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A NOTE ON SOME QUALITATIVE RESULTS FOR BEAMS AND RODS

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Abstract. The small parameter technique of perturbation theory is used to argue that torsional buckling effects are of the same order of magnitude as beam column effects in the analysis of three-dimensional frames. Applications to the computer analysis of nonlinear structures are discussed.

Introduction. The qualitative result that

> *In the case of truss analysis, geometric stiffness effects are of the same order-of-magnitude as linear effects when prestress is present*

helps explain the fact that while the two-bar truss of Fig. 1 is geometrically unstable (can't be solved using linear truss theory), the same structure performs well when prestressed adequately. This kind of result, the predominance of effects (in this case forces due to small deformations) which are commonly thought to be of second order, is of course the driving force of Biot's classic book [1]. More recently Green, et al. [2] have pursued the application of this kind of analysis to problems of three-dimensional beam theory.

Qualitative results can be important. The one cited above, for example, implies that any incremental truss analysis should include geometric stiffness effects and thus influences the selection of appropriate computer programs. Put another way, qualitative results can hlep you decide what effects should be built into computer programs when you write them.

The interest here is primarily three-dimensional beams. While we now have good understanding of two-dimensional problems and even three-dimensional cable nets, three-dimensional beams are still the subject of some discussion. The principle result obtained below, then, has to do with the role of torsional buckling in three-dimensional frames. Other results for trusses and plane frames are obtained along the way.

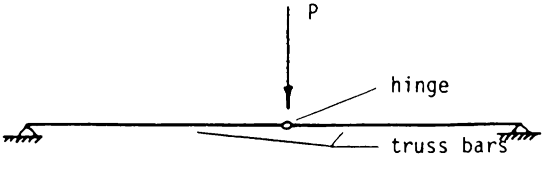

FIG. 1. The effect of prestress.

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The analysis below is straightforward, if cumbersome. The starting point is undeformed but prestressed equilibrium in the sense that the initial configuration has internal stress-resultants associated with it. (No concern is given to the manner in which the initial configuration has been reached.) A small deformation is then superimposed upon the initial configuration and zero- and first-order solutions constructed with products of small terms neglected. Assuming that the initial configuration is straight and undeformed, the first-order solutions then return the expected theories for beam-column and torsional buckling. It is the order comparison which is of interest for all cases considered.

Small parameter analysis. Given an arbitrary beam, its equilibrium in its initial and perturbed configurations can be described as (2)

$$\begin{array}{ll} \mathbf{P}^{0\prime} + \mathbf{p}^{0} = 0, & \mathbf{P}^{1\prime} + \mathbf{p}^{1} = 0, \\ \mathbf{M}^{0\prime} + \mathbf{m}^{0} + \mathbf{t}^{0} \times \mathbf{P}^{0} = 0, & \mathbf{M}^{1\prime} + \mathbf{m}^{1} + \mathbf{t}^{1} \times \mathbf{P}^{1} = 0. \\ \text{Initial Configuration} & \text{Perturbed Configuration} \end{array} \tag{1}$$

Here $\mathbf{P}$ and $\mathbf{M}$ are the usual force and moment stress-resultants, with $\mathbf{p}$ and $\mathbf{m}$ the applied forces and moments, the superscripts refer to the configuration, and the prime symbol refers to the differentiation with respect to arc length. The vector $\mathbf{t}$ is of course the unit tangent vector which will also be referred to as $\mathbf{i}$ in the applications cited below. In component form

$$\begin{aligned} \mathbf{P} &= P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}, \\ \mathbf{M} &= M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k}, \end{aligned} \tag{2}$$

and the base vectors $\mathbf{i}\,\mathbf{j}\,\mathbf{k}$ selected so that P_x and M_x represent thrust and torque, respectively. Going from the initial to the perturbed configuration, changes are indicated as

$$\begin{aligned} \mathbf{P}^{0} &\rightarrow \mathbf{P}^{1} = \mathbf{P}^{0} + \varepsilon\overline{\mathbf{P}}, \\ \mathbf{M}^{0} &\rightarrow \mathbf{M}^{1} = \mathbf{M}^{0} + \varepsilon\overline{\mathbf{M}}, \\ \mathbf{i}^{0} &\rightarrow \mathbf{i}^{1} = \mathbf{i}^{0} + \varepsilon\bar{\mathbf{i}}, \end{aligned} \tag{3}$$

where ε is the anticipated small parameter and the bar is used to distinguish the perturbation term. The perturbed terms of Eq. (2) are now inserted into the eqilibrium equations of the perturbed configuration given in Eq. (1). In component form these equations are

$$P_x^{1\prime}\mathbf{i}^1 + P_x^1\mathbf{i}^{1\prime} + P_y^{1\prime}\mathbf{j}^1 + P_y^1\mathbf{j}^{1\prime} + P_z^{1\prime}\mathbf{k}^1 + P_z^1\mathbf{k}^{1\prime} + p_x^1\mathbf{i}^1 + p_y^1\mathbf{j}^1 + p_z^1\mathbf{k}^1 = 0, \tag{4}$$

$$\begin{aligned} M_x^{1\prime}\mathbf{i}^1 + M_x^1\mathbf{i}^{1\prime} + M_y^{1\prime}\mathbf{j}^1 + M_y^1\mathbf{j}^{1\prime} + M_z^{1\prime}\mathbf{k}^1& \\ + M_z^1\mathbf{k}^{1\prime} + m_x^1\mathbf{i}^1 + m_y^1\mathbf{j}^1 + m_z^1\mathbf{k}^1 + \mathbf{t}^1 \times \mathbf{P}^1 &= 0. \end{aligned} \tag{5}$$

In order to complete the analysis, it is simply a matter of inserting the definition of the perturbations (Eq. (3)) into Eqs. (4–5) and collecting terms in the coordinate directions. Before doing so it is convenient to introduce displacements in the following manner. Let $\boldsymbol{\omega}$ represent the rotation vector associated with any beam element.

If in the initial configuration the beam is straight and lies along the x axis, for the case of small rotations $\boldsymbol{\omega}$ can be written as

$$\boldsymbol{\omega} = \theta_x \mathbf{i} - \delta_{z,x} \mathbf{j} + \delta_{y,x} \mathbf{k} \tag{6}$$

where θ is the torsional rotation, δ_z is the z component of the beam displacement vector, δ_y is the y component of the beam displacement vector, and the comma is used to indicate differentiation. The rotation vector can, of course, be used to determine the changes of the base vectors as

$$\begin{aligned} \mathbf{i}^0 &\to \mathbf{i}^1 = \mathbf{i}^0 + \varepsilon\bar{\mathbf{i}} = \mathbf{i}^0 + \boldsymbol{\omega} \times \mathbf{i}^0, \\ \mathbf{j}^0 &\to \mathbf{j}^1 = \mathbf{j}^0 + \varepsilon\bar{\mathbf{j}} = \mathbf{j}^0 + \boldsymbol{\omega} \times \mathbf{j}^0, \\ \mathbf{k}^0 &\to \mathbf{k}^1 = \mathbf{k}^0 + \varepsilon\bar{\mathbf{k}} = \mathbf{k}^0 + \boldsymbol{\omega} \times \mathbf{k}^0. \end{aligned} \tag{7}$$

Clearly

$$\begin{aligned} \boldsymbol{\omega} \times \mathbf{i}^0 &= \delta_{y,x}\mathbf{j}^0 + \delta_{z,x}\mathbf{k}^0, \\ \boldsymbol{\omega} \times \mathbf{j}^0 &= -\delta_{y,x}\mathbf{i}^0 + \boldsymbol{\theta}_x\mathbf{k}^0, \\ \boldsymbol{\omega} \times \mathbf{k}^0 &= -\delta_{z,x}\mathbf{i}^0 - \theta_x\mathbf{j}^0. \end{aligned} \tag{8}$$

The first-order equations are collected in Table 1 and will be discussed in the next section. The zero-order equations are assumed to be satisfied by the equilibrium of the initial configuration and will not be discussed here. The equations of Table 1 are obtained by writing the equilibrium equations in the perturbed configuration and keeping terms which are linear in the small parameter ε. For convenience this parameter can then be set to one in which case the terms indicated by bars (Eq. (3)) then represent the full perturbation.

It should be noted that the assumptions of small displacement theory have been invoked above to allow arc length differentiation to be replaced by differentiation with respect to the space variable x.

Some applications. The equations of Table 1 are rather awkward to deal with in their most general form. Here only some special cases of practical interest will be discussed:

Case 1. *An Originally Straight Cable.* Consider the case of a taut, undeformed, cable under an initial tension $P_x^0 = \text{const.}$: For the cable to be initially straight it is necessary that $p_y^0 = p_z^0 = 0$ (for convenience p_x^0 is taken to be 0); since a cable supports no moments it follows that $M_x^0 = M_y^0 = M_z^0 = 0 = \overline{M}_x = \overline{M}_y = \overline{M}_z$; and for practical purposes let $\overline{m}_x = \overline{m}_y = \overline{m}_z = 0$. The equations of Table 1 then degenerate to

Force Equilibrium

$$\begin{aligned} \overline{P}'_x + \overline{p}_x &= 0, \\ \overline{P}'_y + \overline{p}_y + P_x^0\delta_{y,xx} &= 0, \\ \overline{P}'_z + \overline{p}_z + P_x^0\delta_{z,xx} &= 0. \end{aligned} \tag{9}$$

TABLE 1. First-order equations.

Force Equilibrium
i comp: $\overline{P}'_x + \overline{p}_x - P^{0\prime}_y\delta_{y,x} - P^{0\prime}_z\delta_{z,x} - P^0_y\delta_{y,xx} - P^0_z\delta_{z,xx} - p^0_y\delta_{y,x} - p^0_z\delta_{z,x} = 0$
y comp: $\overline{P}'_y + \overline{p}_y + P^{0\prime}_x\delta_{y,x} - P^{0\prime}_z\theta_x + P^0_x\delta_{y,xx} - P^0_z\theta_{x,x} + p^0_x\delta_{y,x} - p^0_z\theta_x = 0$
k comp: $\overline{P}'_z + \overline{p}_z + P^{0\prime}_x\delta_{z,x} + P^{0\prime}_y\theta_x + P^0_x\delta_{z,xx} + P^0_y\theta_{x,x} + p^0_x\delta_{z,x} + p^0_y\theta_x = 0$

Moment Equilibrium
i comp: $\overline{M}'_x + \overline{m}_x - M^{0\prime}_y\delta_{y,x} - M^{0\prime}_z\delta_{z,x} - M^0_y\delta_{y,xx} - M^0_z\delta_{z,xx}$ $- m^0_y\delta_{y,x} - m^0_z\delta_{z,x} + P^0_z\delta_{y,k} - P^0_y\delta_{z,x} = 0$
y comp: $\overline{M}'_y + \overline{m}_y + M^{0\prime}_x\delta_{y,x} - M^{0\prime}_z\theta_x + M^0_x\delta_{y,xx} - M^0_z\theta_{x,x}$ $+ m^0_x\delta_{y,x} - m^0_z\theta_x - \overline{P}_z - P^0_y\theta_x = 0$
k comp: $\overline{M}'_z + \overline{m}_z + M^{0\prime}_x\delta_{z,x} + M^{0\prime}_y\theta_x + M^0_x\delta_{z,xx} + M^0_y\theta_{x,x}$ $+ m^0_x\delta_{z,x} + m^0_y\theta_x + \overline{P}_y - P^0_z\theta_x = 0$

Moment Equilibrium

$$
\begin{aligned}
0 &= 0, \\
\overline{P}_z &= 0, \\
\overline{P}_y &= 0.
\end{aligned}
\tag{10}
$$

Setting $\overline{P}_y = \overline{P}_z = 0$, three equations remain,

$$
\begin{aligned}
\overline{P}'_x + \overline{p}_x &= 0, \\
\overline{p}_y + P^0_x\delta_{y,xx} &= 0, \\
\overline{p}_\varepsilon + P^0_x\delta_{z,xx} &= 0.
\end{aligned}
\tag{11}
$$

The first of these simply describes equilibrium along the cable; the second and third of these return the well-known equations of a plane cable (string). That is, Eqs. (11) imply that in this case the originally straight cable responds as a plane cable in two directions when lateral loads are applied.

From this analysis it is clear that the lateral response of a prestressed cable is controlled by what are usually thought to be second-order effects (forces produced by small deformations) giving the qualitative result cited above in the introduction.

Case 2. Plane Beams (Beam-Columns). In this case the equations of Table 1 are first transformed to eliminate out-of-plane effects and torsion giving

Force Equilibrium

$$
\begin{aligned}
\overline{P}'_x + \overline{p}_x - P^{0\prime}_y\delta_{y,x} - P^0_y\delta_{y,xx} - p^0_y\delta_{y,x} &= 0, \\
\overline{P}'_y + \overline{p}_y + P^{0\prime}_x\delta_{y,x} + P^0_x\delta_{y,xx} + p^0_x\delta_{y,x} &= 0, \\
0 &= 0.
\end{aligned}
\tag{12}
$$

Moment Equilibrium

$$\begin{aligned} 0 &= 0, \\ 0 &= 0, \\ \overline{M}'_z + \overline{m}_z + \overline{P}_y &= 0. \end{aligned} \tag{13}$$

If the shear force $\overline{P}_y$ is eliminated by combining the second of Eq. (12) with the third of Eq. (13), it follows that

$$-\overline{M}''_z - \overline{m}'_z + \overline{p}_y + P_x^{0\prime}\delta_{y,x} + P_x^0\delta_{y,xx} + p_x^0\delta_{y,x} = 0. \tag{14}$$

Finally, if the initial configuration is simply one of constant axial load, Eq. (14) returns the well-known beam-column equation

$$-\overline{M}''_z - \overline{m}'_z + \overline{p}_y + P_x^0\delta_{y,xx} = 0. \tag{15}$$

The solution of the system of Eqs. (12–13) can then be completed by using the first of Eq. (12) to find the axial response.

The qualitative result in this two-dimensional case is then

For two-dimensional beams in the presence of axial prestress, the flexural response is controlled by the beam-column equations.

Put another way, any iterative, two-dimensional frame analysis computer program should be written using the beam-column equations to describe flexure.

Case 3. *Three-Dimensional Frames.* As a simple three-dimensional example, consider the case of a beam subjected to an axial load P_x^0 and end moment $M_z^0 = \text{const}$. In this case the equations of Table 1 take the form

Force Equilibrium

$$\begin{aligned} \overline{P}'_x + \overline{p}_x &= 0, \\ \overline{P}'_y + \overline{p}_y + P_x^0\delta_{y,xx} &= 0, \\ \overline{P}'_z + \overline{p}_z + P_x^0\delta_{z,xx} &= 0. \end{aligned} \tag{16}$$

Moment Equilibrium

$$\begin{aligned} \overline{M}'_x + \overline{m}_x - M_z^0\delta_{z,xx} &= 0, \\ \overline{M}'_y + \overline{m}_y - M_z^0\theta_{x,x} - \overline{P}_z &= 0, \\ \overline{M}'_z + \overline{m}_z + \overline{P}_y &= 0. \end{aligned} \tag{17}$$

It is further assumed that the applied loads and moments $\overline{\mathbf{p}} = \overline{\mathbf{m}} = 0$, and this system reduces to the lateral buckling problem of a beam under axial load and an applied end moment which is discussed in Timoshenko's book [3]. (Note that quite generally, the last two of Eqs. (16) can be combined with the last two of Eqs. (17) to eliminate the shears $\overline{P}_z$ and $\overline{P}_y$.

It is of some interest to review the general set of equations in Table 1. In order to eliminate the torsional term θ_x from the system, it is necessary to eliminate the shear terms P_y^0 and P_z^0. Put another way, this provides a final qualitative result

> *In the presence of shear and axial load, lateral buckling effects are of the same order of magnitude as beam-column effects in three-dimensional beams.*

Concluding note. A reviewer has raised the issue of how realistic it is to ignore the deformations accompanying the initial stresses as has been done above. It seems to me that there are two answers to this difficult question. The careful answer is that if the structure is prestressed cleverly there will be nothing to neglect. The practical answer is that in many applications we use methods such as those of this paper in situations where they do not apply exactly and where we are not sure of the level of approximation involved (which can be significant). Perhaps that is the price to be paid for simple results.

REFERENCES

[1] M. A. Biot, *Mechanics of Incremental Deformation*, John Wiley and Sons, New York, 1965

[2] A. E. Green, R. J. Knops, and N. Laws, *Large deformations, superimposed small deformations, and stability of elastic rods*, Internat. Solids and Structures **4**, 555–577, (1968)

[3] S. Timoshenko, *Theory of Elastic Stability*, McGraw Hill, New York, 1936, p. 263