properties (a)–(c) are easily established. To prove the existence and uniqueness of a solution of Problem (II), it is convenient to introduce the new dependent variable $v(s) = s^{-N}\rho(s)$. We then look for a solution of

$$(\mathrm{II}')\begin{cases} v'' + \dfrac{2N+1}{s}v' + v(1 - s^{2N}v^2) = 0, \quad v > 0 \quad \text{for } 0 < s < \infty, & \text{(A.2a)} \\ v(0) = A, \quad v'(0) = 0, & \text{(A.2b)} \end{cases}$$

which has the property that

$$s^N v(s) \to 1 \quad \text{as } s \to \infty \tag{A.2c}$$

for some appropriate value A_0 of A. It is readily established that for each $A > 0$ there exists a unique solution $v(s, A)$ in a neighborhood of $s = 0$.

We begin with a monotonicity property.

LEMMA A.2. We have

$$A_1 < A_2 \Rightarrow v(s, A_1) < v(s, A_2)$$

as long as these solutions exist and are positive.

Proof. For convenience we write $v(s, A_i) = v_i(s)$ $(i = 1, 2)$. Suppose that there exists a first $s_0 > 0$ where $v_1 = v_2$. Then if we multiply the equation for v_1 by $s^{2N+1}v_2'$ and the equation for v_2 by $s^{2N+1}v_1'$, subtract and integrate over $(0, s_0)$, we obtain

$$v_1(s_0)s_0^{2N+1}\{v_2'(s_0) - v_1'(s_0)\} = \int_0^{s_0} s^{4N+1}v_1v_2(v_2^2 - v_1^2)\,ds. \tag{A.3}$$

Since $v_2 > v_1 > 0$ on $[0, s_0)$, the right-hand side is positive and the left-hand side is negative, so that we have a contradiction.

Existence can now be proved by means of a shooting technique. One defines the sets

$$\mathscr{S}^+ = \{A > 0 \colon v(s_0, A) > s_0^{-N} \text{ for some } s_0 > 0\},$$
$$\mathscr{S}^- = \{A > 0 \colon v(s_1, A) = 0 \text{ for some } s_1 > 0\};$$

note that if $A \in \mathscr{S}^-$, then $v(s, A) < s^{-N}$ on $(0, s_1)$. One shows that $A \in \mathscr{S}^-$ if A is sufficiently small and that $A \in \mathscr{S}^+$ if A is sufficiently large. This can be done by means of some elementary estimates. These sets are clearly disjoint and open, so that there must exist an A_0 that does not belong to either $\mathscr{S}^+$ or $\mathscr{S}^-$. Thus we have

$$0 < v(s, A_0) \leq s^{-N} \quad \text{for all } s > 0, \tag{A.4}$$

and it remains to show that $v(s, A_0)$ satisfies (A.2c), i.e.,

$$\rho(s, A_0) = s^N v(s, A_0) \to 1 \quad \text{as } s \to \infty.$$

Since $0 < \rho(s, A_0) \leq 1$ by (A.4) it follows that $\rho' > 0$ and so $\rho(s, A_0)$ must tend to a limit, which can only be 1.

This completes the proof of existence.

From the monotonicity with respect to A we deduce that $\mathscr{S}^- = (0, A^-)$ and $\mathscr{S}^+ = (A^+, \infty)$ for some $0 < A^- \leq A^+$ and it remains to prove that $A^- = A^+$ to establish the uniqueness. This is done by means of an argument similar to that used in the proof of Lemma A.1.

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