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DIRECT BOUNDARY INTEGRAL EQUATION METHOD IN THE THEORY OF ELASTICITY

By

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Abstract. The Direct Boundary Integral Equation Method (BIEM), which leads to second-kind singular integral equations for all types of commonly used boundary-value problems of the theory of elasticity, is formulated. The exposition is applied to anisotropic bodies with arbitrary elastic anisotropy.

1. Introduction. The "direct" in connection with modern terminology [1] are called such formulations of the boundary integral equation method that lead to equations with unknown densities, located on the boundary surface and having real geometric or physical meaning. In contrast to that in indirect approaches, unknown densities do not have any physical or geometric meaning. The advantage of the direct formulation is based on the possibility of determining unknown surface densities just in the process of resolution, and quite often these data are of main interest.

The first direct BIEM formulations in the theory of elasticity are, to all appearances, due to Kupradze and Alexidze [2, 3]. Their approach was based on the Somigliana identity written for the complement $\Omega_- = \mathbf{R}^n \backslash \overline{\Omega}$, where Ω is an open region under consideration:

$$\begin{aligned}\int_{\partial\Omega} \mathbf{u}_0(\mathbf{y}') \cdot \mathbf{T}(\boldsymbol{\nu}_{y'}, \partial_{y'})\mathbf{E}(\mathbf{x}-\mathbf{y}')\,dy' \\ - \int_{\partial\Omega} \mathbf{t}_0(\mathbf{y}') \cdot E(\mathbf{x}-\mathbf{y}')\,dy' = 0, \qquad \mathbf{x} \in \partial U.\end{aligned} \tag{1.1}$$

Here $\mathbf{u}_0$, $\mathbf{t}_0$ are surface displacements and tractions respectively; $\mathbf{T}$ is the operator of surface tractions; $\mathbf{E}$ is Kelvin's fundamental solution (or Kelvin-Boussinesq's for planar problems). The auxiliary surface ∂U in this method was chosen in such a manner that $\partial U \subset \Omega_-$ and $\operatorname{dist}(\partial U, \partial\Omega) > 0$. Afterwards, discrete points $\mathbf{x}_i \in \partial U$ were fixed, for which the identity (1.1) was written. This procedure gave a system of linear equations for unknown discrete values of densities located on the surface $\partial\Omega$.

Mathematical problems involving the solution of the integral equations of the first kind in Kupradze-Alexidze's approach were considered in [4, 5]. Bakushinsky [6]

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suggested using algorithms of Tychonov's regularization, which allow the transfer of the ill-posed first-kind integral equations to the well-posed equations of the second kind. At the same time, numerical experiments carried out by Theocaris et al. [7] revealed that the instability of the first-kind equations is "exaggerated". Location of the surface ∂U in Kupradze-Alexidze's approach is quite arbitrary. Mathon and Johnston [8] in an analogous context proposed using nonlinear optimization algorithms for the optimal disposition of the surface ∂U or, more precisely, points $\mathbf{x}_i \in \partial U$. As mentioned in [8], serious computational problems can arise when different points merge in the process of optimization.

Integral equations of the second kind in the direct approach applied to the second boundary-value problem can be obtained from the Somigliana identity written for the surface $\partial\Omega$:

$$(\mathbf{I}/2 + \mathbf{S})(\mathbf{u}_0)(\mathbf{x}') = \int_{\partial\Omega} \mathbf{t}_0(\mathbf{y}') \cdot \mathbf{E}(\mathbf{x}' - \mathbf{y}')\, dy', \tag{1.2}$$

where $\mathbf{I}$ is the unit diagonal matrix and $\mathbf{S}$ is a matrix singular operator arising when the double-layer potential is restricted to $\partial\Omega$:

$$\mathbf{S}(\mathbf{u}_0)(\mathbf{x}') = \text{P.V.} \int_{\partial\Omega} \mathbf{u}_0(\mathbf{y}') \cdot \mathbf{T}(\boldsymbol{\nu}_{y'}, \partial_{y'})\mathbf{E}(\mathbf{x}' - \mathbf{y}')\, dy'.$$

This approach is due to Rizzo [9]. The first boundary-value problem in this approach leads to the first-kind integral equations with respect to $\mathbf{u}_0$. In this case, kernel $\mathbf{E}$ in the right-hand side of Eq. (1.2) has weak singularity on $\partial\Omega$. Taking into account the compactness of such operators, the regularization technique is needed to obtain numerically stable solutions. Analogous difficulties arise when mixed boundary-value problems are considered.

Second-kind integral equations in the first boundary-value problem can be obtained if the surface-traction operator acts on both sides of Eq. (1.2):

$$(\mathbf{I}/2 + \mathbf{S}^*)(\mathbf{t}_0)(\mathbf{x}') = \mathbf{G}_0(\mathbf{u}_0)(\mathbf{x}'), \tag{1.3}$$

where $\mathbf{S}^*$ is an adjoined operator to $\mathbf{S}$:

$$\mathbf{S}^*(\mathbf{t}_0)(\mathbf{x}') = \text{P.V.} \int_{\partial\Omega} \mathbf{T}(\boldsymbol{\nu}_{x'}, \partial_{x'})\mathbf{E}(\mathbf{x}' - \mathbf{y}')\mathbf{t}_0(y')\, dy'$$

and the operator $\mathbf{G}_0$ is of the form

$$\begin{aligned} \mathbf{G}_0(\mathbf{u}_0)(\mathbf{x}') \equiv & \lim_{x'' \to x'} \mathbf{T}(\boldsymbol{\nu}_{x''}, \partial_{x''}) \\ & \times \int_{\partial\Omega} \mathbf{u}_0(\mathbf{y}') \cdot \mathbf{T}(\boldsymbol{\nu}_{y'}, \partial_{y'})\mathbf{E}(\mathbf{x}'' - \mathbf{y}')\, dy', \qquad \mathbf{x}'' \in \Omega_-. \end{aligned} \tag{1.4}$$

Limits in the left-hand side of formula (1.4) are computed in nontangential directions to $\partial\Omega$. Equations (1.3) are equations of the second kind for which we are looking, but the operator $\mathbf{G}_0$ is strongly singular. Despite the fact that similar operators arise in some mechanical problems, their computational properties are almost unexplored, except for the one-dimensional case which was studied by Kaya and Erdogan [10] and Martin [11].

In this article an approach leading to the second-kind integral equations analogous to Eq. (1.3) is developed for all types of commonly used boundary-value problems of the linear theory of elasticity. Anisotropic homogeneous bodies with arbitrary elastic anisotropy are discussed below.

2. Basic operators and symbols. An initially anisotropic homogeneous elastic medium is regarded, with equations of equilibrium written in the form

$$\mathbf{A}(\partial_x)\mathbf{u} \equiv -\operatorname{div} \mathbf{C} \cdots (\nabla \mathbf{u}) \tag{2.1}$$

where $\mathbf{u}$ is a displacement field and $\mathbf{C}$ is the fourth-order strongly elliptic elasticity tensor. The given anisotropic medium is assumed to be a hyperelastic one; so $\mathbf{C}$ is symmetric with respect to extreme pairs: $C^{ijmn} = C^{mnij}$.

The Fourier transform

$$f^{\vee}(\boldsymbol{\xi}) = \int_{\mathbf{R}^3} f(\mathbf{x}) \exp(-2\pi i \mathbf{x} \cdot \boldsymbol{\xi})\, dx$$

applied to the operator $\mathbf{A}$ gives its symbol:

$$\mathbf{A}^{\vee}(\boldsymbol{\xi}) = 4\pi^2 \boldsymbol{\xi} \cdot \mathbf{C} \cdot \boldsymbol{\xi}. \tag{2.2}$$

Similarly the surface-traction symbol is defined as

$$\mathbf{T}^{\vee}(\boldsymbol{\nu}, \boldsymbol{\xi}) = 2\pi i \boldsymbol{\nu} \cdot \mathbf{C} \cdot \boldsymbol{\xi}. \tag{2.3}$$

Here $\boldsymbol{\nu}$ is the unit normal to a boundary surface $\partial\Omega$.

Using symbol $\mathbf{A}$ the symbol of fundamental solution of Eqs. (2.1) can be represented in the form

$$\mathbf{E}^{\vee}(\boldsymbol{\xi}) = \mathbf{A}_0^{\vee}(\boldsymbol{\xi}) / \det \mathbf{A}^{\vee}(\boldsymbol{\xi}), \tag{2.4}$$

where $\mathbf{A}_0^{\vee}$ is a matrix of algebraic complements of $\mathbf{A}^{\vee}(\boldsymbol{\xi})$. This expression shows that $\mathbf{E}^{\vee}$ is strongly elliptic and positively homogeneous of order -2 with respect to $|\boldsymbol{\xi}|$.

REMARK 2.1. In the three-dimensional case, the analytical Fourier transform inversion of symbols, which gives corresponding fundamental solutions, can be done only for one narrow subclass of elastic orthotropy. That subclass includes isotropic and transversely isotropic materials [12]. For plane problems the situation is better, and analytic formulas are known for elastic materials with arbitrary anisotropy [13].

When three-dimensional problems with arbitrary anisotropy are concerned, only numerical methods can be used for the reconstruction of fundamental solutions from corresponding symbols. These methods can be divided into two groups. The first one is referred to as disintegration of Lebesgue's measure on hyperplanes (Radon transformation) [14]. Apparently, first, this method was applied to static elasticity problems by Lifshitz and Rosenzveig [15], to dynamic transient problems by Burridge [16], and to time-harmonic Green's functions by Willis [17]. Numerical experiments carried out in [18] revealed that for arbitrary anisotropy, additional approximations on spheres are needed to achieve appropriate accuracy in computing values of fundamental solutions under realistic computational time. The second group is based on multipole decompositions of symbols, i.e., decompositions into the series of spherical harmonics. That, by the use of Bochner's inverting formulas [19], provides fundamental solutions in the form of multipole series. For the fundamental solutions of statics, this method was applied in [20, 21]. Some properties of the operators similar to Eq. (1.4) were investigated by multipole decompositions in [22].

Let boundary $\partial\Omega$ be a compact two-dimensional submanifold in $\mathbf{R}^3$ of the class $C^{m,\alpha}$, $m \geq 1$, $\alpha > 0$. An operator of boundary conditions on $\partial\Omega$ can be defined by the formula

$$\mathbf{B}(\boldsymbol{\nu}, \partial_x)\mathbf{u} \equiv (\mathbf{M}\cdot\mathbf{u} + \mathbf{N}\cdot\mathbf{T}(\boldsymbol{\nu}, \partial_x)\mathbf{u})|_{\partial\Omega} = \mathbf{g}, \tag{2.5}$$

where $\mathbf{M}$, $\mathbf{N}$ are square matrices. Operator $\mathbf{B}$, by a single analytical expression, allows different types of boundary conditions to be described, namely: $\mathbf{M} = \mathbf{I}$, $\mathbf{N} = 0$ corresponds to the first boundary-value problem; $\mathbf{M} = 0$, $\mathbf{N} = \mathbf{I}$ corresponds to the second one; $\mathbf{M} = \boldsymbol{\nu}\otimes\boldsymbol{\nu}$, $\mathbf{N} = \mathbf{I} - \boldsymbol{\nu}\otimes\boldsymbol{\nu}$ corresponds to the third boundary-value problem; $\mathbf{M} = \mathbf{I} - \boldsymbol{\nu}\otimes\boldsymbol{\nu}$, $\mathbf{N} = \boldsymbol{\nu}\otimes\boldsymbol{\nu}$ corresponds to the fourth one. In an analogous way, other boundary-value problems can be represented.

REMARK 2.2. The condition $\mathbf{M} + \mathbf{N} = \mathbf{I}$ need not always be satisfied, as can be seen from the boundary-value problems described above. For example, for the so-called sixth boundary-value problem [23], matrix $\mathbf{M}$ is a positive-definite function of $\mathbf{x}' \in \partial\Omega$, while $\mathbf{N}$ is equal to $\mathbf{I}$.

3. Boundary operators. The introduction of vectors $\mathbf{f}$ and $\mathbf{g}$ by

$$\mathbf{g} = \mathbf{M}\cdot\mathbf{u}_0 + \mathbf{N}\cdot\mathbf{t}_0, \qquad \mathbf{f} = \mathbf{N}\cdot\mathbf{u}_0 + \mathbf{M}\cdot\mathbf{t}_0 \tag{3.1}$$

shows that they define correspondingly unknown and known vector densities on $\partial\Omega$ for the four boundary-value problems. Action by the surface-traction operator on both sides of the Somigliana identity (1.2) yields the following formulas, which are direct analogies of Eqs. (1.3), (1.4):

$$\mathbf{K}(\mathbf{f})(\mathbf{x}') = \mathbf{G}(\mathbf{g})(\mathbf{x}'). \tag{3.2}$$

Here $\mathbf{K}$ is a matrix operator:

$$\begin{aligned}\mathbf{K} &= (\mathbf{I}/2 + \mathbf{S})\cdot\mathbf{N} + (\mathbf{I}/2 + \mathbf{S}^*)\cdot\mathbf{M} \\ &= \mathbf{I}/2 + \mathbf{S}\cdot\mathbf{N} + \mathbf{S}^*\cdot\mathbf{M}.\end{aligned} \tag{3.3}$$

The kernel of the integro-differential operator $\mathbf{G}$ is of the form

$$\mathbf{G}(\mathbf{x}', \mathbf{y}') = \mathbf{E}(\mathbf{x}' - \mathbf{y}')\cdot\mathbf{N} + \mathbf{G}_0(\mathbf{x}', \mathbf{y}')\cdot\mathbf{M}, \tag{3.4}$$

while the kernel of the operator $\mathbf{G}_0$ is defined by Eq. (1.4). A direct analysis of expression (3.3) shows

PROPOSITION 3.1. Operator $\mathbf{K}$ is a matrix standard pseudodifferential operator (p.d.o.) of the class S^0 on $\partial\Omega$ [24].

DEFINITION 3.1. The spectrum of the operator $\mathbf{X}$ is a set of (complex) numbers λ, at which the operator $\lambda\mathbf{I} - \mathbf{X}$ is not invertible in a class of continuous endomorphisms, acting in an appropriate functional space.

This definition coincides with one accepted in the spectral theory and differs slightly from the corresponding definition in the theory of integral equations.

Let $H^s(\partial\Omega, R)$ be Sobolev's space of the index $s \geq 0$. We will assume that if $\partial\Omega$ belongs to the class $C^{m,\alpha}$ and $s < 2m + \alpha$, this space is correctly defined on $\partial\Omega$.

LEMMA 3.1. (a) The spectrum of the operator $\mathbf{S}$ is discrete.

(b) $\mathrm{Sp}\,\mathbf{S}$ lies in the circle $|\lambda| \leq 1/2$.

(c) The point $\lambda = -1/2$ belongs to $\mathrm{Sp}\,\mathbf{S}$ and is a simple pole of the resolvent operator.

(d) The spectral space $\mathbf{E}_{-1/2}$ is six-dimensional and consists of rigid displacements.

The proof of this lemma for the general anisotropic case can be found in [25]. From assertion (c) of this lemma follows

PROPOSITION 3.2. In the factor-space $H^s(\partial\Omega, R)\backslash\mathbf{E}_{-1/2}$, the operator $\mathbf{K}$ is invertible.

So, in $H^s(\partial\Omega, R)\backslash\mathbf{E}_{-1/2}$ Eq. (3.2) admits a unique solution:

$$\mathbf{f}(\mathbf{x}') = \mathbf{K}^{-1} \circ \mathbf{G}(\mathbf{g})(\mathbf{x}'). \tag{3.5}$$

It is assumed that $\mathbf{G}(\mathbf{g})(\mathbf{x}')$ belongs to the regarded factor-space. The construction of the operator $\mathbf{K}^{-1}$ can be done with the aid of Neumann's series:

$$\mathbf{K}^{-1} = 2\sum_{n=0}^{\infty}(-2\mathbf{S}\cdot\mathbf{N} - 2\mathbf{S}^*\cdot\mathbf{M})^n. \tag{3.6}$$

Assertions (b) and (c) of Lemma 3.1 imply

PROPOSITION 3.3. Neumann's series (3.6) are absolutely convergent in $H^s(\partial\Omega, \mathbf{R}^3)\backslash \mathbf{E}_{-1/2}$.

REMARK 3.1. For the general anisotropic case and the first and second boundary-value problems, the construction of the inverse operator $\mathbf{K}^{-1}$ in the form of Neumann's series can be found in [25]. In the isotropic case the analogous procedure is due to Pham The Lai [27], who also used analytical prolongation, proposed in [28] for abstract operators analytically depending upon a parameter.

4. Properties of the operator $\mathbf{G}_0$. The main result established in this section is the decomposition of the operator $\mathbf{G}_0$ into the product of two operators, which in its turn allows the exclusion of integration points with strong singularities.

The following formula for the symbol $\mathbf{G}_0^{\sim}$ associated with the operator $\mathbf{G}_0$ can easily be obtained from Eq. (1.4):

$$\begin{gathered}\mathbf{G}_0^{\sim}(\mathbf{x}', \boldsymbol{\xi}') = \lim_{x''\to\pm 0}\int \mathbf{T}^{\vee}(\boldsymbol{\nu}_{x'}, \boldsymbol{\xi})\cdot\mathbf{E}^{\vee}(\boldsymbol{\xi})\cdot\mathbf{T}^{\vee t}(\boldsymbol{\nu}_{x'}, \boldsymbol{\xi})\exp(-2\pi i\xi''x'')\,dx'', \\ \boldsymbol{\xi}' = \boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\boldsymbol{\nu}_{x'})\boldsymbol{\nu}_{x'}, \quad \xi'' = \boldsymbol{\xi}\cdot\boldsymbol{\nu}_{x'}, \quad \mathbf{x}' \in \partial\Omega.\end{gathered} \tag{4.1}$$

Here and throughout, "$\sim$" refers to the Fourier transform on variables belonging to fibers of the cotangent bundle $T^*\partial\Omega$; where there is no risk of confusion, the operator and its symbol or amplitude are denoted by the same letter.

The integral in Eq. (4.1) can be expressed in another form if two obvious equalities

$$\begin{aligned} \mathbf{T}^{\vee}(\boldsymbol{\nu}, \boldsymbol{\xi}) &= -(\mathbf{A}^{\vee}(\boldsymbol{\xi}) - (2\pi)^2 \boldsymbol{\xi}' \cdot \mathbf{C} \cdot \boldsymbol{\xi})/(2\pi i \xi''), \\ \mathbf{T}^{\vee t}(\boldsymbol{\nu}, \boldsymbol{\xi}) &= -(\mathbf{A}^{\vee}(\boldsymbol{\xi}) - (2\pi)^2 \boldsymbol{\xi} \cdot \mathbf{C} \cdot \boldsymbol{\xi}')/(2\pi i \xi'') \end{aligned} \tag{4.2}$$

are taken into consideration:

$$\begin{aligned} \mathbf{G}_0^{\sim}(\boldsymbol{\xi}') = & -\operatorname{sym}[(2\pi i)^3 (\boldsymbol{\xi}' \cdot \mathbf{C} \cdot \boldsymbol{\xi}') \cdot H_{\xi''} \mathbf{E}^{\vee}(\boldsymbol{\xi}', 0) \cdot (\boldsymbol{\xi}' \cdot \mathbf{C} \cdot \boldsymbol{\nu}) \\ & + (2\pi i)^2 (\boldsymbol{\xi}' \cdot \mathbf{C} \cdot \boldsymbol{\nu}) \cdot \mathbf{E}^{\sim}(\boldsymbol{\xi}', 0) \cdot (\boldsymbol{\xi}' \cdot \mathbf{C} \cdot \boldsymbol{\nu}) \\ & + (2\pi i)^2 (\boldsymbol{\xi}' \cdot \mathbf{C} \cdot \boldsymbol{\xi}') \cdot \mathbf{E}^{\sim}(\boldsymbol{\xi}', 0) \cdot (\boldsymbol{\nu} \cdot \mathbf{C} \cdot \boldsymbol{\nu}) \\ & + (2\pi i)(\boldsymbol{\xi}' \cdot \mathbf{C} \cdot \boldsymbol{\nu}) \cdot \partial_{x''} \mathbf{E}^{\sim}(\boldsymbol{\xi}', 0) \cdot (\boldsymbol{\nu} \cdot \mathbf{C} \cdot \boldsymbol{\nu})]. \end{aligned} \tag{4.3}$$

In Eq. (4.3) $H_{\xi''}\mathbf{E}^{\vee}(\boldsymbol{\xi}', 0)$ is the zero value of Hilbert's transformation on ξ'':

$$H_{\xi''}\mathbf{E}^{\vee}(\boldsymbol{\xi}', 0) = \text{P.V.} \int_{-\infty}^{\infty} \frac{\mathbf{E}^{\vee}(\boldsymbol{\xi})}{2\pi i \xi''}\, d\xi''. \tag{4.4}$$

Other symbols in Eq. (4.3) are defined similarly:

$$\mathbf{E}^{\sim}(\boldsymbol{\xi}', 0) = \int_{-\infty}^{\infty} \mathbf{E}^{\vee}(\boldsymbol{\xi})\, d\xi'', \tag{4.5}$$

$$\partial_{x''}\mathbf{E}^{\sim}(\boldsymbol{\xi}', 0) = \text{P.V.} \int_{-\infty}^{\infty} 2\pi i \mathbf{E}^{\vee}(\boldsymbol{\xi}) \xi''\, d\xi''. \tag{4.6}$$

It is essential that integrals in Eqs. (4.4)–(4.6) are correctly defined, since $\mathbf{E}^{\vee}$ as a function of ξ'' belongs to $C^{\infty} \cap L^p$, $p \geq 1$, for any $\boldsymbol{\xi}' \neq 0$. A straightforward analysis of Eq. (4.3) implies

Proposition 4.1. The matrix symbol $\mathbf{G}_0^{\sim}$ is a symbol of the matrix p.d.o. of the class $S^1(\partial\Omega, \mathbf{R}^3 \otimes \mathbf{R}^3)$.

Taking into account this proposition symbol, $\mathbf{G}_0^{\sim}$ can be represented in the form

$$\mathbf{G}_0^{\sim}(\boldsymbol{\xi}') = -(2\pi)^2 \Delta^{\sim}(\boldsymbol{\xi}') \mathbf{V}^{\sim}(\boldsymbol{\xi}'), \tag{4.7}$$

where the weakly-singular operator $\mathbf{V}^{\sim}$ belongs to the class $S^{-1}(\partial\Omega, \mathbf{R}^3 \times \mathbf{R}^3)$, while $\Delta^{\sim}$ denotes a principal symbol of the Beltrami-Laplace operator on $\partial\Omega$.

The Fourier transform inversion of Eq. (4.7) gives

$$\mathbf{G}_0 = \mathbf{V} \circ \Delta + \mathbf{r}, \tag{4.8}$$

where the residual $\mathbf{r}$ is a p.d.o. operator of the class S^0 on $\partial\Omega$.

Proposition 4.2. For any functions $\mathbf{g} \in H^s(\partial\Omega, \mathbf{R}^3)$, $s \geq 1$, the exclusion of the pole vicinity $\omega_{x'}$ in the evaluation of the strongly-singular integral $\mathbf{G}_0(\mathbf{g})(\mathbf{x}')$ gives an error of the order $O(\operatorname{mes}(\omega_{x'}))$.

Proof. The left-hand side of the identity (1.3) shows that $\mathbf{G}_0(\mathbf{g})(\mathbf{x}') = \mathbf{G}_0(\mathbf{g} - \mathbf{g}_{x'})(\mathbf{x}')$, where $\mathbf{g}_{x'}$ is a constant on the $\partial\Omega$ vector field with the value $\mathbf{g}$ in $\mathbf{x}'$. Therefore, the operator $\mathbf{r}$ in Eq. (4.8) is a Calderon-Zygmund operator and it does not contain δ-functions. Since $\mathbf{g}(y') - \mathbf{g}_{x'}|_{y'=x'} = 0$, the assumption that $\mathbf{g}$ is locally constant in a vicinity of $\mathbf{x}'$ assures that the strongly-singular integral $\mathbf{G}_0(\mathbf{g} - \mathbf{g}_{x'})$ is correctly defined in the principal value sense.

SCHOLIUM 4.1. Proposition 4.2 shows that the strongly-singular integral can be evaluated by existing computer programs on numerical integration under the condition that the pole vicinity is excluded from the integration. That makes the strongly-singular integrals similar in computations to weakly-singular or singular integrals.

5. Multipole decompositions. The aim of this section is to present formulas for reconstructing the kernel of the operator $\mathbf{G}_0'$ from the corresponding symbol.

Let $\mathbf{Z}^\vee$ be a symbol of the form

$$\mathbf{Z}^\vee(\boldsymbol{\xi}) = \mathbf{C}\cdots\boldsymbol{\xi}\otimes\mathbf{E}^\vee(\boldsymbol{\xi})\otimes\boldsymbol{\xi}\cdots\mathbf{C}. \tag{5.1}$$

Convolution of $\mathbf{Z}^\vee(\boldsymbol{\xi})$ with vectors $\boldsymbol{\nu}_{x'}, \boldsymbol{\nu}_{y'}$, $x', y' \in \partial\Omega$, gives an amplitude

$$\begin{aligned}\mathbf{G}_0^\vee(\mathbf{x}', \mathbf{y}', \boldsymbol{\xi}) &= \mathbf{T}^\vee(\boldsymbol{\nu}_{x'}, \boldsymbol{\xi})\cdot\mathbf{E}^\vee(\boldsymbol{\xi})\cdot T^{\vee t}(\boldsymbol{\nu}_{y'}, \boldsymbol{\xi}) \\ &= \boldsymbol{\nu}_{x'}\cdot\mathbf{Z}^\vee(\boldsymbol{\xi})\cdot\boldsymbol{\nu}_{y'}.\end{aligned} \tag{5.2}$$

This amplitude in its turn generates the symbol $\mathbf{G}_0^\sim(\mathbf{x}', \boldsymbol{\xi})$ that was introduced in the preceding section.

The decomposition of $\mathbf{Z}^\vee(\boldsymbol{\xi})$ into a multipole series gives

$$\mathbf{Z}^\vee(\boldsymbol{\xi}') = \sum_{p=0,2,\dots}^{\infty}\sum_{k=1}^{2p+1}\mathbf{Z}^{p,k}Y_k^p(\boldsymbol{\xi}'), \qquad \boldsymbol{\xi}' \in S, \tag{5.3}$$

where Y_k^p is a spherical harmonic of the order p and of index k and $\mathbf{Z}^{p,k}$ is a tensorial coefficient, which is determined by integration over the sphere S of unit radius:

$$\mathbf{Z}^{p,k} = (2\pi)^{-2}\int_S \mathbf{Z}^\vee(\boldsymbol{\xi}')Y_k^p(\boldsymbol{\xi}')\,d\xi'.$$

The only presence of the even-order harmonics in Eq. (5.3) is due to the (positive) homogeneity of the symbol $\mathbf{Z}^\vee$. The Fourier transform inversion of Eq. (5.3), based on the generalization of Bochner's inverting formula [20], yields

$$\mathbf{Z}'(\mathbf{x}) = \pi^{-3/2}\sum_{p=2,4,\dots}^{\infty}(-1)^{p/2}\frac{\Gamma((p+3)/2)}{\Gamma(p/2)}\sum_{k=1}^{2p+1}\mathbf{Z}^{p,k}\frac{Y_k^p(\mathbf{x}')}{|x|^3}. \tag{5.4}$$

Here, an item corresponding to the spherical harmonic of the zero order in Eq. (5.3) is omitted, because this function leads to the three-dimensional δ-function under the Fourier transform inversion but disappears when it is restricted to manifolds of lower dimension.

Thus, the multipole decomposition (5.4) defines the strongly-singular kernel $\mathbf{Z}'$ on $\partial\Omega$. The convolution with vectors $\boldsymbol{\nu}_{x'}$, $\boldsymbol{\nu}_{y'}$ gives the amplitude $\mathbf{G}_0$ for which we are looking.

REMARK 5.1. The kernel of the singular operator $\mathbf{S}$ on $\partial\Omega$ can also be constructed by the use of multipole decompositions, in a way similar to what was done for $\mathbf{G}_0$.

REMARK 5.2. A numerically stable method for the summation of multipole series, based on a property of the Beltrami-Laplace operator on a sphere, was proposed in [20].

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