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ON THE CLASSIFICATION OF SOLUTIONS TO THE ZERO-SURFACE-TENSION MODEL FOR HELE-SHAW FREE BOUNDARY FLOWS

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Abstract. We discuss the classification of solutions to the zero-surface-tension model for Hele-Shaw flows in bounded and unbounded regions with suction and injection. We use results from the theory of univalent functions to derive estimates for certain geometric properties of the fluid region in the injection case.

1. Introduction. Recently Howison et al. [8, 9, 11, 12] have proposed a classification of the evolution of the free boundary between fluid and air in a Hele-Shaw cell in the "suction" case, essentially dividing such solutions into those that blow up and those that do not. Our aim is to extend this classification to include some cases missed in the earlier study, to interpret our examples in the light of the Carathéodory theorem of univalent function theory, and finally, to give some estimates for the linear dimension of the fluid region in the injection case.

We use the following simple dimensionless model (Howison [10] and references therein):

$$-\Delta p = Q(t)\delta(x, y), \tag{1}$$

for the pressure p in the fluid region $D(t)$, which is a bounded or unbounded simply-connected open region in the plane, together with the zero-surface-tension dynamic boundary condition

$$p = 0 \tag{2}$$

and the kinematic boundary condition

$$\frac{\partial F}{\partial t} - \nabla F \cdot \nabla p = 0 \tag{3}$$

on the moving boundary $F(x, y, t) = 0$. Here the right-hand term in (1) is a mathematical model for a sink or a source of strength $Q(t)$ at the origin. Finally, the initial fluid region is given by

$$D(0) = D_0. \tag{4}$$

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The above model can be reformulated in terms of a complex potential and an auxiliary mapping of the fluid region onto a fixed canonical domain, the unit disk $U = \{\zeta \in \mathbf{C} : |\zeta| < 1\}$ (see Hohlov [6] and Howison [10] for surveys of the method). Let us denote such a mapping $f: U \to D(t)$ by $z = f(\zeta, t)$, normalized by the conditions $f(0, t) = 0$, $f'(0, t) > 0$. The moving boundary in the model is the boundary of the domain $D(t) = f(U, t)$, where f is a solution of the nonlinear boundary value problem

$$\begin{aligned} \Re\left(\frac{\partial f}{\partial t}\zeta\overline{\frac{\partial f}{\partial \zeta}}\right) &= \frac{Q(t)}{2\pi}, \qquad |\zeta| = 1, \\ f(\zeta, 0) &= f_0(\zeta), \qquad \zeta \in U, \end{aligned} \tag{5}$$

where the initial domain $D_0 = f_0(U)$ is given. We consider only the case when $Q(t) \equiv Q$ is constant, since nonconstant Q can be treated by a change of time variable. We note that $Q < 0$ corresponds to suction and $Q > 0$ to injection, but we also note that the problem is formally time-reversible $(Q \to -Q, p \to -p, t \to -t)$; there is, however, a considerable difference between the two cases in that the injection case is well posed at least in its weak formulation, while the suction case is not (see Howison et al. [11] for discussion of this point, to which we also return below). Here we shall be concerned with the classical solution, which can blow up (via nonanalyticity of the moving boundary) in either case; roughly speaking, in the injection case, the solution can be continued beyond the blow-up time, while in the suction case, this is generally impossible, and the blow-up is terminal.

In Sec. 2, we shall assume that $Q < 0$ (suction), and we shall give several examples illustrating the various kinds of behaviour that can occur when $D(t)$ is finite. In Sec. 3, we discuss these solutions in the context of Carathéodory kernel convergence. Sections 4 and 5 contain some geometric estimates for the well-posed case $(Q > 0)$.

2. Classification of flows in bounded regions. It is possible to construct many explicit solutions to the Hele-Shaw problem using the nonlinear boundary value problem (5). The idea is simply to choose a functional form for $f(\zeta, t)$ depending on one or more parameters that are functions of time and then to find differential equations for these functions from (5). Naturally this procedure does not work for all functional forms, but it can be shown that whenever $f(\zeta, t)$ is a rational function (which includes polynomials as a special case), then it does yield an explicit solution. All the examples we give below are rational functions, and further details of these solutions, together with a review of the literature on exact solutions, are given in (Hohlov [6], Howison [10]); thus, here we simply cite the formulae (which can be verified using (5)).

A. *Solutions removing all the fluid from a finite region.* This case is straightforward because the problem is time-reversible.

THEOREM 1. Let D_0 be a bounded domain and suppose that for some T_* the area of the domain $D(T_*)$ is equal to zero:

$$|D(T_*)| = 0;$$

then the initial domain is a disk centred on the sink. This is the only such case.

The solution for the initial domain $D_0 = \{z \in \mathbf{C}: |z| < R_0\}$ is given by the mapping

$$f(\zeta, t) = \sqrt{R_0^2 + Qt/\pi}\,\zeta,$$

and there is a natural stopping time for the model. This is the time $T_* = -|D_0|/Q$ when the limiting domain $D(T_*) = f(U, T_*)$ is the point $z = 0$ and the linear measure of the residual region of fluid is equal to zero.

B1. *Solutions that develop singularities in the moving boundary in finite time and before it reaches the sink.* This was in fact the first type of nontrivial solution for Hele-Shaw flows in bounded regions to be constructed by complex analytic methods (see the review in Hohlov [7]). The ill-posed character of the model with suction means that the solution develops a singularity, frequently a cusp, in the moving boundary in finite time; the velocity of the cusp point is infinite. The minimum distance from the moving boundary to the sink may or may not vanish at the blow-up time (the latter case is discussed in Sec. B2).

The classical solution ceases to exist at the time when the cusp forms, but it is possible to subdivide these solutions into (a) solutions that cannot be continued beyond the blow-up time, such as the cardioid solution of Polubarinova-Kochina (1945) which we describe below, and (b) solutions with a free boundary that is regular analytic both before and after the cusp formation.

Polubarinova-Kochina's example is the quadratic mapping $f(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2$ where a_1 and a_2 satisfy the nonlinear system of algebraic equations

$$a_1^2(t)a_2(t) = a_1^2(0)a_2(0),$$
$$a_1^2(t) + 2a_2^2(t) = a_1^2(0) + 2a_2^2(0) + Qt/\pi.$$

If, for example, $a_1(0) = 1$, $a_2(0) = 1/4$, then blow-up occurs at $t = T_* = -9\pi(1 - 2^{-2/3}/3)/Q$ when $a_1 = 2^{-4/3}$, $a_2 = 2^{-7/3}$. At this time a 3/2-power cusp has appeared at the image of $\zeta = -1$, and the solution cannot be continued for $t \geq T_*$ without some regularization of the model (see Fig. 1 on p. 780).

For our second example, we consider a cubic mapping $f(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2 + a_3(t)\zeta^3$. In general this function produces a 3/2-power cusp corresponding to a simple zero of $\partial f/\partial\zeta$, but with the appropriate choice of $a_i(0)$, it is possible to arrange for the cusp to be of 5/2-power type, and it is shown in Howison [8] that in this case the solution is regular analytic both immediately before and after the blow-up. We choose the cusp to occur at the image of $\zeta = 1$, and then blow-up occurs when $a_1(T^*) + 2a_2(T^*) + 3a_3(T^*) = 0$. If in addition $a_1(T^*) + 8a_2(T^*) + 27a_3(T^*) = 0$, then the cusp is a 5/2-power one. It is straightforward to arrange this by suitable choice of the initial data (see Figs. 2 and 3 on pp. 780 and 781, where the captions give details of the parameters used).

Solutions of this type can be constructed using higher-degree polynomials to obtain higher-degree cusps (7/2-power from quartic $f(\zeta, t)$, 9/2-power from quintic, and so on), although as seen above, the initial values of the a_i are very special. Continuation beyond the blow-up time occurs only for $(2N + 1)/2$-power cusps (Howison

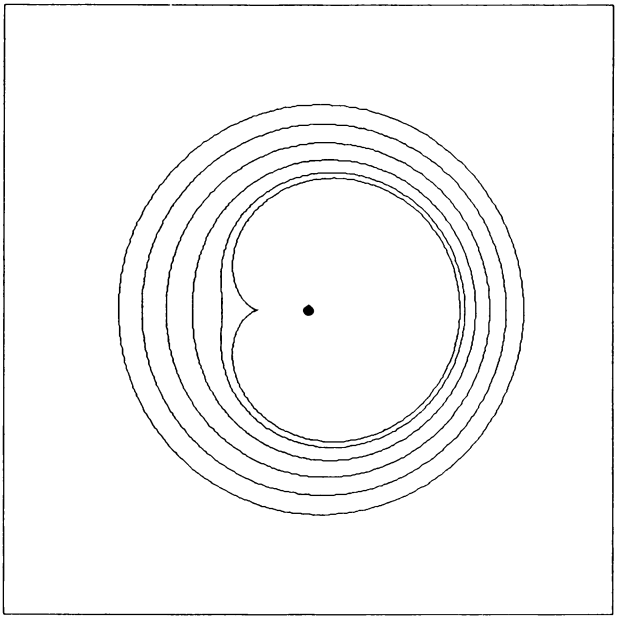

FIG. 1. The cardioid solution.

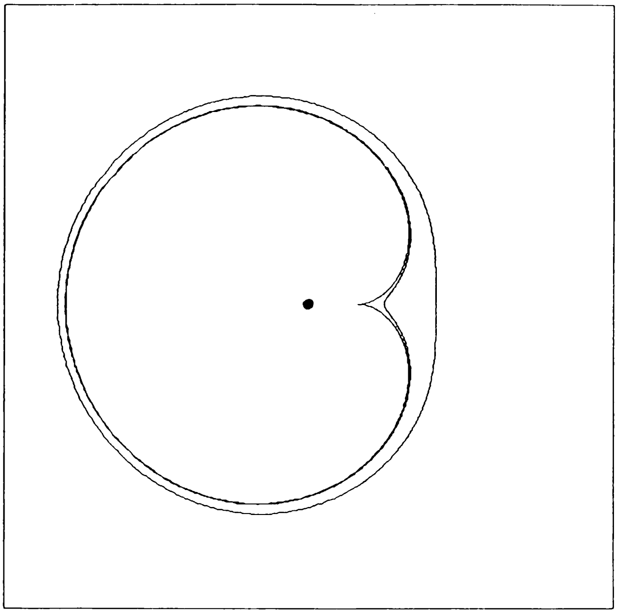

FIG. 2. The cubic solution, showing a 5/2-power cusp at $t = T^*$, when $a_1 = 1$, $a_2 = -4/5$, $a_3 = 1/5$.

[8]), although for the polynomial $f(\zeta, t)$ eventual blow-up without continuation is guaranteed before all the fluid has been extracted and before the moving boundary reaches the sink (Fig. 3). This fact was known to Galin [3], but his proof was incorrect. A correct proof is as follows.

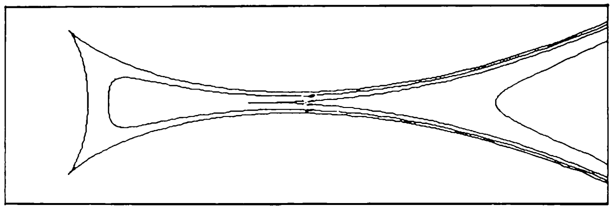

FIG. 3. The cubic solution near the cusp. From right to left, $t = T^* - 0.00042|Q/\pi|$, T^* (the 5/2-power cusp), $T^* + 0.0010|Q/\pi|$, $T^* + 0.0019|Q/\pi|$. At the last of these, $a_1 = (3/5)^{1/4}$, and two 3/2-power cusps have formed.

THEOREM 2. When the initial mapping function is a polynomial of degree $N \geq 2$, cusp formation is guaranteed before the moving boundary reaches the sink.

Proof. Suppose that the polynomial is $f_N(\zeta) = \sum_{n=1}^{N} a_n(t)\zeta^n$, where $a_N(0) \neq 0$; then substitution into (5) yields N ordinary differential equations for the coefficients $a_n(t)$, of which the first yields the area relation

$$\frac{d}{dt}\sum_{n=1}^{N} n|a_n|^2 = Q/\pi,$$

and the last relates a_1 and a_N via

$$\frac{d}{dt}(a_1^N \overline{a}_N) = 0.$$

Now if the area of $D(t)$ reaches zero or the moving boundary reaches the sink at the stopping time T_*, then $|a_1(T_*)| = 0$ (see Theorem 3 below). However, this contradicts the fact, which follows from the second of the differential equations, that $a_1(t)^N \overline{a}_N(t) = a_1(0)^N \overline{a}_N(0) \neq 0$.

Before moving on, we remark that it is likely that blow-up can occur by means other than a zero-angle cusp in the moving boundary. Since problem (1)–(4) (or (5)) is formally time-reversible, one may speculate that injection into a domain with, for example, piecewise analytic boundary (such as a square) might in the right circumstances produce a sequence of domains with analytic boundaries whose time reversal would blow up without a cusp. However, very little seems to be known about even the existence of classical solutions with this behaviour.

B2. *Solutions in which the moving boundary reaches the sink leaving residual fluid in a finite region.* A time reversal argument, involving reversal of the sequence of domains created by injection at a source initially situated at the boundary, suggests that this situation is possible and indeed there is a variety of exact solutions for this case. We describe three rational solutions of this type. In the first, the residual domain has an analytic boundary (a circle) containing the sink; in the second, it is a cardioid with the sink at the tip of the cusp, illustrating blow-up with a cusp at the sink; in the third, it consists of two circles touching at the sink, a case in which the connectivity of $D(t)$ changes at $t = T_*$.

In the first example, the mapping is a special case of the three-parameter rational function $a(t)\zeta(1 + b(t)\zeta)/(1 + c(t)\zeta)$, chosen to have the correct form at $t = T_*$.

It is

$$f(\zeta, t) = \zeta \frac{(2\alpha^4 - \alpha^2 + Qt/\pi) - (\alpha^3 + \alpha Qt/\pi)\zeta}{2\alpha^2(\zeta - \alpha)},$$

where $\alpha = \alpha(t)$ is the root of the algebraic equation

$$2\alpha^6 - (2Qt/\pi + 5)\alpha^4 + Q^2t^2/\pi^2 = 0,$$

satisfying the condition $\lim_{t \to -\pi/Q} \alpha(t) = -1$. The initial domain $D_0 = f(U, 0)$ is given by the mapping $f(\zeta, 0) = \zeta(4 - \sqrt{5/2}\zeta)/2(\zeta - \sqrt{5/2})$, and the stopping time T_* is $-\pi/Q$. At this moment the moving boundary reaches the sink and the residual region of fluid is the disk $\{z \in \mathbf{C}: |z+1| < 1\}$; see Fig. 4.

Our second example is obtained by composing the example just given with the cardioid solution yielding a rational function which is the ratio of a quartic to a quadratic. The details are complicated, but the upshot is sketched in Fig. 5; the residual domain is a cardioid with the sink at the tip of its cusp. This example thus illustrates blow-up by a 3/2-power cusp reaching the sink.

Our third example is a special case of the rational mapping $a\zeta(1+b^2\zeta^2)/(1+c^2\zeta^2)$ with the parameters chosen so that the final domain consists of two equal circles touching at $z = 0$. It is

$$f(\zeta, t) = \frac{\zeta}{2\alpha^{5/2}} \left[\sqrt{(1-\alpha^2)^2 + 16\alpha^3 - 4Qt/\pi} - 1 + \alpha^2 + (1-\alpha^2)(\sqrt{(1-\alpha^2)^2 + 16\alpha^3} - 1 + \alpha^2)/2(1 - \alpha\zeta^2) \right], \tag{6}$$

where $\alpha = \alpha(t)$ is the unique root of the equation

$$3 + \alpha^2 - \sqrt{(1-\alpha^2)^2 + 16\alpha^3} - 2\sqrt{(1-\alpha^2)^2 + 16\alpha^3 + 4Qt/\pi} = 0$$

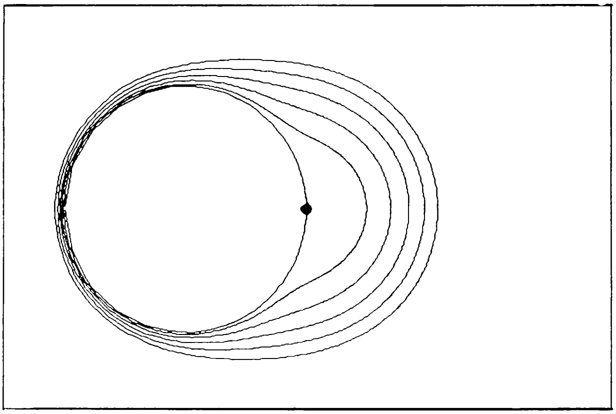

FIG. 4. A solution for which the moving boundary reaches the sink while remaining analytic.

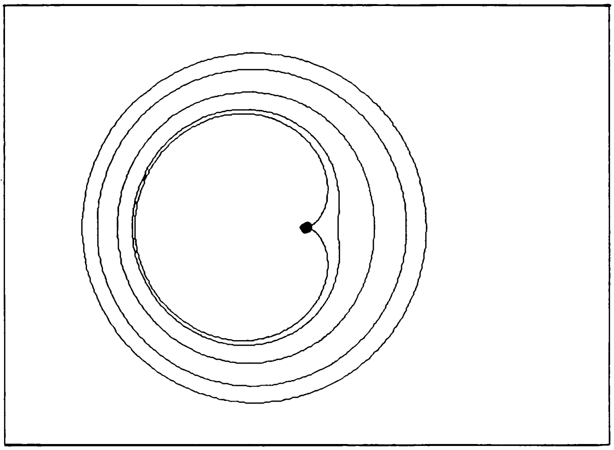

FIG. 5. A 3/2-power cusp forms just as the moving boundary reaches the sink.

at the interval $(0, 1)$, which tends to 1 as $t \to T_* = -4\pi/Q$. The initial domain $D_0 = f(U, 0)$ is given by the mapping

$$f(\zeta, 0) = \frac{\zeta(3 - \alpha_0^2 - 2\alpha_0\zeta^2)}{6\sqrt{\alpha_0}(1 - \alpha_0\zeta^2)}, \qquad \alpha_0 = \sqrt{84} - 9.$$

The stopping time T_* is $-4\pi/Q$. At this moment two cusps form at the sink and the residual region of fluid is the union of the two unit disks $\{z \in \mathbf{C}: |z \pm 1| \leq 1\}$, and the sink is at the common point of their closures (Fig. 6).

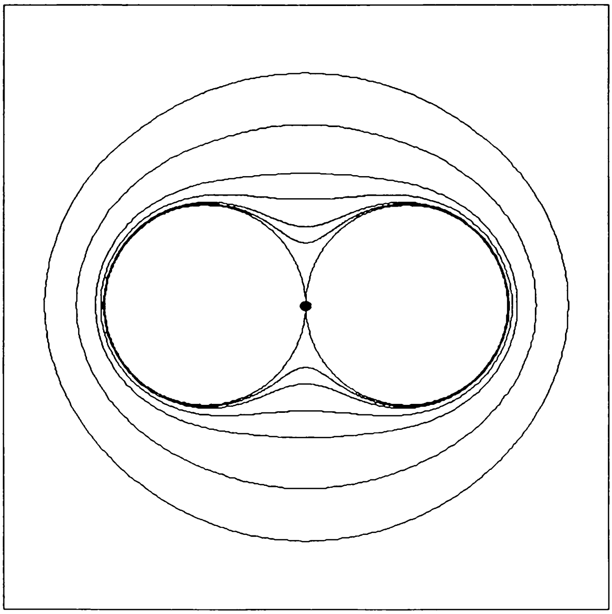

FIG. 6. A solution leaving two discs as the moving boundary reaches the sink.

Note that this solution is different from that of Richardson [17], who described injection at equal rates from the points $z = \pm 1$, thereby generating two expanding discs which touched at the origin; thereafter the fluid region was simply connected.

3. Hele-Shaw flows and the Carathéodory theorem of kernel convergence. In this section we relate our examples to the concept of Carathéodory kernel convergence. We recall the following definition (see Aleksandrov [1], Duren [4]):

Let $\{D(t)\}_{t\in[a,b]}$, $0 \le a < b < \infty$, be a continuous one-parameter family of domains in the complex plane. Then if the origin is an interior point of $\bigcap_{t\in[a,b]}\{D(t)\}$ for t sufficiently close to, but not equal to, $T_* \in [a, b]$, the *kernel* of the family $\{D(t)\}_{t\in[a,b]}$ with respect to the origin is defined to be the largest domain D_* containing the origin such that, for any compact subset $\mathscr{K}$ of D_*, there exists $\delta = \delta(\mathscr{K})$ such that $\mathscr{K} \subset D(t)$ for all $t \in [a, b]$, $0 < |t - T_*| < \delta$. If $0 \in \bigcap_{t\in[a,b]}\{D(t)\}$ for all $t \in [a, b]$, $0 < |t - T_*| < \delta$, but there is no neighbourhood of the origin that lies in $\bigcap_{t\in[a,b]}\{D(t)\}$, the kernel is defined to be $\{0\}$; in this case the kernel is termed *degenerate*. Note that in general the kernel D_* does not coincide with the domain D_{T_*}.

Let $\{D_n\}_{n\in\mathbf{N}}$ be a sequence of domains in the complex plane The sequence $\{D_n\}$ converges to D_* in the sense of Carathéodory if every subsequence of $\{D_n\}$ has the same kernel. The continuous family $\{D(t)\}$ converges to D_* for $t \to T_*$ in the sense of Carathéodory ($D(t) \to D_*$ for $t \to T_*$) if every discrete subsequence $\{D(t_n)\}_{t_n\in[a,b]}$ of $\{D(t)\}$ has the same kernel.

The *Carathéodory Convergence Theorem* is the following: Suppose that $\{D(t)\}$ is a continuous family of simply-connected domains, each of which contains the origin and none of which is the whole plane; let D_* be its kernel. Let f_t map the unit disc $|\zeta| < 1$ onto $D(t)$ with $f_t(0) = 0$, $f_t'(0) > 0$. Then $f_t \to f_*$ uniformly on each compact subset of $|\zeta| < 1$ if and only if $D(t) \to D_* \ne \mathbf{C}$ for $t \to T_*$. Moreover, $f_*(U) = D_*$. In the case of convergence, if $D_* = \{0\}$, then $f_* \equiv 0$ and vice versa.

This theorem enables us to state a general result for Hele-Shaw flows in which the moving boundary reaches the sink.

THEOREM 3. If the moving boundary $\partial D(t)$ of the continuous family $\{D(t)\}$, $t \in [0, T_*]$, of solutions of the Hele-Shaw flow moving boundary problem (1)–(3) reaches the sink at $t = T_*$, the kernel D_* of $\{D(t)\}$ is degenerate, i.e., $D_* = \{0\}$ and $f(\zeta, T_*) \equiv 0$.

Proof. Let $f(\zeta, t) = a_1(t)\zeta + \sum_{n\ge2} a_n(t)\zeta^n$ be the univalent mapping of the unit disk U onto $D(t)$.

Let us suppose that at time T_* a point z_* of the moving boundary $\partial D(T_*)$, the image of ζ_* on $|\zeta| = 1$ reaches the sink $z = 0$.

Applying the Koebe theorem (see Golusin [5] or Pommerenke [16]) we have the estimate

$$\tfrac{1}{4}|a_1(T_*)| = \tfrac{1}{4}|f'(0, T_*)| \le |f(\zeta_*, T_*)| = 0.$$

On the other hand, for any univalent function f the Bieberbach estimates (de Branges [2]) for the coefficients,

$$|a_n(t)| \le n \cdot |a_1(t)|,$$

imply that

$$\lim_{t\to T_*} |a_n(t)| = 0 \quad \text{for any } n = 2, 3, \dots .$$

Hence, $f(\zeta, T_*) \equiv 0$, and we can apply the Carathéodory theorem above to conclude that the continuous family $\{D(t)\}_{t\in[0, T_*]}$ has the degenerate kernel $D_* = \{0\}$.

In our examples above, the disc with centre at the sink was the only case in which $D(T_*) = D_* = \{0\}$. In all other cases, either D_* is more than just $\{0\}$ (for example, the cardioid solution), or $D_* = \{0\} \subsetneq D(T_*)$, as when the remaining region is the disc $|z+1| < 1$. In the latter situation, although $f(\zeta, T_*) \equiv 0$, $f(\zeta, t)$ has a singularity (in the above example it is a pole) which approaches $\zeta = -1$ as $t \uparrow T_*$; $f(\zeta, t)$ tends to zero uniformly inside $|\zeta| < 1$ but not on any subset of $|\zeta| \le 1$ that contains the point $\zeta = -1$.

4. A geometric estimate for flows in bounded regions. We now move on to give some estimates for the growth of the fluid region in the injection (well-posed) case. Thus, we now assume that $Q > 0$.

THEOREM 4. Let D_0 be a bounded domain such that a classical solution of the Hele-Shaw problem with injection from a source of strength Q at the origin and initial domain D_0 exists for any $t \in [0, T]$. Then the distance from the moving boundary $\partial D(t)$ to the source is bounded below by

$$\operatorname{dist}(D(t), 0) \ge \tfrac{1}{4}\sqrt{R^2(D_0) + Qt/\pi},$$

where $R(D_0)$ is the conformal radius of the initial domain with respect to the origin.

Proof. Let $f\colon U \to D(t)$, $f(0, t) = 0$, $f'(0, t) > 0$ be a Riemann mapping function of U onto $D(t)$ such that for any $t \in [0, T]$ the classical solution of the Hele-Shaw problem exists.

Let us note that from the boundary condition (5) it is easy to obtain the integro-differential representation

$$\frac{d}{dt}\frac{f(\zeta, t)}{\zeta} = \frac{Q}{2\pi} f'(\zeta, t) \frac{1}{2\pi}\int_{\partial U} |f'(e^{i\theta}, t)|^{-2} \frac{e^{i\theta} - \zeta}{e^{i\theta} + \zeta}\, d\theta.$$

Using the assumption that $f'(0, t) > 0$ and taking the limit $\zeta \to 0$, we obtain the equation

$$\frac{d}{dt}|f'(0, t)| = \frac{Q}{2\pi}|f'(0, t)| \frac{1}{2\pi}\int_{\partial U} |f'(e^{i\theta}, t)|^{-2}\, d\theta.$$

Let us write $\phi(t) = |f'(0, t)|^2$ and $\tau = Qt/\pi$; then we obtain the equation

$$\frac{d\phi}{d\tau} = \chi(\phi) = \phi J(\phi),$$

where

$$J(\phi) = \frac{1}{2\pi}\int_{\partial U} |f'(e^{i\theta}, \tau)|^{-2}\, d\theta > 0.$$

Using the mean value theorem for analytic functions in $\overline{U}$, we have the following estimate:

$$J(\phi) \ge \frac{1}{2\pi}\left|\int_{\partial U} [f'(e^{i\theta}, \tau)]^{-2}\, d\theta\right| = |f'(0, \tau)|^{-2} = \phi^{-1}.$$

Therefore, $\chi(\phi) \geq 1$, and so we have the differential inequality

$$\frac{d\phi}{d\tau} \geq 1,$$

which means that $\phi(\tau) \geq \tau + \phi(0)$, and consequently

$$|f'(0, t)|^2 \geq Qt/\pi + |f'(0, 0)|^2.$$

Finally, we have the conclusion of the theorem from the Koebe one-quarter theorem (see Golusin [5] or Pommerenke [16]).

Remark 1. There is no result corresponding to the one-quarter theorem of Koebe for estimates of $\text{dist}(D(t), 0)$ from above. To obtain such an estimate for the suction case, we would need to bound

$$\frac{1}{2\pi}\int_0^{2\pi} |f'(e^{i\theta})|^{-2}\, d\theta$$

from above, and this is impossible because the derivative f' on the boundary may vanish in the case of loss of conformality (cusp formation of the moving boundary which, as remarked above, can occur even in the well-posed case).

Remark 2. The restriction to "classical solutions" means that Theorem 3 does not apply directly to solutions that develop a cusp in their moving boundary but nevertheless can be continued, as mentioned in Sec. 2B. In such cases a time reversal of a suction problem shows that an initially analytic moving boundary can develop a cusp, albeit of a restricted kind, while remaining regular analytic before and after the cusp formation. The bound in the theorem would then have to be split into two stages, one before and one after the cusp formation.

5. Flows in unbounded regions. When the fluid region is unbounded we have a similar classification of solutions to the Hele-Shaw problem. We need a more precise formulation of the notion of "unbounded" domain because there are many possibilities; two commonly studied ones are the exterior of a bounded simply-connected domain as in the "bubble" problem (Howison [9]) and a domain with the moving boundary going to infinity as in the Saffman-Taylor finger solution for Hele-Shaw flows in a channel (Taylor and Saffman [18]). Here we only consider the bubble problem. The equations in this case are as before except that (1) is replaced by

$$-\Delta p = 0$$

in the fluid region $D(t)$, with

$$p \sim Q\log|z| \quad \text{as } |z| \to \infty.$$

A classification of solutions similar to that given above for finite regions is possible. Here we remark that Sec. 2B1 suggests the possibility that in the bubble case too the moving boundary may reach the sink, which in this case is at infinity, in finite time. Such a bubble would shoot out a long finger of finite area to infinity; unfortunately no such example is known.

In the injection case (shrinking bubble), we can derive an estimate on the dimensions of the bubble similar to that given above for finite regions. In fact, we have

THEOREM 5. Let a solution of the bubble problem with $Q > 0$ and initial domain B_0 exist for any $t \in [0, T]$. Then the linear measure of the shrinking bubble at time t is estimated by the following bound:

$$\operatorname{diam} B(t) \leq 4\sqrt{C^2(B_0) - Qt/\pi},$$

where $C(B_0)$ is the capacity of the initial bubble.

Proof. Let $F: U^- \to D(t)$ be a Riemann mapping function of U^- onto $D(t)$ such that for any $t \in [0, T]$ the classical solution of the bubble problem exists. The function F has an expansion near infinity of the form

$$F(\zeta, t) = F'(\infty, t)\zeta + \alpha_0(t) + \alpha_1(t)\zeta^{-1} + \cdots,$$

where $F'(\infty, t) > 0$ without loss of generality.

Let us suppose that for any $t \in [0, T]$ the bubble $B(t)$ contains a fixed point z_0, that is, $F(\zeta, t) \neq z_0$. Then the function

$$f(\zeta^*, t) = F'(\infty, t) \Big/ \left(F\left(\frac{1}{\zeta^*}, t\right) - z_0\right) = \zeta^* + \frac{z_0 - \alpha_0(t)}{F'(\infty, t)}\zeta^{*2} + \cdots$$

is a univalent function in the unit disk with the required normalization. Using the Bieberbach estimate $|a_2| \leq 2$, we obtain the inequality

$$|z_0 - \alpha_0(t)| \leq 2|F'(\infty, t)|,$$

which means that for any t the linear measure of the bubble is bounded by

$$\operatorname{diam} B(t) \leq 4F'(\infty, t).$$

Now we estimate $F'(\infty, t)$ from above. Let us note that from the boundary condition (6) it is easy to obtain the integro-differential representation

$$\frac{d}{dt}\frac{F(\zeta, t)}{\zeta} = \frac{Q}{2\pi}F'(\zeta, t)\left(-\frac{1}{2\pi}\int_{\partial U^-} |F'(e^{i\theta}, t)|^{-2}\frac{e^{i\theta} - \zeta}{e^{i\theta} + \zeta}\, d\theta\right).$$

Using the assumption that $F'(\infty, t) > 0$ and taking the limit $\zeta \to \infty$, we obtain the equation

$$\frac{d}{dt}|F'(\infty, t)| = \frac{Q}{2\pi}|F'(\infty, t)|\left(\frac{-1}{2\pi}\int_{\partial U^-} |F'(e^{i\theta}, t)|^{-2}\, d\theta\right).$$

Let us write $\psi(t) = |F'(\infty, t)|^2$ and $\tau = Qt/\pi$; then we obtain the equation

$$d\psi/d\tau = v(\psi) = -\psi J(\psi),$$

where

$$J(\psi) = \frac{1}{2\pi}\int_{\partial U^-} |F'(e^{i\theta}, \tau)|^{-2}\, d\theta > 0.$$

Using the mean value theorem for analytic functions in $\overline{U^-}$, we have

$$J(\psi) \geq \left|\frac{1}{2\pi}\int_{\partial U^-} [F'(e^{i\theta}, \tau)]^{-2}\, d\theta\right| = |F'(\infty, \tau)|^{-2} = \psi^{-1}.$$

Now from the estimate

$$v(\psi) \leq -1$$

for the vector field ψ, we have the differential inequality

$$\frac{d\psi}{d\tau} \leq -1,$$

which means that $\psi(\tau) \leq -\tau + \psi(0)$ and consequently

$$|F'(\infty, t)|^2 \leq -Qt/\pi + |F'(\infty, 0)|^2.$$

Finally, we have the estimate

$$\operatorname{diam} B(t) \leq 4\sqrt{C^2(B_0) - Qt/\pi},$$

where $C(B_0) = |F'(\infty, 0)|$ is the capacity of the initial bubble.

6. Conclusion. We have reviewed the classification of finite simply-connected Hele-Shaw flows driven by a single sink or source. When the driving mechanism is a sink, there is always a time T^* at which the classical solution with an analytic boundary ceases to exist; this stopping time is always bounded by the ratio of the initial area of fluid to the strength of the sink, i.e., the time to remove all the fluid, but this bound is only achieved when $D(0)$ is a disc centered on the origin. In all other cases, blow-up occurs earlier, and we have given examples in which the blow-up occurs through a cusp in the moving boundary. The moving boundary may or may not reach the sink at the blow-up time; in the latter case (which we have proved to be generic for polynomials of degree greater than one), it is possible in special circumstances to have a classical solution for times greater than the initial blow-up time by continuation after formation of a $(2N + 1)/2$-power cusp, although terminal blow-up is assured later (and still before $|D(0)|/|Q|$). In the latter case, which has not been discussed in previous reviews of this kind, the moving boundary may or may not be analytic when it reaches the sink; we have given examples of both types.

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