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VARIATIONAL FORMULATIONS FOR THE VIBRATION OF A PIEZOELECTRIC BODY

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Abstract. This paper presents a systematic discussion on the variational principles for the vibration of a piezoelectric body. It is shown that there exist four types of variational formulations depending on the internal energy, electric enthalpy, mechanical enthalpy, and enthalpy, respectively. The one depending on the internal energy is in a positive-definite form which immediately leads to a few important properties of the lowest resonant frequency.

1. Introduction. In classical elasticity, there are two types of variational principles for the free vibration of an elastic body. One is associated with potential energy, the other with complementary energy [1-3]. A variational formulation for the free vibration of a piezoelectric body is given in [4], which is related to the electric enthalpy. In this paper, three other variational formulations are given for the free vibration of a piezoelectric body which are related to the internal energy, mechanical enthalpy, and enthalpy, respectively. The one depending on the electric enthalpy in [4] is presented in a more general form in this paper. These variational principles can be considered as generalizations of the corresponding variational formulations in classical elasticity. Because of the presence of the electric fields, there can be four generalizations for the two formulations in classical elasticity. They each have a different set of independent arguments, which allow different but equivalent formulations of the same eigenvalue problem. The variational principles are given without constraints. They can be reduced to various constraint variational principles. The constraint internal energy formulation is in a positive-definite form, which can be used to show a few properties of the lowest resonant frequency.

2. The eigenvalue problem. Let the region occupied by the piezoelectric body be V, the boundary surface of V be S, the unit outward normal of S be n_i, and S be partitioned in the following way:

$$\begin{aligned} S_u \cup S_T = S_\phi \cup S_D = S, \\ S_u \cap S_T = S_\phi \cap S_D = \varnothing. \end{aligned} \tag{1}$$

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Then the eigenvalue problem for the resonance of a linear piezoelectric body is [5]

$$\begin{aligned}
&-T_{ji,j} = \rho\omega^2 u_i \quad \text{in } V,\\
&-D_{i,i} = 0 \quad \text{in } V,\\
&-S_{ij} + \tfrac{1}{2}(u_{i,j} + u_{j,i}) = 0 \quad \text{in } V,\\
&E_i + \phi_{,i} = 0 \quad \text{in } V,\\
&-T_{ij} + \frac{\partial H}{\partial S_{ij}} = 0 \quad \text{in } V,\\
&D_i + \frac{\partial H}{\partial E_i} = 0 \quad \text{in } V,\\
&-u_i = 0 \quad \text{on } S_u,\\
&T_{ji}n_j = 0 \quad \text{on } S_T,\\
&-\phi = 0 \quad \text{on } S_\phi,\\
&D_i n_i = 0 \quad \text{on } S_D,
\end{aligned} \tag{2}$$

where u_i is displacement, S_{ij} strain, T_{ji} stress, ϕ electric potential, E_i electric field, D_i electric displacement, ρ mass density, and ω resonant frequency. $H = H(S_{ij}, E_i)$ is the electric enthalpy function for the piezoelectric material. Since we are only considering linear materials, H and the corresponding linear constitutive relations assume the following form:

$$H = \tfrac{1}{2}c_{ijkl}S_{ij}S_{kl} - \tfrac{1}{2}\varepsilon_{ij}E_iE_j - e_{ijk}E_iS_{jk}, \tag{3}$$

$$\begin{aligned}
T_{ij} &= c_{ijkl}S_{kl} - e_{kij}E_k,\\
D_i &= \varepsilon_{ij}E_j + e_{ijk}S_{jk},
\end{aligned} \tag{4}$$

where the elastic moduli c_{ijkl}, electric permitivity ε_{ij}, and piezoelectric constants e_{ijk} are material constants with the following symmetry properties:

$$\begin{aligned}
&c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij},\\
&e_{ijk} = e_{ikj}, \quad \varepsilon_{ij} = \varepsilon_{ji},
\end{aligned} \tag{5}$$

and for any nonzero symmetric tensor a_{ij} and vector b_i

$$c_{ijkl}a_{ij}a_{kj} > 0, \quad \varepsilon_{ij}b_ib_j > 0. \tag{6}$$

Given ρ and H, values of ω^2 are sought corresponding to which nontrivial solutions of u_i, S_{ij}, T_{ij}, ϕ, E_i, and D_i exist.

3. The electric enthalpy $H(S_{ij}, E_i)$ formulation. We first give the formulation involving the electric enthalpy $H(S_{ij}, E_i)$ in a form that generalizes what is in [4]. Let

$$\Lambda_1(u_i, S_{ij}, T_{ij}, \phi, E_i, D_i) = \int_V [T_{ij}u_{i,j} + D_i\phi_{,i} + H(S_{ij}, E_i) + E_iD_i - T_{ij}S_{ij}]dV - \int_{S_u} T_{ji}n_j u_j\, dS - \int_{S_\phi} D_in_i\phi\, dS,$$

$$\Gamma_1(u_i) = \int_V \frac{1}{2}\rho u_i u_i dV,$$

$$\Pi_1(u_i, S_{ij}, T_{ij}, \phi, E_i, D_i) = \frac{\Lambda_1(u_i, S_{ij}, T_{ij}, \phi, E_i, D_i)}{\Gamma_1(u_i)}. \tag{7}$$

To obtain the stationary conditions of Π_1 with all its arguments as independent variables, we begin with

$$\begin{aligned}\delta\Pi_1 &= \frac{1}{\Gamma_1^2}(\Gamma_1\delta\Lambda_1 - \Lambda_1\delta\Gamma_1) \\ &= \frac{1}{\Gamma_1}\left(\delta\Lambda_1 - \frac{\Lambda_1}{\Gamma_1}\delta\Gamma_1\right).\end{aligned} \tag{8}$$

Therefore $\delta\Pi_1 = 0$ implies

$$\delta\Lambda_1 - \frac{\Lambda_1}{\Gamma_1}\delta\Gamma_1 = 0. \tag{9}$$

With integration by parts, we can obtain

$$\begin{aligned}\delta\Lambda_1 = &\int_V \left\{-T_{ji,j}\delta u_i - D_{i,i}\delta\phi + \left(\frac{\partial H}{\partial S_{ij}} - T_{ij}\right)\delta S_{ij} + \left(\frac{\partial H}{\partial E_i} + D_i\right)\delta E_i \right.\\ &\left. + \left[\frac{1}{2}(u_{i,j} + u_{j,i}) - S_{ij}\right]\delta T_{ij} + (E_i + \phi_{,i})\delta D_i\right\} dV \\ &- \int_{S_u} u_i\delta T_{ji}n_j\, dS + \int_{S_T} T_{ji}n_j\delta u_i\, dS - \int_{S_\phi} \phi\delta D_in_i\, dS + \int_{S_D} D_in_i\delta\phi\, dS,\end{aligned} \tag{10}$$

$$\delta\Gamma_1 = \int_V \rho u_i\delta u_i\, dV.$$

Since all the variations of $\delta u_i, \delta S_{ij}$, δT_{ij}, $\delta\phi$, δE_i, and δD_i are independent, $\delta\Pi_1 = 0$ implies

$$
\begin{aligned}
&-T_{ji,j} = \rho\frac{\Lambda_1}{\Gamma_1}u_i \quad \text{in } V, \\
&-D_{i,i} = 0 \quad \text{in } V, \\
&-S_{ij} + \tfrac{1}{2}(u_{i,j} + u_{j,i}) = 0 \quad \text{in } V, \\
&E_i + \phi_{,i} = 0, \\
&-T_{ij} + \frac{\partial H}{\partial S_{ij}} = 0 \quad \text{in } V, \\
&D_i + \frac{\partial H}{\partial E_i} = 0 \quad \text{in } V, \\
&-u_i = 0 \quad \text{on } S_u, \\
&T_{ji}n_j = 0 \quad \text{on } S_T, \\
&-\phi = 0 \quad \text{on } S_\phi, \\
&D_i n_i = 0 \quad \text{on } S_D.
\end{aligned} \tag{11}
$$

Comparing (11) to (2), we have the following variational principle: the stationary condition of Π_1 gives the eigenvalue problem (2), with the stationary value of Π_1 as ω^2.

The above variational principle has no constraints. If we choose our admissible functions to satisfy

$$
\begin{aligned}
-S_{ij} + \tfrac{1}{2}(u_{i,j} + u_{j,i}) &= 0 \quad \text{in } V, \\
E_i + \phi_{,i} &= 0, \\
-T_{ij} + \frac{\partial H}{\partial S_{ij}} &= 0 \quad \text{in } V, \\
D_i + \frac{\partial H}{\partial E_i} &= 0 \quad \text{in } V, \\
-u_i &= 0 \quad \text{on } S_u, \\
-\phi &= 0 \quad \text{on } S_\phi,
\end{aligned} \tag{12}
$$

then Π_1 reduces to

$$
\Pi_1 = \frac{\int_V H dV}{\int_V \frac{1}{2}\rho u_i u_i dV}, \tag{13}
$$

and the stationary condition of Π_1 becomes

$$
\begin{aligned}
&-T_{ji,j} = \rho\frac{\Lambda_1}{\Gamma_1}u_i \quad \text{in } V, \\
&-D_{i,i} = 0 \quad \text{in } V, \\
&T_{ji}n_j = 0 \quad \text{on } S_T, \\
&D_i n_i = 0 \quad \text{on } S_D.
\end{aligned} \tag{14}
$$

This constraint version of Λ_1 is equivalent to what is given in [4]. The above H formulation can be considered as a generalization of the potential energy formulation for the vibration problem in classical elasticity.

4. The mechanical enthalpy $M(T_{ij}, D_i)$ formulation. The mechanical enthalpy M can be introduced through Legendre transform from H as

$$M = M(T_{ij}, D_i) = H + E_i D_i - T_{ij} S_{ij}, \tag{15}$$

which generates the following constitutive relations:

$$S_{ij} = -\frac{\partial M}{\partial T_{ij}}, \quad E_i = \frac{\partial M}{\partial D_i}. \tag{16}$$

For this formulation, we need to introduce (when $\omega \neq 0$)

$$a_i = -\omega^2 u_i, \quad \psi = -\omega^2 \phi. \tag{17}$$

We note that the physical meaning of a_i is acceleration. Let

$$\begin{aligned}
\Lambda_2(a_i, T_{ij}, \psi, D_i) &= \int_V (-T_{ji,j} a_i + \tfrac{1}{2}\rho a_i a_i - D_{i,i}\psi) dV + \int_{S_T} T_{ji} n_j a_i \, dS \\
&\quad + \int_{S_D} D_i n_i \psi \, dS, \\
\Gamma_2(T_{ij}, D_i) &= \int_V M(T_{ij}, D_i) dV, \\
\Pi_2(a_i, T_{ij}, \psi, D_i) &= \frac{\Lambda_2(a_i, T_{ij}, \psi, D_i)}{\Gamma_2(T_{ij}, D_i)},
\end{aligned} \tag{18}$$

where independent arguments are now a_i, T_{ij}, ψ, and D_i. Since

$$\begin{aligned}
\delta\Lambda_2 &= \int_V [(-T_{ji,j} + \rho a_i)\delta a_i - D_{i,i}\delta\psi + \tfrac{1}{2}(a_{i,j} + a_{j,i})\delta T_{ij} + \psi_{,i}\delta D_i] dV \\
&\quad - \int_{S_u} a_i \delta T_{ji} n_j \, dS + \int_{S_T} T_{ji} n_j \delta a_i dS - \int_{S_\phi} \psi \delta D_i n_i dS + \int_{S_D} D_i n_i \delta\psi dS,
\end{aligned} \tag{19}$$

$$\delta\Gamma_2 = \int_V \left(\frac{\partial M}{\partial T_{ij}} \delta T_{ij} + \frac{\partial M}{\partial D_i} \delta D_i \right) dV,$$

$\delta\Pi_2 = 0$ implies

$$\begin{aligned}
&- T_{ji,j} + \rho a_i = 0 \quad \text{in } V, \\
&- D_{i,i} = 0 \quad \text{in } V, \\
&\tfrac{1}{2}(a_{i,j} + a_{j,i}) = \frac{\Lambda_2}{\Gamma_2} \frac{\partial M}{\partial T_{ij}} \quad \text{in } V, \\
&\psi_{,i} = \frac{\Lambda_2}{\Gamma_2} \frac{\partial M}{\partial D_i} \quad \text{in } V, \\
&- a_i = 0 \quad \text{on } S_u, \\
&T_{ji} n_j = 0 \quad \text{on } S_T, \\
&- \psi = 0 \quad \text{on } S_\phi, \\
&D_i n_i = 0 \quad \text{on } S_D,
\end{aligned} \tag{20}$$

which is an equivalent system of the original eigenvalue problem (2) (when $\omega \neq 0$). Equations $(20)_{3,4}$ can be obtained by multiplying both sides of (16) by ω^2 and by substituting $(2)_{3,4}$ and (17). Hence, the stationary condition of the functional Π_2 gives the eigenvalue problem (20) with the stationary value of Π_2 as ω^2. If we choose the admissible functions to satisfy

$$\begin{aligned} -T_{ji,j} + \rho a_i &= 0 \quad \text{in } V, \\ -D_{i,i} &= 0 \quad \text{in } V, \\ T_{ji} n_j &= 0 \quad \text{on } S_T, \\ D_i n_i &= 0 \quad \text{on } S_D, \end{aligned} \tag{21}$$

then Π_2 reduces to

$$\Pi_2 = \frac{\int_V -\frac{1}{2}\rho a_i a_i \, dV}{\int_V M dV}, \tag{22}$$

and the stationary condition of Π_2 under constraints (21) gives

$$\begin{aligned} &\tfrac{1}{2}(a_{i,j} + a_{j,i}) = \frac{\Lambda_2}{\Gamma_2}\frac{\partial M}{\partial T_{ij}} \quad \text{in } V, \\ &\psi_{,i} = \frac{\Lambda_2}{\Gamma_2}\frac{\partial M}{\partial D_i} \quad \text{in } V, \\ &-a_i = 0 \quad \text{on } S_u, \\ &-\psi = 0 \quad \text{on } S_\phi, \end{aligned} \tag{23}$$

with the stationary value of Π_2 as ω^2.

This M formulation can be considered as a generalization of the complementary energy formulation for the vibration problem in classical elasticity.

5. The enthalpy $G(T_{ij}, E_i)$ formulation. The enthalpy G can be obtained from H as

$$G = G(T_{ij}, E_i) = H - T_{ij} S_{ij}, \tag{24}$$

which generates the following constitutive relations:

$$S_{ij} = -\frac{\partial G}{\partial T_{ij}}, \qquad D_i = -\frac{\partial G}{\partial E_i}. \tag{25}$$

We introduce

$$a_i = -\omega^2 u_i, \qquad \mathscr{D}_i = -\omega^2 D_i, \tag{26}$$

and let

$$\begin{aligned} \Lambda_3(a_i, T_{ij}, \phi, E_i, \mathscr{D}_i) &= \int_V [-T_{ji,j} a_i + \tfrac{1}{2}\rho a_i a_i + \mathscr{D}_i \phi_{,i} + \mathscr{D}_i E_i] dV \\ &\quad + \int_{S_T} T_{ji} n_j a_i \, dS - \int_{S_\phi} \mathscr{D}_i n_i \phi \, dS, \\ \Gamma_3(T_{ij}, E_i) &= \int_V G(T_{ij}, E_i) dV, \\ \Pi_3(a_i, T_{ij}, \phi, E_i, \mathscr{D}_i) &= \frac{\Lambda_3(a_i, T_{ij}, \phi, E_i, \mathscr{D}_i)}{\Gamma_3(T_{ij}, E_i)}. \end{aligned} \tag{27}$$

We have

$$\begin{aligned}\delta\Lambda_3 &= \int_V [(-T_{ji,j}+\rho a_i)\delta a_i - \mathscr{D}_{i,i}\delta\phi + \tfrac{1}{2}(a_{i,j}+a_{j,i})\delta T_{ij} + (E_i+\phi_{,i})\delta\mathscr{D}_i + \mathscr{D}_i\delta E_i]dV \\ &\quad - \int_{S_u} a_i\delta T_{ji}n_j\,dS + \int_{S_T} T_{ji}n_j\,\delta a_i\,dS - \int_{S_\phi} \phi\delta\mathscr{D}_i n_i dS + \int_{S_D} \mathscr{D}_i n_i \delta\phi\,dS\,, \\ \delta\Gamma_3 &= \int_V \left(\frac{\partial G}{\partial T_{ij}}\delta T_{ij} + \frac{\partial G}{\partial E_i}\delta E_i\right) dV\,.\end{aligned} \tag{28}$$

Hence $\delta\Pi_3 = 0$ implies

$$\begin{aligned} &-T_{ji,j} + \rho a_i = 0 \quad \text{in } V\,, \\ &-\mathscr{D}_{i,i} = 0 \quad \text{in } V\,, \\ &E_i + \phi_{,i} = 0 \quad \text{in } V\,, \\ &\tfrac{1}{2}(a_{i,j}+a_{j,i}) = \frac{\Lambda_3}{\Gamma_3}\frac{\partial G}{\partial T_{ij}} \quad \text{in } V\,, \\ &\mathscr{D}_i = \frac{\Lambda_3}{\Gamma_3}\frac{\partial G}{\partial E_i} \quad \text{in } V\,, \\ &-a_i = 0 \quad \text{on } S_u\,, \\ &T_{ji}n_j = 0 \quad \text{on } S_T\,, \\ &-\phi = 0 \quad \text{on } S_\phi\,, \\ &\mathscr{D}_i n_i = 0 \quad \text{on } S_D\,, \end{aligned} \tag{29}$$

which is another equivalent system of the original eigenvalue problem (2). Hence the stationary condition of Π_3 gives the eigenvalue problem (29), with the stationary value of Π_3 as ω^2. If we choose our admissible functions to satisfy

$$\begin{aligned} -T_{ji,j} + \rho a_i &= 0 \quad \text{in } V\,, \\ E_i + \phi_{,i} &= 0\,, \\ T_{ji}n_j &= 0 \quad \text{on } S_T\,, \\ -\phi &= 0 \quad \text{on } S_\phi\,, \end{aligned} \tag{30}$$

then Π_3 reduces to

$$\Pi_3 = \frac{\int_V -\frac{1}{2}\rho a_i a_i dV}{\int_V G dV}\,, \tag{31}$$

and the stationary condition of Π_3 becomes

$$\begin{aligned} &-\mathscr{D}_{i,i} = 0 \quad \text{in } V\,, \\ &\tfrac{1}{2}(a_{i,j}+a_{j,i}) = \frac{\Lambda_3}{\Gamma_3}\frac{\partial G}{\partial T_{ij}} \quad \text{in } V\,, \\ &\mathscr{D}_i = \frac{\Lambda_3}{\Gamma_3}\frac{\partial G}{\partial E_i} \quad \text{in } V\,, \\ &-a_i = 0 \quad \text{on } S_u\,, \\ &\mathscr{D}_i n_i = 0 \quad \text{on } S_D\,. \end{aligned} \tag{32}$$

This G formulation can be considered as another generalization of the complementary energy formulation for the vibration problem in classical elasticity.

6. The internal energy $U(S_{ij}, D_i)$ formulation. The internal energy U can be obtained from H as

$$U = U(S_{ij}, D_i) = H + E_i D_i, \tag{33}$$

which generates the following constitutive relations:

$$T_{ij} = \frac{\partial U}{\partial S_{ij}}, \qquad E_i = \frac{\partial U}{\partial D_i}. \tag{34}$$

Let

$$\begin{aligned}
\Lambda_4(u_i, S_{ij}, T_{ij}, \phi, D_i) &= \int_V [T_{ij} u_{i,j} - D_{i,i}\phi + U(S_{ij}, D_i) - T_{ij} S_{ij}]\, dV \\
&\quad - \int_{S_u} T_{ji} n_j u_i \, dS + \int_{S_D} D_i n_i \phi \, dS, \\
\Gamma_4(u_i) &= \int_V \tfrac{1}{2}\rho u_i u_i \, dV, \\
\Pi_4(u_i, S_{ij}, T_{ij}, \phi, D_i) &= \frac{\Lambda_4(u_i, S_{ij}, T_{ij}, \phi, D_i)}{\Gamma_4(u_i)}.
\end{aligned} \tag{35}$$

Then

$$\begin{aligned}
\delta\Lambda_4 = \int_V \Big\{ &-T_{ji,j}\delta u_i - D_{i,i}\delta\phi + \left(\frac{\partial U}{\partial S_{ij}} - T_{ij}\right)\delta S_{ij} + \left(\frac{\partial U}{\partial D_i} + \phi_{,i}\right)\delta D_i \\
&+ \left[\frac{1}{2}(u_{i,j} + u_{j,i}) - S_{ij}\right]\delta T_{ij}\Big\} dV \\
&- \int_{S_u} u_i \delta T_{ji} n_j \, dS + \int_{S_T} T_{ji} n_j \delta u_i \, dS - \int_{S_\phi} \phi \delta D_i n_i \, dS + \int_{S_D} D_i n_i \delta\phi \, dS, \\
\delta\Gamma_4 = \int_V &\rho u_i \delta u_i dV.
\end{aligned} \tag{36}$$

Hence $\delta\Pi_4 = 0$ implies

$$\begin{aligned}
&- T_{ji,j} = \rho \frac{\Lambda_4}{\Gamma_4} u_i \quad \text{in } V, \\
&- D_{i,i} = 0 \quad \text{in } V, \\
&- S_{ij} + \tfrac{1}{2}(u_{i,j} + u_{j,i}) = 0 \quad \text{in } V, \\
&- T_{ij} + \frac{\partial U}{\partial S_{ij}} = 0 \quad \text{in } V, \\
&\phi_{,i} + \frac{\partial U}{\partial D_i} = 0 \quad \text{in } V, \\
&- u_i = 0 \quad \text{on } S_u, \\
&T_{ji} n_j = 0 \quad \text{on } S_T, \\
&- \phi = 0 \quad \text{on } S_\phi, \\
&D_i n_i = 0 \quad \text{on } S_D,
\end{aligned} \tag{37}$$

which is also an equivalent form of the original eigenvalue problem (2). The stationary condition of Π_4 gives the eigenvalue problem (37), with the stationary value of Π_4 as ω^2. If we choose admissible functions to satisfy

$$\begin{aligned} -D_{i,i} &= 0 \quad \text{in } V, \\ -S_{ij} + \tfrac{1}{2}(u_{i,j} + u_{j,i}) &= 0 \quad \text{in } V, \\ -u_i &= 0 \quad \text{on } S_u, \\ D_i n_i &= 0 \quad \text{on } S_D, \end{aligned} \tag{38}$$

then Π_4 reduces to

$$\Pi_4 = \frac{\int_V U dV}{\int_V \frac{1}{2}\rho u_i u_i dV}, \tag{39}$$

and the stationary condition of Π_4 becomes

$$\begin{aligned} &-T_{ji,j} = \rho\frac{\Lambda_4}{\Gamma_4}u_i \quad \text{in } V, \\ &-T_{ij} + \frac{\partial U}{\partial S_{ij}} = 0 \quad \text{in } V, \\ &\phi_{,i} + \frac{\partial U}{\partial D_i} = 0 \quad \text{in } V, \\ &T_{ji}n_j = 0 \quad \text{on } S_T, \\ &-\phi = 0 \quad \text{on } S_\phi. \end{aligned} \tag{40}$$

This U formulation can be considered as another generalization of the potential energy formulation for the vibration problem in classical elasticity.

Finally, we note that from (33), (3), and (4) the internal energy can be expressed in a positive-definite form in terms of S_{ij} and E_i as follows:

$$\begin{aligned} U &= H + E_i D_i \\ &= \frac{1}{2}c_{ijkl}S_{ij}S_{kl} - \frac{1}{2}\varepsilon_{ij}E_iE_j - e_{ijk}E_iS_{jk} + E_i(\varepsilon_{ij}E_j + e_{ijk}S_{jk}) \\ &= \frac{1}{2}c_{ijkl}S_{ij}S_{kl} + \frac{1}{2}\varepsilon_{ij}E_iE_j. \end{aligned} \tag{41}$$

Since the internal energy function U is positive definite, the constraint Π_4 is bounded from below. Therefore, the lowest resonant frequency must be a minimum. Following some standard arguments in variational analysis [6], we have the following immediate properties.

The lowest resonant frequency will increase if any of the following happens:

(i) S_u increases;
(ii) S_D increases;
(iii) ρ decreases;
(iv) c_{ijkl} increases to c'_{ijkl} such that $(c'_{ijkl} - c_{ijkl})a_{ij}a_{kl} > 0$ for any nonzero symmetric a_{ij};
(v) ε_{ij} increases to ε'_{ij} such that $(\varepsilon'_{ij} - \varepsilon_{ij})b_ib_j > 0$ for any nonzero b_i.

Some of the above properties may be considered as generalizations of the corresponding properties in classical elasticity.

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