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NONLINEAR GROUPS OF GRAVITY-CAPILLARY WAVES

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Abstract. Nonlinear groups of gravity-capillary waves in deep water are investigated by a systematic direct approach that can be applied to nonlinear groups of other dispersive waves. Two formulas in closed form expressing the variations of the phase velocity c of the basic waves and of their group velocity c_g with the amplitude of the waves are obtained. These are in terms of the wavenumber ε of the envelope and ε^2 can be determined by the present approach as a power series in a^2, if $2a$ represents the amplitude of the waves. To the order of approximation achieved here, ε^2 is determined as a multiple of a^2. If k is the wavenumber of the basic waves, g is the gravitational acceleration, ρ is the density of the fluid, $\widehat{T}$ is surface tension, and $\beta = \widehat{T}k^2/\rho g$, then wave groups are possible for

$$0 \leq \beta < 0.1547 \quad \text{or} \quad \beta > \frac{1}{2},$$

although the analysis is valid only when β is not near $\frac{1}{2}$. The phase velocity increases with the amplitude in the former interval for β and decreases with the amplitude in the latter interval. The group velocity c_g decreases with the amplitude in the former interval for β, or for $\frac{1}{2} < \beta < 1$, but increases with the amplitude if $\beta > 1$. When the results of this paper are compared with the results of previous authors, wherever comparison is possible, complete agreement is found. (Previous authors did not give the variation of c_g with the amplitude.)

1. Introduction. In a previous paper [1] I have given a direct calculation of nonlinear gravity-wave groups that can be carried out to any order of approximation. The results of that analysis were compared with previous results obtained by other authors by a different approach, and wherever comparison could be made agreement was found. However, since my calculation was carried to the order $O(a^3\varepsilon)$, where $2a$ is the amplitude of the velocity potential and ε the wavenumber, so to speak, of the envelope, I was able to give the variation of the group velocity with amplitude. Furthermore, the cubic Schrödinger equation that previous investigators derived and used to obtain the group envelope is invariably based on one particular wavenumber of the basic waves, and for that reason cannot be used to investigate the interaction

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of two wave groups of different wavenumbers for the basic waves, although it can be used to study the interaction of two wave groups of the same basic wavenumber but different amplitudes. My approach allows the investigation of two interacting wave groups of different wavenumbers for the basic waves. Moreover, my approach, while rather cumbersome, as nonlinear calculations often are, is systematic and can be carried to any order of approximation.

For these reasons I have judged it desirable and worthwhile to use my approach to study nonlinear groups of gravity-capillary waves in deep water. Here, however, the procedure in [1] must be considerably modified, because the Bernoulli equation, applied to the free surface, now contains terms involving the surface tension and derivatives of the surface displacement. This makes it impossible to eliminate η (the surface displacement) between the dynamic condition and the kinematic condition for the free surface by a simple substitution of η obtained from the former into the latter, as was possible in [1]. A new systematic approach is herein devised, which will have a wide application to other nonlinear groups of dispersive waves.

The calculation is carried out to $O(a^3\varepsilon)$, where, as in [1], $2a$ is the basic amplitude (that is, of the linear part) of the velocity potential and ε is the wavenumber of the envelope. The main results are the determination of ε^2 as a multiple of a^2 and two formulas, in closed form, for the phase velocity c of the basic waves and for their group velocity c_g as functions of the amplitude of the waves. These formulas are in terms of ε^2, which can be determined as a power series of a^2 by higher approximations. Interaction of wave groups is briefly discussed.

When the results of this paper are compared with the results of Ablowitz and Segur [2] and of Djordjević and Redekopp [3], complete agreement is found wherever comparison is possible. (These authors did not give the variation of the group velocity with amplitude.) In view of the vast difference in approach between this work and the work of these authors, and in view of the tremendous amount of calculations necessary in this work and in theirs, the complete agreement is remarkable and gratifying.

2. Formulation of the differential system. Consider a train of gravity-capillary waves with the amplitude vanishing at infinity. The fluid motion is assumed two-dimensional and irrotational, and the depth of the fluid is assumed infinite. The free surface at $x = \pm\infty$ is at $y = 0$, which is the x-axis, and the individual waves and their envelope are supposed to propagate in the direction of increasing x.

Since the flow is assumed irrotational, there exists a velocity potential ϕ, the gradient of which is the velocity. The equation of continuity then gives the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0, \tag{1}$$

where subscripts indicate partial differentiation. At the free surface,

$$y = \eta(x, t), \tag{2}$$

where η is the displacement of that surface from its undisturbed position. The kinematic condition at the free surface, requiring the equality of two expressions of the vertical velocity, is

$$\eta_t + \phi_x \eta_x = \phi_y. \tag{3}$$

The Bernoulli equation is, at the free surface,

$$\phi_t + g\eta + \frac{\phi_x^2 + \phi_y^2}{2} + \frac{p}{\rho} = 0, \tag{4}$$

where g is the gravitational acceleration, t is the time, ρ is the density of the fluid, and p is the pressure just below the free surface. Equation (4) implies that the pressure at the free surface for $x = \pm\infty$ is taken to be zero for convenience. The pressure just above the free surface is then everywhere zero, with any variation of it due to air motion neglected. Then the p in (4) is given by

$$p = -\widehat{T}\eta_{xx}(1 + \eta_x^2)^{-3/2}, \tag{5}$$

in which $\widehat{T}$ denotes surface tension.

The calculation will be carried to the fourth order of the amplitude at most. Indeed, it will be carried out to the orders $O(a^3)$ and $O(a^3\varepsilon)$, where $2a$ is the amplitude of the velocity potential, and ε is the wavenumber of the envelope of the wave train, which is assumed small and will be shown to be of the order $O(a)$. Thus one can combine (4) and (5) into

$$g\eta = -L\phi + \sigma\eta_{xx}\left(1 - \frac{3}{2}\eta_x^2\right), \tag{6}$$

where

$$L\phi = \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2), \qquad \sigma = \frac{\widehat{T}}{\rho}. \tag{7}$$

Equation (6) is to be used in conjunction with (3), which will be rewritten in the form

$$\eta_t = \phi_y - \phi_x\eta_x. \tag{8}$$

The final boundary condition is

$$\phi \to 0 \quad \text{as } y \to -\infty. \tag{9}$$

Note that (4) ignores viscous effects, which require a boundary layer at the free surface, called the stress layer. The p in (4) then is the pressure just at the lower edge of that layer. Since the stress layer is thin (because both the phase velocity of the waves and their group velocity are large enough to make the Reynolds number large if the fluid is water), its effects are negligible, and one can proceed on the assumption of irrotational motion for the entire fluid.

The differential system governing the dynamics of the wave train then consists of (1), (6), (8), and (9), with $L\phi$ and σ defined by (7).

3. The solution of the differential system. Since ϕ satisfies (1), one can assume it to be the real part of an analytic function of the complex variable

$$z = x + iy,$$

and, as in [1], it is convenient to use the symbols $\widehat{X}$ and $\widehat{Y}$ defined by

$$\widehat{X} = \varepsilon(z - c_g t), \qquad \widehat{Y} = k(z - ct), \tag{10}$$

in which k is the wavenumber of the basic waves and ε that of their envelope, c is the phase velocity of the waves and c_g their group velocity. The value of c and c_g given by the linear theory are

$$c_0 = (gk^{-1} + \sigma k)^{1/2}, \qquad (c_g)_0 = \frac{1}{2c_0}\left(\frac{g}{k} + 3\sigma k\right). \tag{11}$$

Given k, the main task is to determine c, c_g, and ε as functions of a, which is half the amplitude of ϕ. The expansions used are, as in [1],

$$\phi = a\phi_1 + a^2\phi_2 + a^3\phi_3 + \cdots , \tag{12}$$

$$\varepsilon^2 = a^2(\alpha_1 + \alpha_2 a^2 + \alpha_3 a^4 + \cdots), \tag{13}$$

$$c^2 = c_0^2(1 + \beta_1\varepsilon^2 + \beta_2\varepsilon^4 + \cdots), \tag{14}$$

$$2cc_g = 2c_0 c_{g0}(1 + \gamma_1\varepsilon^2 + \gamma_2\varepsilon^4 \cdots). \tag{15}$$

As it will be shown later, the amplitude of ϕ_1 is 2. Since a, when made dimensionless (i.e., expressed in units of $c_0 k^{-1}$), is supposed to be small, $2a$ is the principal part of the amplitude of ϕ, and is considered its nominal amplitude.

If σ is assumed zero, (6) gives an expression of η in terms of ϕ, which, when substituted into (8), provides a single free-surface condition in terms of ϕ. Upon expanding ϕ in a McLaurin series in powers of η and repeatedly using (6), with $\widehat{T} = 0$, one obtains a single free-surface condition in terms ϕ and its derivatives, all evaluated at $y = 0$. This was done in [1], but the simplicity of this procedure is lost when surface tension is taken into account. One now needs an additional expansion

$$\eta = a\eta_1 + a^2\eta_2 + \cdots , \tag{16}$$

and proceeds systematically from the lowest-order terms in a to higher and higher orders.

From (6) and (8), the free-surface condition will now be developed in a form that can be used for calculation to any order of approximation. Using (6), one obtains

$$g\phi_y - \phi_x(g\eta_x) = g\phi_y + \phi_x(L\phi)_x - \sigma\phi_x\left\{\eta_{xx}\left(1 - \frac{3}{2}\eta_x^2\right)\right\}_x . \tag{17}$$

On the other hand, (6) gives

$$g\eta_t = -(L\phi)_t + \sigma\left\{\eta_{xx}\left(1 - \frac{3}{2}\eta_x^2\right)\right\}_t . \tag{18}$$

The left-hand side of (17) and (18) are equal on account of (8). Hence

$$g\phi_y + (L\phi)_t + \phi_x(L\phi)_x - \sigma Q = 0, \tag{19}$$

where

$$Q = \eta_{txx} - \frac{3}{2}(\eta_x^2\eta_{xx})_t + \phi_x\eta_{xxx}. \tag{20}$$

A term

$$-\frac{3}{2}\phi_x(\eta_x^2\eta_{xx})_x \tag{21}$$

has been omitted from Q because it is of $O(a^4)$, and because it will not affect the results to the desired order of approximation.

In four sub-sections that follow one will develop the four terms on the left-hand side of (19), which will be the main equation one works with.

3.1. *The term* $g\phi_y$. A power-series expansion gives

$$\phi_y = \phi_y(0) + \phi_{yy}(0)\eta + \frac{1}{2}\phi_{yyy}(0)\eta^2 + \cdots, \tag{22}$$

in which $\phi_y(0)$ means ϕ_y evaluated at $y = 0$ for any x and t. On account of (1), (12), and (16) this can be written as

$$\begin{aligned}\phi_y = {} & a\phi_{1y} + a^2(\phi_{2y} - \phi_{1xx}\eta_1) \\ & + a^3\left\{\phi_{3y} + (\phi_{2yy}\eta_1 + \phi_{1yy}\eta_2) - \frac{1}{2}\phi_{1yxx}\eta_1^2\right\},\end{aligned} \tag{23}$$

in which all the derivatives of ϕ on the right-hand side are now evaluated at $y = 0$.

3.2. *The terms in* $(L\phi)_t$. Before one differentiates $L\phi$ with respect to t, one expands the terms in it, in order to evaluate $L\phi$ at $y = \eta$. Thus,

$$\phi_t = a\phi_{1t} + a^2(\phi_{2t} + \phi_{1ty}\eta_1) + a^3\left(\phi_{2ty}\eta_1 + \phi_{1ty}\eta_2 - \frac{1}{2}\phi_{1txx}\eta_1^2\right), \tag{24}$$

in which

$$-\phi_{txx} = \phi_{tyy},$$

on account of (1). All the derivatives of ϕ on the right-hand side of (24) are evaluated at $y = 0$. Furthermore,

$$\frac{1}{2}(\phi_x^2+\phi_y^2) = \frac{1}{2}\{a^2(\phi_{1x}^2+\phi_{1y}^2)+2a^3(\phi_{1x}\phi_{1xy}\eta_1-\phi_{1y}\phi_{1xx}\eta_1+\phi_{1x}\phi_{2x}+\phi_{1y}\phi_{2y})\}, \tag{25}$$

in which again the derivatives of ϕ on the right-hand side are evaluated at $y = 0$, and (1) has again been applied. The reason for preferring $-\phi_{xx}$ to ϕ_{yy} is that derivatives with respect to x can be performed after y has been put to zero—a great convenience, as subsequent calculations will show.

Combining (24) and (25), one has

$$\begin{aligned}(L\phi)_t = {} & a\phi_{1tt} + a^2\left\{\phi_{2t} + \phi_{1ty}\eta_1 + \frac{1}{2}(\phi_{1x}^2 + \phi_{1y}^2)\right\}_t \\ & + a^3\left(\phi_{2ty}\eta_1 + \phi_{1ty}\eta_2 - \frac{1}{2}\phi_{1txx}\eta_1^2 + \phi_{1x}\phi_{1xy}\eta_1\right. \\ & \left. - \phi_{1y}\phi_{1xx}\eta_1 + \phi_{1x}\phi_{2x} + \phi_{1y}\phi_{2y}\right)_t.\end{aligned} \tag{26}$$

3.3. *The terms in* $\phi_x(L\phi)_x$. Using (24) and (25) again, and

$$\phi_x = a\phi_{1x} + \alpha^2(\phi_{2x} + \phi_{1xy}\eta_1) + \cdots,$$

one obtains

$$\begin{aligned}\phi_x(L\phi)_x &= a^2\phi_{1x}\phi_{1xt} + a^3\Big[\phi_{2x}\phi_{1xt} + \phi_{1xt}\phi_{1xy}\eta_1 \\ &\qquad + \phi_{1x}\Big\{\phi_{2t} + \phi_{1ty}\eta_1 + \frac{1}{2}(\phi_{1x}^2 + \phi_{1y}^2)\Big\}_x\Big] \\ &= a^2\phi_{1x}\phi_{1xt} + a^3\Big[(\phi_{1x}\phi_{2x})_t + (\phi_{1x}\phi_{1yx})_t\eta_1 \\ &\qquad + \phi_{1x}\phi_{1ty}\eta_{1x} + \frac{1}{2}\phi_{1x}(\phi_{1x}^2 + \phi_{1y}^2)_x\Big].\end{aligned} \tag{27}$$

Again, all terms on the right-hand side of (27) are evaluated at $y = 0$.

3.4. *The terms in Q*. The first term on the right-hand side of (20) is, by virtue of (8),

$$\eta_{txx} = (\phi_y - \phi_x\eta_x)_{xx} = \phi_{yxx} - (\phi_x\eta_x)_{xx}, \tag{28}$$

where all the derivatives of ϕ are still evaluated at $y = \eta$. Expansion of ϕ_{yxx} gives

$$\begin{aligned}\phi_{yxx} &= a\phi_{1yxx} + a^2\{\phi_{2y} - (\phi_{1xx}\eta_1)\}_{xx} \\ &\quad + a^3\Big\{\phi_{3y} - \phi_{1xx}\eta_2 - \phi_{2xx}\eta_1 - \frac{1}{2}\phi_{1yxx}\eta_1^2\Big\}_{xx},\end{aligned} \tag{29}$$

in which (1) is applied whenever applicable, and all the derivatives of ϕ on the right-hand side are evaluated at $y = 0$. As to the last term in (28), one has

$$\begin{aligned}-(\phi_x\eta_x)_{xx} &= -a^2(\phi_{1x}\eta_{1x})_{xx} \\ &\quad - a^3(\phi_{1x}\eta_{2x} + \phi_{2x}\eta_{1x} + \phi_{1xy}\eta_1\eta_{1x})_{xx}.\end{aligned} \tag{30}$$

Combining (29) with (30), one obtains the expression for η_{txx}. But one need not write this out, since later the terms in Q as given by (20) will be combined.

The next term in Q is simply, to $O(a^3)$,

$$-\frac{3}{2}(\eta_x^2\eta_{xx})_t = -\frac{a^3}{2}(\eta_{1x}^3)_{xt}, \tag{31}$$

and the last term is

$$\phi_x\eta_{xxx} = a^2\phi_{1x}\eta_{1xxx} + a^3(\phi_{2x}\eta_{1xxx} + \phi_{1x}\eta_{2xxx} + \phi_{1xy}\eta_1\eta_{1xxx}). \tag{32}$$

Combining (28) with (31) and (32), one obtains, with the help of (29) and (30),

$$Q = aQ_1 + a^2Q_2 + a^3Q_3, \tag{33}$$

where

$$Q_1 = \phi_{1yxx}, \tag{34}$$

$$\begin{aligned}Q_2 &= -\{(\phi_{1xx}\eta_1)_{xx} + \phi_{1xxx}\eta_{1x} + 2\phi_{1xx}\eta_{1xx}\} + \phi_{2yxx} \\ &= -\{(\phi_{1xx}\eta_1)_{xx} + (\phi_{1xx}\eta_{1x})_x + \phi_{1xx}\eta_{1xx}\} + \phi_{2yxx},\end{aligned} \tag{35}$$

$$\begin{aligned}Q_3 = -\Big\{&\Big(\phi_{1x}\eta_2 + \phi_{2x}\eta_1 + \frac{1}{2}\phi_{1yx}\eta_1^2\Big)_{xxx} + \frac{1}{2}(\eta_{1x}^3)_{xt} \\ &- (\phi_{1x}\eta_{2xxx} + \phi_{2x}\eta_{1xxx} + \phi_{1xy}\eta_1\eta_{1xxx})\Big\} + \phi_{3yxx}.\end{aligned} \tag{36}$$

4. Calculation for terms containing a as a factor. The first calculation is for terms that contain a as a factor. The terms may be of orders $O(a)$, $O(a\varepsilon)$, $O(a\varepsilon^2)$, or $O(a\varepsilon^3)$. The working equation for all the calculations for all orders is always (19), with its four groups of terms given by (23), (26), (27), and (33), supplemented by (34), (35), and (36). For the starting calculation, one collects terms containing a as a factor in (23), (26), and (33), and obtains

$$L_0\phi_1 = g\phi_{1y} + \phi_{1tt} - \sigma\phi_{1yxx}. \tag{37}$$

In addition to (10), one now defines

$$X = \varepsilon(x - c_g t), \qquad Y = k(x - ct), \tag{38}$$

and

$$\widehat{S} = \operatorname{sech}\widehat{X}, \qquad \widehat{T} = \tanh\widehat{X}, \qquad S = \operatorname{sech}X, \qquad T = \tanh X. \tag{39}$$

Note that it is immediately clear that $\widehat{S}$ and $\widehat{T}$ have singularities at the points $\widehat{X} = \pm i(2n+1)\pi/2$. Those corresponding to the $+$ sign are outside the liquid, and cause no difficulty. The remaining singularities can be removed, as shown in Yih [4]. This removal has very little effect on the free-surface conditions if k/ε is large compared with 1, as is the case for the phenomenon under investigation here. Higher singularities created at higher approximations can be similarly removed [4]. Now let

$$\phi_1 = \widehat{S}e^{-i\widehat{Y}} + \widehat{S}^* e^{i\widehat{Y}^*}, \tag{40}$$

so that, at $y = 0$,

$$\begin{aligned}
\phi_1 &= 2S\cos Y, \\
\phi_{1y} &= 2kS\cos Y - 2\varepsilon ST\sin Y, \\
\phi_{1x} &= -2kS\sin Y - 2\varepsilon ST\cos Y, \\
\phi_{1tt} &= (-2k^2c^2 + 2\varepsilon^2c_g^2)S\cos Y + 4k\varepsilon cc_g ST\sin Y \\
&\qquad - 4(\varepsilon c_g)^2 S^3\cos Y, \\
\phi_{1yxx} &= (-2k^3 + 6k\varepsilon^2)S\cos Y + (6k^2 - 2\varepsilon^2)\varepsilon ST\sin Y \\
&\qquad - 12k\varepsilon^2S^3\cos Y + 12\varepsilon^3S^3\sin Y.
\end{aligned} \tag{41}$$

Then

$$\begin{aligned}
L_0\phi_1 = {} & 2AS\cos Y - 2\varepsilon BST\sin Y + \varepsilon^2(12\sigma k - 4c_g^2)S^3\cos Y \\
& - 12\sigma\varepsilon^3S^3T\sin Y,
\end{aligned} \tag{42}$$

where

$$A = gk + \sigma k^3 - k^2c^2 + \varepsilon^2(c_g^2 - 3\sigma k), \tag{43}$$

$$B = g + 3\sigma k^2 - 2kcc_g - \sigma\varepsilon^2. \tag{44}$$

In (41) and (42), some higher-order terms which are unnecessary for the main purpose of this paper are exhibited, to give an indication of why they are unnecessary. For instance, it will become clear that the $2\varepsilon^2 c_g^2$ term in ϕ_{1tt} and the term $6k\varepsilon^2$ in ϕ_{1yxx} in (41) merely lead to higher-order determination of ε^2 in terms of a^2. Subsequent development will show that $A = 0 = B$.

To obtain η_1, one uses

$$\eta_{1t} = \phi_{1y}$$

and integrates by parts repeatedly to obtain

$$\eta_1 = 2(E_1 S \sin Y + \varepsilon E_2 ST \cos Y + \varepsilon^2 E_3 S^3 \sin Y) + O(\varepsilon^3), \tag{45}$$

in which

$$E_1 = -\frac{1}{c} - \frac{\varepsilon^2}{k^2 cr}(1-r), \qquad E_2 = \frac{1}{kc}(1-r), \qquad E_3 = \frac{2}{k^2 cr}(1-r),$$

where

$$r = c_g/c. \tag{46}$$

Again some unnecessary terms are included in η_1. In subsequent calculations, one needs only to take

$$\eta_1 = \frac{2}{c}\left\{-S \sin Y + \frac{\varepsilon}{k}(1-r)ST \cos Y\right\}. \tag{47}$$

5. Calculation for ϕ_2 and η_2. Taking terms containing a^2 as a factor in (21), (26), (27), and (33), and applying (19), one has, with all terms on the right-hand side evaluated at $y = 0$,

$$L_0 \phi_2 = T_{21} + T_{22} + T_{23} + T_{24} + T_{25} + T_{26} + T_{27}, \tag{48}$$

where

$$L_0 = g\frac{\partial}{\partial y} + \frac{\partial^2}{\partial t^2} - \sigma \frac{\partial^3}{\partial y \partial x^2}, \tag{49}$$

$$\begin{aligned} T_{21} &= g\phi_{1xx}\eta_1, & T_{22} &= -(\phi_{1ty}\eta_1)_t, \\ T_{23} &= -\tfrac{1}{2}(\phi_{1x}^2 + \phi_{1x}^2)_t, & T_{24} &= -\phi_{1x}\phi_{1xt}, \\ T_{25} &= -\sigma(\phi_{1xx}\eta_1)_{xx}, & T_{26} &= -\sigma(\phi_{1xx}\eta_{1x})_x, \\ T_{27} &= -\sigma\phi_{1xx}\eta_{1xx}. \end{aligned}$$

From (41) and (46),

$$\begin{aligned} T_{21} &= \frac{2gk^2}{c}\left[S^2 \sin 2Y - \frac{\varepsilon}{k}S^2 T\{1 + r - (3-r)\cos 2Y\}\right], \\ T_{22} &= -4k^3 c\left[S^2 \sin 2Y + \frac{\varepsilon}{k}S^2 T\{-r + (2+r)\cos 2Y\}\right], \\ T_{23} &= -4k^2 \varepsilon c r S^2 T, \\ T_{24} &= 2k^3 c\left[S^2 \sin 2Y - \frac{\varepsilon}{k}S^2 T\{r - (2+r)\cos 2Y\}\right], \\ T_{25} &= \frac{8k^4\sigma}{c}\left[S^2 \sin 2Y - \frac{\varepsilon}{k}(5-r)S^2 T \cos 2Y\right], \\ T_{26} &= \frac{4k^4\sigma}{c}\left[S^2 \sin 2Y + \frac{\varepsilon}{k}S^2 T\{1 + (5-r)\cos 2Y\}\right], \\ T_{27} &= \frac{2k^4\sigma}{c}\left[S^2 \sin 2Y - \frac{\varepsilon}{k}S^2 T\{-1 + r + (-5+r)\cos 2Y\}\right]. \end{aligned}$$

Upon collection of terms, (48) becomes

$$L_0\phi_2 = C_{21}S^2 \sin 2Y + C_{22}S^2 T + C_{23}S^2 T \cos 2Y\,, \tag{50}$$

where

$$C_{21} = \frac{2k^2}{c}(-kc^2 + g + 7\sigma k^2)\,,$$
$$C_{22} = \frac{2\varepsilon k}{c}\{(3-r)\sigma k^2 - g(1+r) - kc^2 r\}\,,$$
$$C_{23} = \frac{2\varepsilon k}{c}\{7(5-r)\sigma k^2 - (2kc^2 + kc^2 r - 3g + gr)\}.$$

Before using (50) for ϕ_2, it is advantageous to simplify the C's. This is done by anticipating the results

$$A = 0 = B\,, \tag{51}$$

where A and B are given by (43) and (44). For the purpose at hand it is permissible and desirable to ignore the terms of $O(\varepsilon^2)$ in (43) and (44), which will give terms of $O(a^3\varepsilon^2)$ when the final step in determining ε^2 in terms of a^2 is taken. Then, using (51), one obtains, after some substitutions,

$$C_{21} = \frac{12\sigma k^4}{c}\,, \tag{52}$$

$$C_{22} = \frac{4\varepsilon k g}{c}\,, \tag{53}$$

$$C_{23} = \frac{12\varepsilon\sigma k^3}{c}(5-r) = \frac{6\varepsilon k g}{c}\frac{\beta(9+7\beta)}{1+\beta}\,, \tag{54}$$

where

$$\beta = \frac{\sigma k^2}{g}. \tag{55}$$

One now proceeds to calculate ϕ_2. For this purpose define

$$\phi_{20} = -i(\widehat{S}^2 - \text{c.c.}) \tag{56}$$

$$\phi_{21} = -i\{\widehat{S}^2 \exp(-i2\widehat{Y}) - \text{c.c.}\}\,, \tag{57}$$

$$\phi_{22} = \{\widehat{S}^2\widehat{T}\exp(-i2\widehat{Y}) + \text{c.c.}\}\,, \tag{58}$$

where c.c. denotes the complex conjugate of the preceding quantity.

Direct calculations show that, at $y = 0$ and with terms of $O(\varepsilon^2)$ neglected,

$$L_0\phi_{20} = -4\varepsilon g S^2 T\,, \tag{59}$$

$$L_0\phi_{21} = 4kg(1-2\beta)S^2 \sin 2Y + 4\varepsilon g(1-6\beta)S^2 T\cos 2Y\,, \tag{60}$$

$$L_0\phi_{22} = -4kg(1-2\beta)S^2 T\cos 2Y + O(\varepsilon). \tag{61}$$

With these results,

$$\phi_2 = \frac{k}{c}\left(\phi_{20} + \alpha_{21}\phi_{21} + \frac{\varepsilon}{k}\alpha_{22}\phi_{22}\right)\,, \tag{62}$$

where

$$\alpha_{21} = \frac{3\beta}{1-2\beta}, \tag{63}$$

$$\alpha_{22} = -\frac{3\beta(7-\beta-2\beta^2)}{2(1+\beta)(1-2\beta)^2}. \tag{64}$$

Note that the last term in (60) adds to the burden of the last term in (50), when the first term on the right-hand side of (50) is taken care of. That is why (64) is what it is, to account for that additional burden and to take care of the last term in (50). The factor ε in the last term of (62) shows why it is permissible to drop the term of $O(\varepsilon)$ in (61).

With ϕ_2 defined by (62), one is now in a position to calculate η_2. For that purpose, and for the purpose of later calculations, one lists some useful results below, evaluated at $y=0$:

ϕ_{20} and all its derivatives with respect to x or t are zero,

$$\begin{aligned}
(\phi_2)_y &= -4\varepsilon S^2 T, \\
(\phi_{20})_{yxx} &= O(\varepsilon^3), \\
\phi_{21} &= -2S^2 \sin 2Y, \\
(\phi_{21})_y &= -4kS^2 \sin 2Y - 4\varepsilon k S^2 T \cos 2Y, \\
(\phi_{21})_x &= -4kS^2 \cos 2Y + 4\varepsilon S^2 T \sin 2Y, \\
(\phi_{21})_{xx} &= 8k^2S^2 \sin 2Y + 16\varepsilon k S^2 T \cos 2Y, \\
(\phi_{21})_{xt} &= -8k^2cS^2 \sin 2Y - 8\varepsilon kc(1+r)S^2 T \cos 2Y, \\
\phi_{22} &= 2S^2 T \cos 2Y, \\
(\phi_{22})_y &= 4kS^2 T \cos 2Y, \\
(\phi_{22})_x &= -4kS^2 T \sin 2Y, \\
(\phi_{22})_{xx} &= -8kS^2 T \cos 2Y, \\
(\phi_{22})_{xt} &= 8kcS^2 T \cos 2Y.
\end{aligned}$$

Again, terms of $O(\varepsilon)$ in ϕ_{22} and its derivatives are omitted, because they ultimately produce terms of $O(a^3\varepsilon^2)$ in the crucial calculation for ε^2 in terms of a^2. Note the coefficient of ϕ_{22} in (62).

One now calculates η_2 from

$$\eta_{2t} = \phi_{2y} + \phi_{1yy}\eta_{1x} - \phi_{1x}\eta_{1x} = \phi_{2y} - (\phi_{1x}\eta_1)_x \tag{65}$$

which comes from (8). With the results for ϕ_{20}, ϕ_{21}, and ϕ_{22} given in the foregoing, (65) becomes

$$\eta_{2t} = -\frac{4k^2}{c}(\alpha_{21}+1)S^2 \sin 2Y + \frac{2k\varepsilon}{c}\theta S^2 T \cos 2Y, \tag{66}$$

where

$$\theta = 2\alpha_{22} - 2\alpha_{21} - 6 + 2r. \tag{67}$$

The solution of (66) is, by successive approximation,

$$\eta_2 = -\frac{2k}{c^2}(\alpha_{21} + 1)S^2 \cos 2Y - \frac{\varepsilon}{c^2}\{\theta + 2(\alpha_{21} + 1)r\}S^2 T \sin 2Y. \tag{68}$$

For brevity, one will write this as

$$\eta_2 = \frac{2k}{c^2}\left(E_{21}S^2 \cos 2Y + \frac{\varepsilon}{k}E_{22}S^2 T \sin 2Y\right), \tag{69}$$

where

$$E_{21} = -(\alpha_{21} + 1), \qquad E_{22} = -\left(\frac{\theta}{2} + r(\alpha_{21} + 1)\right). \tag{70}$$

6. Calculation of terms containing a^3 as a factor. Consider now the terms that contain a^3 or $a^3\varepsilon$ as a factor in (19). Again, for clarity, it is desirable to consider these terms in each of the terms on the left-hand side of (19). The calculation will be more detailed for the first term of (19), to show how the detailed calculation is done. Once the way of calculation is illustrated, the presentation will be much briefer.

6.1. *Terms from* $g\phi_y$. Leaving the term ϕ_{3y} aside for the moment, the first term containing a^3 as a factor is

$$\phi_{2yy}\eta_1 = -\phi_{2xx}\eta_1,$$

since ϕ_2 does satisfy the Laplace equation. As can be readily verified, on $y = 0$,

$$\phi_{2xx} = \frac{8k^3}{c}\left\{\alpha_{21}S^2 \sin 2Y + \frac{\varepsilon}{k}(2\alpha_{21} - \alpha_{22})S^2 T \cos 2Y\right\},$$

so that

$$\begin{aligned} -\phi_{2xx}\eta_1 = \frac{8k^3}{c}[\alpha_{21}S^3(\cos Y - \cos 3Y) \\ + \frac{\varepsilon}{k}S^3 T\{\alpha_{21}(1 - r)(\sin 3Y + \sin Y) \\ + (2\alpha_{21} - \alpha_{22})(\sin 3Y - \sin Y)\}]. \end{aligned}$$

Of this, the terms involving $\sin Y$ or $\cos Y$ are

$$[[-\phi_{2xx}\eta_1]] = \frac{8k^3}{c}[\alpha_{21}S^3 \cos Y + \frac{\varepsilon}{k}\{\alpha_{22} - \alpha_{21}(1 + r)\}S^3 T \sin Y], \tag{71}$$

in which the double brackets on the left-hand side mean "terms containing $\sin Y$ or $\cos Y$ in the quantity indicated in the brackets." Equation (71) is tabulated in Table 1, for convenience. The other terms in (23), except ϕ_{3y}, are also entered in Table 1 after multiplication by g.

6.2. *Terms from* $(L\phi)_t$. One now considers the terms in the last parenthesis in (26). The method of calculation being illustrated in Sec. 6.1, one now merely records the results. In Table 2, the results for $L\phi$ are recorded.

In forming the sum in Table 2, the definition of θ by (67) has been used, so that

$$\alpha_{21}(2r + 1) + \frac{\theta}{2} = 2r\alpha_{21} + \alpha_{22} + 3r - 2.$$

TABLE 1. Terms associated with a^3 in $g\phi_y$.

	$S^3 \cos Y$	$S^3 T \sin Y$
$g[[-\phi_{2xx}\eta_1]]$	$\frac{8k^3 g}{c^2}\alpha_{21}$	$\frac{8k^2\varepsilon g}{c^2}\{\alpha_{22} - \alpha_{21}(1+r)\}$
$g[[-\phi_{1xx}\eta_2]]$	$\frac{-2k^3 g}{c^2}(\alpha_{21}+1)$	$-\frac{2k^2\varepsilon g}{c^2}\{\frac{\alpha_{22}}{2} + (1+r)\alpha_{21} - 1 + 2r\}$
$\frac{g}{2}[[-\phi_{1yxx}\eta_1^2]]$	$\frac{k^3 g}{c^2}$	$-\frac{k^2\varepsilon g}{c^2}(7+2r)$
Sum	$\frac{k^3 g}{c^2}(6\alpha_{21} - 1)$	$\frac{k^2\varepsilon g}{c^2}\{7\alpha_{22} - 10(1+r)\alpha_{21} - 5 - 6r\}$

TABLE 2. Terms associated with a^3 in $L\phi$.

	$S^3 \sin Y$	$S^3 T \cos Y$
$[[\phi_{2ty}\eta_1]]$	$\frac{8k^3}{c}\alpha_{21}$	$\frac{4k^2\varepsilon}{c}(4\alpha_{21}r - 2\alpha_{22})$
$[[\phi_{1ty}\eta_2]]$	$\frac{2k^3}{c}(\alpha_{21}+1)$	$-\frac{2k^2\varepsilon}{c}\{(\alpha_{21}+1)(2r+1) + \frac{\theta}{2}\}$
$[[\phi_{1x}\phi_{2x} + \phi_{1y}\phi_{2y}]]$	$-\frac{8k^3}{c}\alpha_{21}$	$-\frac{2k^2\varepsilon}{c}(4 - 4\alpha_{22})$
$\frac{1}{2}[[\phi_{1txx}\eta_1^2]]$	$\frac{3k^3}{c}$	$\frac{2k^2\varepsilon}{c}(2 - \frac{r}{2})$
$\frac{1}{2}[[(\phi_{1x}^2 + \phi_{1y}^2)_y\eta_1]]$	$-\frac{8k^3}{c}$	$-\frac{8k^2\varepsilon}{c}(1-r)$
Sum	$\frac{k^3}{c}(2\alpha_{21} - 3)$	$\frac{2k^2\varepsilon}{c}(6r\alpha_{21} - \alpha_{22} + \frac{1}{2}r - 4)$

One now has

$$
\begin{aligned}
[[(L\phi)_t]] &= \left[\frac{k^3}{c}(2\alpha_{21} - 3)S^3 \sin Y \right.\\
&\quad \left. + \frac{2k^2\varepsilon}{c}\left(6r\alpha_{21} - \alpha_{22} + \frac{1}{2}r - 4\right) S^3 T \cos Y\right]_t \\
&= -k^4(2\alpha_{21} - 3)S^3 \cos Y \\
&\quad + k^3\varepsilon(18r\alpha_{21} - 2\alpha_{22} - 8r - 8)S^3 T \sin Y \\
&= k^4\left[(3 - 2\alpha_{21})S^3 \cos Y \right.\\
&\quad \left. + \frac{\varepsilon}{k}(18r\alpha_{21} - 2\alpha_{22} - 8r - 8)S^3 T \sin Y\right].
\end{aligned} \tag{72}
$$

6.3. *Terms from* $\phi_x(L\phi)_x$. Results for these terms, shown in (27), are summarized in Table 3.

6.4. *Terms in* Q. The terms in Q defined by (20) which have a^3 as a factor are in Q_3 defined in (36). Again neglecting ϕ_3 for the moment, one computes the terms in the braces in (36). First, the terms within the first parenthesis in (36) are summarized in Table 4.

Let

$$
I = \left[\left[\phi_{1x}\eta_2 + \phi_{2x}\eta_1 + \frac{1}{2}\phi_{1yx}\eta_1^2\right]\right].
$$

TABLE 3. Terms associated with a^3 in $\phi_x(L\phi)_x$.

	$S^3\cos Y$	$S^3T\sin Y$
$\left[\left[\frac{1}{2}\phi_{1z}(\phi_{1x}^2+\phi_{1y}^2)_x\right]\right]$	0	$8k^3\varepsilon$
$[[(\phi_{1x}\phi_{1xy})_t\eta_1]]$	$4k^4$	$-8k^3\varepsilon(2r+1)$
$[[(\phi_{1x}\phi_{2x})_t]]$	$4k^4\alpha_{21}$	$-4k^3\varepsilon(3r\alpha_{21}-\alpha_{22})$
$[[\phi_{1x}\phi_{1yt}\eta_{1x}]]$	$2k^4$	$8k^3\varepsilon(r-1)$
Sum	$k^4(6+4\alpha_{21})$	$-4k^3\varepsilon(2r+2+3r\alpha_{21}-\alpha_{22})$

TABLE 4. Terms in the first parenthesis of (36).

	$S^3\sin Y$	$S^3T\cos Y$
$[[\phi_{1x}\eta_2]]$	$-\frac{2k^2}{c^2}(\alpha_{21}+1)$	$\frac{2k\varepsilon}{c^2}\left\{(\alpha_{21}+1)(1+r)+\frac{\theta}{2}\right\}$
$[[\phi_{2x}\eta_1]]$	$-\frac{4k^2}{c^2}\alpha_{21}$	$\frac{4k\varepsilon}{c^2}(\alpha_{22}-\alpha_{21}r)$
$\frac{1}{2}[[\phi_{1yx}\eta_1^2]]$	$-\frac{3k^2}{c^2}$	$-\frac{k\varepsilon}{c^2}(4-2r)$
Sum	$-\frac{k^2}{c^2}(6\alpha_{21}+5)$	$\frac{k\varepsilon}{c^2}\{4r-2+2(1-r)\alpha_{21}+4\alpha_{22}+\theta\}$

Then

$$\begin{aligned} I_{xxx} = -\frac{k^5}{c^2}\Big[& -(6\alpha_{21}+5)S^3\cos Y \\ & +\frac{\varepsilon}{k}\{3(6\alpha_{21}+5)(-3S^2T)(-\sin Y)\} \\ & +\frac{\varepsilon}{k}\{2(r-1)\alpha_{21}-4\alpha_{22}+2-4r-\theta\}S^3T\sin Y\Big] \\ = \frac{k^5}{c^2}\Big[& (6\alpha_{21}+5)S^3\cos Y \\ & -\frac{\varepsilon}{k}\{(52+2r)\alpha_{21}-4\alpha_{22}+47+4r-\theta\}S^3T\sin Y\Big]. \end{aligned}$$

Upon use of (67), this becomes, finally,

$$I_{xxx} = \frac{k^5}{c^2}\left[(6\alpha_{21}+5)S^3\cos Y - \frac{\varepsilon}{k}\{(54+2r)\alpha_{21}-6\alpha_{22}+53-6r\}S^3T\sin Y\right],$$

and this is recorded in Table 5, where the other terms in $[[-Q_3]]$ are also recorded.

Summarizing the results in the last lines of Tables 1, 3, and 5, and in (72), one has the final Table 6.

In Table 6 c^2 can be identified with the c_0^2 defined in (11), which is

$$c_0^2 = \frac{g}{2}(1+\beta),$$

where β is defined by (55). Then, with the help of (63) and (64), a straightforward calculation gives

$$S_1 = \frac{k^4}{(1+\beta)(1-2\beta)}(8+\beta+2\beta^2). \tag{73}$$

There is no need to carry out S_2 in more detail for the purpose at hand.

TABLE 5. Terms in $-Q_3$.

	$S^3 \cos Y$	$S^3 T \sin Y$
I_{xxx}	$\frac{k^5}{c^2}(6\alpha_{21}+5)$	$\frac{\varepsilon k^4}{c^2}\{(54+2r)\alpha_{21}-6\alpha_{22}+53-6r\}$
$\frac{1}{2}[[(\eta_{1x}^3)_{xt}]]$	$-\frac{3k^5}{c^2}$	$\frac{3\varepsilon k^4}{c^2}(5+2r)$
$[[-\phi_{1x}\eta_{2xxx}]]$	$-\frac{16k^5}{c^2}(\alpha_{21}+1)$	$\frac{8\varepsilon k^4}{c^2}\{(6-2r)\alpha_{21}-2\alpha_{22}+10-4r\}$
$[[-\phi_{2x}\eta_{1xxx}]]$	$\frac{4k^5}{c^2}\alpha_{21}$	$\frac{4\varepsilon k^4}{c^2}(-\alpha_{21}+\alpha_{22}+4-r)$
$[[-\phi_{1xy}\eta_1\eta_{1xxx}]]$	$-\frac{2k^5}{c^2}$	$-\frac{\varepsilon k^4}{c^2}(6-4r)$
Sum $=[[-Q_3]]$	$\frac{k^5}{c^2}(6\alpha_{21}+16)$	$-\frac{\varepsilon k^4}{c^2}\{(10+18r)\alpha_{21}+6\alpha_{22}-52+20r\}$

TABLE 6. Summary of results associated with a^3.

	$S^3 \cos Y$	$S^3 T \sin Y$
$[[g\phi_y]]$	$\frac{k^3 g}{c^2}(6\alpha_{21}-1)$	$\frac{\varepsilon k^2 g}{c^2}\{-10(1+r)\alpha_{21}+7\alpha_{22}-5-6r\}$
$[[(L\phi)_t]]$	$k^4(-2\alpha_{21}+3)$	$\varepsilon k^3\{18r\alpha_{21}-2\alpha_{22}-8-8r\}$
$[[\phi_x(L\phi)_x]]$	$k^4(4\alpha_{21}+6)$	$-4\varepsilon k^3\{3r\alpha_{21}-4\alpha_{22}+2+2r\}$
$[[-\sigma Q]]$	$-\frac{\sigma k^5}{c^2}(6\alpha_{21}+16)$	$-\frac{\varepsilon\sigma k^4}{c^2}\{(10+18r)\alpha_{21}+6\alpha_{22}-52+20r\}$
Sum	S_1	S_2

7. Determination of ε^2 as a function of a^2. The third-order terms of the left-hand side of (19) that contains the factors $S^3 \cos Y$ and $S^3 T \sin Y$ have now been determined to be

$$a^3 S_1 S^3 \cos Y + a^3 S_2 S^3 T \sin Y, \tag{74}$$

and one is in a position to determine ε^2 in terms of a^2. For this purpose one returns to (42), which, after multiplication by a, gives the terms in (19) containing a as a factor, as can be seen from (37). It is then clear that (74) should be combined with the last two terms in (42), after (42) is multiplied by a. (See (12).) Collecting the terms containing the factor $S^3 \cos Y$ in (74) and (42), after the factor a is added to it, one has

$$a\varepsilon^2(12\sigma k - 4c_g^2) + a^3 S_1 = 0. \tag{75}$$

In this one can set c_g^2 equal to $(c_g)_0^2$, committing discrepancies of higher orders to be taken care of in later approximations if necessary. Then, using (11), one obtains

$$\varepsilon^2 = \frac{a^2 k^5}{g}\frac{8+\beta+2\beta^2}{(1-6\beta-3\beta^2)(1-2\beta)}, \tag{76}$$

and

$$\varepsilon^2(c_g^2 - 3\sigma k) = \frac{a^2 k^4}{4}\frac{8+\beta+2\beta^2}{(1+\beta)(1-2\beta)}, \tag{77}$$

which can be used in (43) to calculate the variation of c^2 with a^2, to the present order of approximation. Since ε^2 must be positive, it is evident that the solution for wave groups considered here is restricted to those values of β that make it so in (76).

Collecting the terms containing the factor $S^3 T \sin Y$ in (42), after it has been multiplied by a, and in (74), one has

$$(-12a\varepsilon^3\sigma + a^3 S_2)S^3 T \sin Y. \tag{78}$$

The question immediately arises: Why is (78) not set equal to zero to determine ε^2? This question does not arise for pure gravity waves, since $\sigma = 0$ for that case. But it does now. And the two determinations of ε^2, from (77) and from (78), would not give the same results. The answer to this question is that the term (78) can be annihilated by the same procedure as in [1], whereas if an attempt is made to annihilate a term with the factor $S^3 \cos Y$ and the coefficient equal to the left-hand side of (75), one is obliged to add a term to ϕ that has the factor $S \cos Y$. In other words ϕ_1 would change its amplitude, and that amplitude is not to be tampered with. Therefore (75) is used to determine ε^2, and (78) is to be annihilated.

At this juncture it is prudent to check (77) and (78) against the corresponding results for pure gravity waves, by putting σ or β equal to zero in S_1 and S_2 in (74), and compare it with (60) of [1], which will be denoted by [1, (60)] for convenience. Doing so, one finds, upon putting c^2 equal to c_0^2,

$$S_1 = 8k^4, \qquad S_2 = -32\varepsilon k^3. \tag{79}$$

The result for S_1 agrees with [1, (60)], but the number corresponding to 32 in (79) is 56 in [1, (60)]. Close examination of [1, (59)] shows that a factor of $\frac{1}{2}$ in [1, (58)] had been overlooked, and the number 48 in [1, (59)] should be 24. Correction of that error changes the number 56 in [1, (60)] to 32, in agreement with (79). The oversight has no effect whatever on the main results and all the conclusions of [1], but I take this opportunity to present the corrigenda for the errors that arose from that oversight. Apart from the corrections mentioned above, all the numbers divisible by 7 in [1, (64)] down to [1, (68)], as well as in [1, (77)] and the equation in [1] following [1, (77)], should be multiplied by $\frac{4}{7}$. In [1, (78)], 17 should be 11. Corrections for some unrelated misprints in [1] are: (a) In the last term of the line 6 from the bottom of p. 170 of [1], a plus sign should be added between ϕ_x^2 and ϕ_y^2. (b) In the last term of [1, (25)], g^2 should be c_g^2. (c) In [1, (37)], the factor a should be added after the first two equality signs, and $\varepsilon^3 a^3$ should be $\varepsilon^2 a^3$. (d) In [1, (63)], ε should be ε^2.

The annihilation of the term (78) is similar to the same process in [1]. It involves adding a term like Z in [1] to ϕ, and results in (51). Consequently (75) or (76) is not affected by this annihilation. The main results are then (76) and

$$k^2c^2 = gk + \sigma k^3 + \varepsilon^2(c_g^2 - 3\sigma k), \tag{80}$$

$$2kcc_g = g + 3\sigma k^2 - \sigma\varepsilon^2. \tag{81}$$

Equation (76) determines ε^2 to the order of approximation achieved in this paper.

Higher-order approximations will determine ε^2 as a power series in a^2. But (80) and (81) in terms of ε^2 will remain valid to any order of approximation.

As in [1], the terms of $O(a^3)$ in (19) containing $S^3 \cos 3Y$ or $S^3 T \sin 3Y$, as appear in the equation before (71) and in the calculation for Secs. 6.2, 6.3, and 6.4 though not recorded in those sections, can be annihilated without affecting (80) and (81). Indeed, higher approximations will leave (80) and (81) intact, although they will improve (76) and determine ε^2 as a power series of a^2. This is the reason why it is not necessary to determine ϕ_3 for the purposes of this paper.

8. Comparison with existing results. The results obtained in this paper can be compared with the results obtained by Ablowitz and Segur [2], which was referred to as AS in [1], and the results of Djordjević and Redekopp [3], referred to as DR in [1]. The abbreviations will continue to be used in this paper.

First, AS (p. 697, last two lines) noted that their results were equivalent to those of DR, except for the correction of a misprint. Examination of (AS 2.24d) also reveals a misprint: The factor $(2-\sigma^2)$ in that equation should be $(3-\sigma^2)$. However, that misprint was truly a mere misprint; the results on p. 698 of AS are correct. One then needs only compare the results of this paper with those of AS. For this purpose focus on (AS 3.4), in which, as AS said, b can be taken to be zero.

When ω_0/ω in the definition of λ in AS (equation for λ_∞ on p. 698 of AS) is equated to my $(1+\beta)^{1/2}$,

$$\lambda = -\frac{1}{8}\frac{1-6\beta-3\beta^2}{(1+\beta)^{3/2}}, \tag{82}$$

where my β has replaced $\tilde{T}$ in AS. Furthermore (AS 3.4) also shows

$$|2\lambda/\nu|^{1/2} a = A_{\max}, \tag{83}$$

in which ν is the X_∞ on p. 698 of AS, and

$$\frac{2\lambda}{\nu} = -\frac{(1-6\beta-3\beta^2)(1-2\beta)}{8+\beta+2\beta^2}. \tag{84}$$

The solution (AS 3.4) is valid only if $2\lambda/\nu$ is negative. This agrees with the requirement that the ε^2 in (76) be positive, and limits the possibility of having a gravity-capillary wave train to the β-values satisfying

$$0 \le \beta < \frac{-3+\sqrt{12}}{3} = 0.1547 \quad \text{or} \quad \beta > \frac{1}{2}. \tag{85}$$

When either of these inequalities is satisfied, my ϕ_1 has the same envelope as the AS solution (AS 3.4). This can be seen by noting the definition of ξ in (AS 2.22), and that the $a\varepsilon$ in AS corresponds to my ε. The former is, in AS notation entirely,

$$a\varepsilon = \frac{A_{\max}\varepsilon}{|2\lambda/\nu|^{1/2}} \tag{86}$$

and my ε is given by (76). Taking into account that the velocity scale used in AS is $(g/k)^{1/2}$ and the length scale is k^{-1}, and that their $A_{\max}\varepsilon$ is dimensionless and corresponds with my dimensional a, one has (the right-hand in my notation)

$$A_{\max}\varepsilon = \frac{ak^{3/2}}{g}. \tag{87}$$

Then, in view of (84), with $2\lambda/\nu$ negative, one sees that, apart from being dimensional, the $a\varepsilon$ in AS is exactly my ε given by (76), showing agreement of the form of the envelope of ϕ_1 with the envelope of A in (AS 3.4).

Now one looks at the variation of the phase velocity c with the amplitude. In the AS solution (AS 3.4), the last factor can be combined with the factor

$$\exp(i\theta) = \exp[i(kx - \omega t)] \tag{88}$$

in (AS 2.15a) to obtain, in view of the definition of τ in (AS 2.22),

$$c = \frac{\omega}{k}\left\{1 - \lambda a^2\varepsilon^2\frac{(gk)^{1/2}}{\omega}\right\}. \tag{89}$$

In (89), c is in my notation and the rest in AS notation, except one has equated k with κ in AS, because only two-dimensional waves are considered. Recalling (85), and hence that λ is negative, as is $2\lambda/\nu$, and using (84), (86), and (β and c_0 in my notation)

$$\frac{\omega}{k} = c_0, \qquad \frac{(gk)^{1/2}}{\omega} = (1+\beta)^{-1/2},$$

one can write (89) as

$$c = c_0\left(1 + \frac{A_{\max}^2\varepsilon^2}{8}\frac{8+\beta+2\beta^2}{(1+\beta)^2(1-2\beta)}\right), \tag{90}$$

or, in view of (87),

$$c = c_0\left(1 + \frac{a^2k^3}{8g}\frac{8+\beta+2\beta^2}{(1+\beta)^2(1-2\beta)}\right), \tag{91}$$

now entirely in my notation.

On the other hand c is given by (80) in my analysis, and is

$$c = c_0\left\{1 + \frac{\varepsilon^2(c_g^2 - 3\sigma k)}{k^2c_0^2}\right\}. \tag{92}$$

When $(c_g)_0$ is used for c_g, committing an error of $O(\varepsilon^4)$, this is, upon use of (11) and (77),

$$c = c_0\left(1 + \frac{a^2k^3}{8g}\frac{8+\beta+2\beta^2}{(1+\beta)^2(1-2\beta)}\right), \tag{93}$$

which agrees with (91) exactly. This agreement is remarkable in view of the vast difference between the approach of AS and mine, and in view of the tremendous amount of detailed calculation in their work and in mine. From (95), it can be seen that c increases or decreases with the amplitude accordingly as $\beta < \frac{1}{2}$ or $B > \frac{1}{2}$. The analysis is not valid in the neighborhood of $\beta = \frac{1}{2}$.

As to the group velocity, one first uses (11) to reduce (92) to the form

$$c = c_0 \left(1 + \frac{\varepsilon^2}{k^2} \frac{1 - 6\beta - 3\beta^2}{8(1+\beta)^2} \right), \tag{94}$$

and then uses (94) in (81) to obtain

$$c_g = (c_g)_0 \left[1 - \frac{\varepsilon^2}{k^2} \left(\frac{1 - 6\beta - 3\beta^2}{8(1+\beta)^2} + \frac{\beta}{1+3\beta} \right) \right]. \tag{95}$$

One emphasizes that (81) is good to any order of approximation. To $O(\varepsilon^2)$, one has (95). When the first inequality in (85) holds, c_g always *decreases* with the amplitude. Since the parenthesis containing β is equal to

$$\frac{(1-\beta)(\beta^2 + 6\beta + 1)}{8(3\beta^3 + 7\beta^2 + 5\beta + 1)}, \tag{96}$$

it is clear that c_g decreases with ε^2 for

$$\frac{1}{2} < \beta < 1,$$

but increases with β when $\beta > 1$.

Since AS and DR did not go beyond the third order in the amplitude in their calculation, their approach has not produced any information on the variation of the group velocity with amplitude.

9. Interaction of gravity-capillary wave trains. Consider normal (that is, not oblique) interaction between two trains of gravity-capillary waves. The cases of interaction are: overtaking of one wave train by another, head-on collision of one wave train with another, and reflection of a wave train from a vertical wall normal to its path. The last case is a special case of head-on collision. By using the arguments in [1], one concludes that in all cases of normal interaction each of the two wave trains involved will regain its integrity after interaction, without even a shift of phase either in basic waves or in their envelope.

10. Conclusions. From the foregoing the following conclusions may be drawn:

a. A systematic approach has been devised to construct gravity-capillary wave trains to any degree of approximation.

b. Whenever comparison is possible, the results of this paper have been compared with the results of previous workers (AS and DR) who treated the same subject, and complete agreement has been found. The areas of comparison include the criterion of existence of gravity-capillary wave trains (when $0 \le \beta < 0.1547$ or $\beta > \frac{1}{2}$), the shape of the envelope, and the variation of the phase velocity of the basic waves with amplitude. The agreement is remarkable in view of the vast difference of my approach from theirs, and in view of the great amount of detailed calculation in my work and in theirs.

c. Two closed formulas, (80) and (81), give the variation of the phase velocity c of the basic waves and the group velocity c_g with the wavenumber ε of the envelope. The quantity ε^2 can be expressed as a power series in a^2 and determined by successive approximations.

d. Gravity-capillary wave trains exist for β-values satisfying

$$0 \leq \beta < 0.1547 \quad \text{or} \quad \beta > \frac{1}{2}.$$

e. The phase velocity c of the basic waves increases with amplitude when $\beta < 0.1547$, but decreases with amplitude when $\beta > \frac{1}{2}$.

f. The group velocity c_g decreases with amplitude when

$$\beta < 0.1547 \quad \text{or} \quad \frac{1}{2} < \beta < 1,$$

but increases with amplitude when $\beta > 1$.

g. Previous investigators of gravity-capillary wave trains have carried their calculations to the third order of the amplitude, and therefore have not been able to produce any results on the variation of the group velocity with amplitude.

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