QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLVIII, NUMBER 3
SEPTEMBER 1990, PAGES 555-567

A METHOD OF MULTIPLE-PARAMETER PERTURBATIONS WITH AN APPLICATION TO DROP OSCILLATIONS IN AN ELECTRIC FIELD

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Abstract. A formulation of a perturbation technique with several small parameters is presented. As a particular example, this method is used to study the axisymmetric oscillations of an inviscid conducting drop in an electric field. It is shown that all characteristic frequencies of the axisymmetric oscillations decrease as the electric field strength increases, which is in general agreement with simplified models based on the assumption of spheroidal deformations.

1. Introduction. Systematic methods of perturbations with respect to one small parameter have been used as the foremost analytic techniques in approximating solutions of complex nonlinear problems that preclude exact solutions, as summarized in the book of Nayfeh and Mook [1]. Many physical systems are, however, influenced by several essentially independent factors. For instance, the shape of an oscillating drop in an electric field is related to both the quiescent distortion due to a nonuniform distribution of electrostatic stress at the interface and the oscillations that can exist in the absence of an electric field. Therefore, in more general cases, it is appropriate to make use of a method of perturbations with respect to several small parameters.

This paper outlines a method of multiple-parameter perturbations, which combines the domain perturbation technique detailed by Joseph [2] for free-boundary problems in fluid mechanics and the method of multiple time scales [1] for typical nonlinear oscillations, and extends them to account for more than one small parameter. The single-parameter version of this perturbation technique reduces to that used by Tsamopoulos and Brown [3] in the analysis of nonlinear resonances of charged drops. The relationship between the single- and multiple-parameter perturbation methods is parallel to that of the Taylor expansions with respect to one variable and several variables. In order to make it easily accessible, the mathematical method is presented with an application to the calculation of the small-amplitude oscillations of an inviscid conducting drop in an electric field. The results of this calculation will be of interest in a wide variety of scientific and engineering applications such as cloud physics [4,5], spray generation [6], and materials processing [7].

Received October 16, 1989.

Drop oscillations in an electric field have been studied mainly by simplified models based on the assumption of spheroidal deformations [5,8]. This assumption restricts the study to only the lowest oscillation mode. Relatively little theoretical work has been done to investigate, in a general way, the modification of the modes and characteristic frequencies arising from the quiescent distortion of a drop in an electric field. The attempts of Sample et al. [9] and Morrison et al. [10] led to quantitatively different results. In both papers Lagrange's equation was solved using approximate expressions for the kinetic energy and potential energy without clear scaling of the small parameters, so that terms of the same order were not systematically retained.

The method of multiple-parameter perturbations presented in this work makes the solution procedure more transparent and consistent. Once the small parameters are defined, the solutions to each order of approximation can be obtained in a systematic way. In some cases, one may designate a scale relation between these small parameters so that the problem becomes a single-parameter one. Such a procedure, however, would not practically simplify the solution procedure and, if performed at the beginning, would reduce the generality of the method. Following the method of multiple-parameter perturbations, the equation set and solution for each order problem are independent of any scale relations. This fact turns out to be a major advantage of the multiple-parameter method over the single-parameter one; the latter method would result in different perturbation formulations depending upon each specified scale relation.

2. Governing equations. In this section, the governing equations are formulated for the irrotational and incompressible motion of an electrically conducting drop with volume $\frac{4}{3}\pi R^3$, density ρ, uniform interfacial tension σ, and zero net electric charge, situated in an externally applied uniform electrostatic field $\mathbf{E}_0^*$. For simplicity, only the axisymmetric case is considered and the effects of gravity and viscosity are ignored. Also, the medium surrounding the drop is assumed to be either a vacuum or a tenuous insulating gas, so that its hydrodynamical and electrodynamical effects can be neglected.

The surface of the drop is described by $RF(\theta, t)$, where $F(\theta, t)$ is the dimensionless shape function of the drop and θ is the meridional angle in spherical coordinates measured from the axis of symmetry, chosen parallel to the direction of the externally applied electrostatic field $\mathbf{E}_0^*$. The nondimensionalized equation set may be obtained by defining the dimensionless radial coordinate $r_0 = r^*/R$, time $t = t^*\sqrt{\sigma/(\rho R^3)}$, velocity potential $\Phi = \Phi^*\sqrt{\rho/(\sigma R)}$, and normal stress terms such as pressure and electric stress $(\Delta p_0, p, N_e) = (R/\sigma)(\Delta p_0^*, p^*, N_e^*)$, with the asterisk denoting the corresponding dimensional variables.

In terms of these dimensionless variables, the governing equation for the velocity potential Φ is written as

$$\nabla^2\Phi = 0 \qquad (0 \le r_0 \le F(\theta, t)). \tag{2.1}$$

The condition for a finite radial velocity at the center of the drop takes the form

$$\frac{\partial\Phi}{\partial r_0} \ne \infty \qquad (r_0 = 0). \tag{2.2}$$

Bernoulli's equation for the pressure everywhere in the drop can be expressed as

$$p + \frac{\partial \Phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial r_0}\right)^2 + \left(\frac{1}{r_0}\frac{\partial \Phi}{\partial \theta}\right)^2\right] = \Delta p_0 \qquad (0 \le r_0 \le F(\theta, t)), \tag{2.3}$$

where Δp_0 is a "constant" of integration which may be a function of time but is spatially uniform, and can be determined by satisfying the constraint of volume conservation.

The kinematic and the normal stress conditions on the drop surface are of the form

$$\frac{\partial \Phi}{\partial r_0} = \frac{\partial F}{\partial t} + \frac{1}{r_0^2}\frac{\partial \Phi}{\partial \theta}\frac{\partial F}{\partial \theta} \qquad (r_0 = F(\theta, t)), \tag{2.4}$$

and

$$p + N_e = \nabla \cdot \mathbf{n} \qquad (r_0 = F(\theta, t)), \tag{2.5}$$

where N_e denotes the electric normal stress induced by the externally applied electric field and the unit normal vector of the surface can be written as

$$\mathbf{n} = \frac{\mathbf{e}_r F - \mathbf{e}_\theta \partial F/\partial \theta}{\sqrt{F^2 + (\partial F/\partial \theta)^2}}. \tag{2.6}$$

In addition, the solution for the drop shape should also satisfy the constraint that the volume of the drop be constant

$$\int_0^\pi F^3(\theta, t)\sin\theta\, d\theta = 2, \tag{2.7}$$

and the constraint that the center of mass of the drop remain at the origin

$$\int_0^\pi F^4(\theta, t)\cos\theta \sin\theta\, d\theta = 0. \tag{2.8}$$

In restricting this study to the oscillations of an uncharged conducting drop in an externally applied uniform electrostatic field $\mathbf{E}_0^*$, it is convenient to define the dimensionless electric potential as

$$V = \frac{1}{E_0^* R} V^*. \tag{2.9}$$

Since the electric stress on the surface of a conductor is normal to the surface and, when written in SI units, is of the form $N_e^* = \frac{1}{2}\varepsilon_m E^{*2}$ according to Landau and Lifshitz [11], the dimensionless normal stress in (2.5) can be expressed as

$$N_e = \frac{1}{2}\frac{\varepsilon_m R E_0^{*2}}{\sigma}\left[\left(\frac{\partial V}{\partial r_0}\right)^2 + \left(\frac{1}{r_0}\frac{\partial V}{\partial \theta}\right)^2\right]_{r_0 = F(\theta, t)} \tag{2.10}$$

where ε_m is the permitivity of the medium surrounding the drop, which is assumed to be electrically insulating.

In dimensionless form, the electric potential is governed by the Laplace equation

$$\nabla^2 V = 0 \qquad (F(\theta, t) \le r_0 < \infty), \tag{2.11}$$

the far-field condition is a uniform electric field

$$V = -r_0 \cos\theta \qquad (r_0 \to \infty), \tag{2.12}$$

the equation of charge conservation (no net charge on the drop in our case) is of the form

$$\int_0^\pi (\mathbf{n}\cdot\nabla V)_{r_0=F} \sqrt{F^2 + \left(\frac{\partial F}{\partial\theta}\right)^2}\, F \sin\theta \, d\theta = 0, \tag{2.13}$$

and the continuity of the tangential component of the electric field across the interface is guaranteed by

$$\mathbf{t}\cdot\nabla V = 0 \qquad (r_0 = F(\theta, t)), \tag{2.14}$$

where the unit tangential vector to the drop surface is defined as

$$\mathbf{t} = \frac{\mathbf{e}_r \partial F/\partial\theta + \mathbf{e}_\theta F}{\sqrt{F^2 + (\partial F/\partial\theta)^2}}. \tag{2.15}$$

Equations (2.13) and (2.14) are based on the assumption of electrostatic equilibrium, which means that charge is confined to the interface and equilibrates in a time much shorter than the time scale of the fluid motion. Even for the case of distilled water, the characteristic time for conduction is smaller than the time for typical drop oscillation by at least several orders of magnitude when $R \geq 0.1$ cm, as shown in [3].

3. Method of multiple-parameter perturbations. The exact solution to the equation set (2.1)–(2.15) is intractable because of the nonlinearities arising from capillarity, inertia, the coupling of the surface kinematics to the velocity field and the coupling of the surface deformations to the electric field, rendering the velocity and electric potentials dependent on the shape of the mathematical domain which, in turn, is determined by those field values. However, for a nearly spherical drop, it seems appropriate to use a domain perturbation technique to transform the drop shape into the unit sphere. This can be done by introducing the change of coordinates

$$r_0 \equiv rF(\theta, t), \tag{3.1}$$

mapping the interface of complex configuration $r_0 = F(\theta, t)$ into a simple domain $r = 1$.

In the present study, two small parameters ε_1 and ε_2 are considered, where $\varepsilon_1 = \varepsilon_m R E_0^{*2}/\sigma$ is used to scale the quiescent deformation of the drop surface in the presence of electric field $\mathbf{E}_0^*$ and ε_2 is used to measure the magnitude of the oscillations which result from physical factors independent of ε_1. Thus, with respect to these small parameters, each dependent variable may be expanded in a Taylor series. For instance, an arbitrary function $f(r_0, \theta, t; \varepsilon_1, \varepsilon_2)$ may be written in the form of the expansion as

$$f(r_0, \theta, t; \varepsilon_1, \varepsilon_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{n!\,m!} \varepsilon_1^n \varepsilon_2^m f^{[n,m]}(r, \theta, t), \tag{3.2}$$

where

$$f^{[n,m]}(r,\theta,t) \equiv \left[\left(\frac{\partial}{\partial\varepsilon_1} + r\frac{\partial F}{\partial\varepsilon_1}\frac{\partial}{\partial r_0}\right)^n \left(\frac{\partial}{\partial\varepsilon_2} + r\frac{\partial F}{\partial\varepsilon_2}\frac{\partial}{\partial r_0}\right)^m f(r_0,\theta,t;\varepsilon_1,\varepsilon_2)\right]_{\substack{\varepsilon_1=0\\ \varepsilon_2=0\\ r_0=r}} . \tag{3.3}$$

Moreover, in the following derivations it is more convenient to use the notation

$$f^{\langle n,m\rangle}(r,\theta,t) \equiv \left[\frac{\partial^{n+m} f(r,\theta,t;\varepsilon_1,\varepsilon_2)}{\partial\varepsilon_1^n\partial\varepsilon_2^m}\right]_{\substack{\varepsilon_1=0\\ \varepsilon_2=0}} , \tag{3.4}$$

because the terms of $f^{[n,m]}(r,\theta,t)$ can be explicitly written in terms of $f^{\langle n,m\rangle}(r,\theta,t)$ as

$$f^{[0,0]}(r,\theta,t) \equiv f^{\langle 0,0\rangle}(r,\theta,t),$$

$$f^{[1,0]}(r,\theta,t) \equiv f^{\langle 1,0\rangle} + rF^{\langle 1,0\rangle}\frac{\partial f^{\langle 0,0\rangle}}{\partial r},$$

$$f^{[0,1]}(r,\theta,t) \equiv f^{\langle 0,1\rangle} + rF^{\langle 0,1\rangle}\frac{\partial f^{\langle 0,0\rangle}}{\partial r},$$

$$f^{[1,1]}(r,\theta,t) \equiv f^{\langle 1,1\rangle} + rF^{\langle 1,1\rangle}\frac{\partial f^{\langle 0,0\rangle}}{\partial r} + rF^{\langle 1,0\rangle}\frac{\partial f^{\langle 0,1\rangle}}{\partial r} + rF^{\langle 0,1\rangle}\frac{\partial f^{\langle 1,0\rangle}}{\partial r} + r^2F^{\langle 1,0\rangle}F^{\langle 0,1\rangle}\frac{\partial^2 f^{\langle 0,0\rangle}}{\partial r^2},$$

$$f^{[2,0]}(r,\theta,t) \equiv f^{\langle 2,0\rangle} + rF^{\langle 2,0\rangle}\frac{\partial f^{\langle 0,0\rangle}}{\partial r} + 2rF^{\langle 1,0\rangle}\frac{\partial f^{\langle 1,0\rangle}}{\partial r} + r^2F^{\langle 1,0\rangle^2}\frac{\partial^2 f^{\langle 0,0\rangle}}{\partial r^2},$$

$\ldots$.

Accordingly, the expansions of the governing equations (2.1) and (2.11) in terms of ε_1 and ε_2 yield

$$\nabla^2\Phi^{\langle n,m\rangle} = 0 \quad (0 \le r \le 1) \quad \text{and} \quad \nabla^2 V^{\langle n,m\rangle} = 0 \quad (1 \le r < \infty), \tag{3.5}$$

and the expansions of the natural boundary conditions (2.2) and (2.12) lead to

$$\frac{\partial\Phi^{\langle n,m\rangle}}{\partial r} \ne \infty \qquad (r=0),$$
$$V^{\langle n,m\rangle} = -\delta_{n0}\delta_{m0} r\cos\theta \qquad (r\to\infty), \tag{3.6}$$

where δ_{nm} denotes the Kronecker delta.

Hence it is expected that the zeroth-order solution for this system takes the form

$$\begin{bmatrix} F^{\langle 0,0\rangle} \\ \Phi^{\langle 0,0\rangle} \\ V^{\langle 0,0\rangle} \\ \Delta p_0^{\langle 0,0\rangle} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -\left(r-\frac{1}{r^2}\right)\cos\theta \\ 2 \end{bmatrix}, \tag{3.7}$$

which recovers the solution for a spherical drop in a uniform electric field, and

$$\begin{bmatrix} \Phi^{\langle n,m\rangle}(r,\theta,t) \\ V^{\langle n,m\rangle}(r,\theta,t) \end{bmatrix} = \sum_{l=0}^{\infty} \begin{bmatrix} r^l\beta_l^{\langle n,m\rangle}(t) \\ r^{-l-1}\xi_l^{\langle n,m\rangle}(t) \end{bmatrix} P_l(\theta) \quad \text{for } n\ne 0 \text{ and } m\ne 0, \tag{3.8}$$

where $P_l(\theta)$ are the Legendre polynomials. Since any physically measurable effect of the spatially uniform term $\beta_0^{\langle n,m\rangle}(t)$ can be absorbed into Δp_0, $\beta_0^{\langle n,m\rangle}(t)$ are simply set to zero in the following analysis.

For convenience, the shape function at each order of ε_1 and ε_2 is also expanded as

$$F^{\langle n,m\rangle}(\theta, t) = \sum_{l=0}^{\infty} \alpha_l^{\langle n,m\rangle}(t) P_l(\theta). \tag{3.9}$$

Hence, the solution is presumed known in the reference domain, which is the unit sphere in this problem, as Joseph [2] has pointed out. The changes in the domain shape are accounted for only through the corrections to the boundary conditions at each order of ε_1 and ε_2.

As will be seen, separating terms of equal order ε_1 and ε_2 in the expansions of the governing equations given in Sec. 2 leads to a sequence of linear, inhomogeneous problems. The inhomogeneities at a given order are determined from the solutions of the lower-order problems, with the leading-order equations being homogeneous. The solutions of the homogeneous leading-order equations can usually be constructed by a sum of the normal modes. Therefore, nonlinear effects can be calculated from the higher-order problems by successive substitution.

In higher-order inhomogeneous problems, the appearance of secular terms, i.e., terms that have the same spatial form and frequency as one of the linear modes, will lead to difficulties in the perturbation approach. Such secular terms give rise to solutions with a polynomial growth in time that render the ordering assumptions of the perturbation theory invalid after short times. In order to properly avoid secular terms, it is usually convenient to make use of a method of multiple time scales similar to that described in [1]. Formally, the time-dependent variables are assumed to be functions of time scales related to the actual time t and small parameters ε_1 and ε_2 as

$$T_{(n,m)} \equiv \varepsilon_1^n \varepsilon_2^m t. \tag{3.10}$$

Hence, for example, we write $\alpha_l^{\langle n,m\rangle}(T_{(0,0)}, T_{(1,0)}, T_{(0,1)}, T_{(1,1)}, \dots)$ instead of $\alpha_l^{\langle n,m\rangle}(t)$. This means that not only does $\alpha_l^{\langle n,m\rangle}$ depend on t $(= T_{(0,0)})$, but also on the combinations $\varepsilon_1 t$, $\varepsilon_2 t$, $\varepsilon_1\varepsilon_2 t$, $\varepsilon_1^2 t$, $\varepsilon_2^2 t$, ...

It is seen that $T_{(n,m)}$ represent different time scales because ε_1 and ε_2 are small parameters, which means the terms of higher order of ε_1 and ε_2 represent slower time variations. The different time scales are introduced into the dynamic equations (2.3) and (2.4) by expanding the partial derivative with respect to time as

$$\begin{aligned}\frac{\partial}{\partial t} &\equiv \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \varepsilon_1^n \varepsilon_2^m \frac{\partial}{\partial T_{(n,m)}} \\ &= \frac{\partial}{\partial T_{(0,0)}} + \varepsilon_1 \frac{\partial}{\partial T_{(1,0)}} + \varepsilon_2 \frac{\partial}{\partial T_{(0,1)}} + \varepsilon_1\varepsilon_2 \frac{\partial}{\partial T_{(1,1)}} + \dots .\end{aligned} \tag{3.11}$$

4. Perturbation solutions. By the definition given in this paper, $\varepsilon_2 = 0$ means that there is no oscillatory motion in the drop, but the externally applied electrostatic field would produce a quiescent deformation on the drop surface. Therefore, both the time derivative and velocity potential $\Phi^{\langle n,0\rangle}$ disappear when ε_2 is zero. On the other hand, setting $\varepsilon_1 = 0$ implies there is no electric field. Thus the solutions to each order of ε_2 should yield the results obtained by Tsamopoulos and Brown [12] for nonlinear oscillations of inviscid drops when ε_1 becomes zero. However, because of the way in which the electric potential was nondimensionalized, in general $V^{\langle 0,m\rangle}$ are not zero, although they would not affect the drop shape when ε_1 is zero.

Since the solution form for the terms of each order of ε_1 and ε_2 is presumed known (given by (3.7)–(3.9)), what need to be determined are the expansion coefficients for the drop shape $(\alpha_l^{\langle n,m\rangle})$, velocity potential $(\beta_l^{\langle n,m\rangle})$, electric potential $(\xi_l^{\langle n,m\rangle})$, etc., through the corrections to the boundary conditions at each order of ε_1 and ε_2. In the following derivations, subscripts will be employed to denote most of the partial derivatives in order to shorten some lengthy equations.

4.1. $O(\varepsilon_1)$ *problem (first-order approximation for quiescent deformation)*. The nontrivial equations used in this case are

$$\Delta p_0^{\langle 1,0\rangle} + \frac{1}{2} V_1^{\langle 0,0\rangle^2} = -\left[2F^{\langle 1,0\rangle} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F^{\langle 1,0\rangle}}{\partial\theta}\right)\right] \qquad (r=1), \tag{4.1}$$

$$\int_0^\pi F^{\langle 1,0\rangle}\sin\theta\, d\theta = 0, \tag{4.2}$$

$$\int_0^\pi F^{\langle 1,0\rangle}\cos\theta\sin\theta\, d\theta = 0, \tag{4.3}$$

$$\int_0^\pi (V_r^{\langle 1,0\rangle} + 2F^{\langle 1,0\rangle}V_r^{\langle 0,0\rangle} + F^{\langle 1,0\rangle}V_{rr}^{\langle 0,0\rangle})_{r=1}\sin\theta\, d\theta = 0, \tag{4.4}$$

$$\frac{\partial}{\partial\theta}(V^{\langle 1,0\rangle} + F^{\langle 1,0\rangle}V_r^{\langle 0,0\rangle}) = 0 \qquad (r=1). \tag{4.5}$$

The shape function $F^{\langle 1,0\rangle}$ and $\Delta p_0^{\langle 1,0\rangle}$ may be obtained directly from (4.1)–(4.3), since $V^{\langle 0,0\rangle}$ is known from (3.7). The nonzero coefficients are found to be

$$\alpha_2^{\langle 1,0\rangle} = \frac{3}{4} \quad\text{and}\quad \Delta p_0^{\langle 1,0\rangle} = -\frac{3}{2}. \tag{4.6}$$

Once $F^{\langle 1,0\rangle}$ is determined, the coefficients for the electric potential can be obtained from (4.4) and (4.5), giving

$$\xi_1^{\langle 1,0\rangle} = \frac{9}{10}, \qquad \xi_3^{\langle 1,0\rangle} = \frac{27}{20}, \quad\text{and}\quad \xi_l^{\langle 1,0\rangle} = 0 \quad \text{for } l \neq 1, 3. \tag{4.7}$$

The solution shows that, with respect to the equatorial plane $(\theta = \pi/2)$, the drop shape is symmetric whereas the electric potential is antisymmetric. To the first order in ε_1 the shape function consists only of a two-lobed Legendre function. Adornato and Brown [13] used an asymptotic method to calculate the equilibrium shapes of electrostatically levitated drops. The result obtained here is identical to their result in the limit of zero charge.

4.2. $O(\varepsilon_2)$ *problem* (*first-order oscillatory motion about the spherical shape*). The equations used for this problem are

$$\Phi_r^{\langle 0,1\rangle} = \frac{\partial F^{\langle 0,1\rangle}}{\partial T_{(0,0)}} \qquad (r=1), \tag{4.8}$$

$$\Delta p_0^{\langle 0,1\rangle} - \frac{\partial \Phi^{\langle 0,1\rangle}}{\partial T_{(0,0)}} = -2F^{\langle 0,1\rangle} - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F^{\langle 0,1\rangle}}{\partial\theta}\right) \qquad (r=1), \tag{4.9}$$

$$\int_0^\pi F^{\langle 0,1\rangle}\sin\theta\, d\theta = 0, \tag{4.10}$$

$$\int_0^\pi F^{\langle 0,1\rangle}\cos\theta\sin\theta\, d\theta = 0, \tag{4.11}$$

$$\int_0^\pi (V_r^{\langle 0,1\rangle} + 2F^{\langle 0,1\rangle}V_r^{\langle 0,0\rangle} + F^{\langle 0,1\rangle}V_{rr}^{\langle 0,0\rangle})_{r=1}\sin\theta\, d\theta = 0, \tag{4.12}$$

$$\frac{\partial}{\partial\theta}(V^{\langle 0,1\rangle} + F^{\langle 0,1\rangle}V_r^{\langle 0,0\rangle}) = 0 \qquad (r=1). \tag{4.13}$$

The solution takes the form

$$\Delta p_0^{\langle 0,1\rangle} = \alpha_0^{\langle 0,1\rangle} = \alpha_1^{\langle 0,1\rangle} = 0$$

and

$$\alpha_l^{\langle 0,1\rangle} = c_l^{\langle 0,1\rangle}\exp[i\omega_l T_{(0,0)}] \qquad (l = 2, 3, 4, \ldots), \tag{4.14}$$

where $c_l^{\langle 0,1\rangle}$ could be functions of slower time scales such as $T_{(1,0)}$, $T_{(0,1)}$, $T_{(0,2)}$, ..., and

$$\omega_l^2 = l(l-1)(l+2). \tag{4.15}$$

The velocity and electric potentials are determined through the relations

$$l\beta_l^{\langle 0,1\rangle} = \frac{\partial\alpha_l^{\langle 0,1\rangle}}{\partial T_{(0,0)}} \quad \text{and} \quad \xi_l^{\langle 0,1\rangle} = 3\left[\frac{l+1}{2l+3}\alpha_{l+1}^{\langle 0,1\rangle} + \frac{l}{2l-1}\alpha_{l-1}^{\langle 0,1\rangle}\right]. \tag{4.16}$$

If the dimensionless electric potential $V^{\langle 0,1\rangle}$ is ignored, the solution of the $O(\varepsilon_2)$ problem corresponds to the linear modes of oscillation analyzed by Rayleigh [14].

4.3. $O(\varepsilon_1\varepsilon_2)$ *problem* (*small-amplitude oscillations of a slightly deformed drop*). Carrying out the expansion up to this order will enable us to see the phenomena arising from the coupling between the oscillations and quiescent deformations. The characteristic frequency shift due to the externally applied electric field will show up by virtue of the solvability condition used to avoid the secular terms.

The equations for this problem are of the form

$$\Phi_r^{\langle 1,1\rangle} + F^{\langle 1,0\rangle}\Phi_{rr}^{\langle 0,1\rangle} = \frac{\partial F^{\langle 1,1\rangle}}{\partial T_{(0,0)}} + \frac{\partial F^{\langle 0,1\rangle}}{\partial T_{(1,0)}} + F_\theta^{\langle 1,0\rangle}\Phi_\theta^{\langle 0,1\rangle} \qquad (r=1), \tag{4.17}$$

$$\Delta p_0^{\langle 1,1\rangle} - \frac{\partial \Phi^{\langle 1,1\rangle}}{\partial T_{(0,0)}} - \frac{\partial \Phi^{\langle 0,1\rangle}}{\partial T_{(1,0)}} - F^{\langle 1,0\rangle}\frac{\partial^2 \Phi^{\langle 0,1\rangle}}{\partial r \partial T_{(0,0)}} + V_r^{\langle 0,0\rangle}(V_r^{\langle 0,1\rangle} + F^{\langle 0,1\rangle}V_{rr}^{\langle 0,0\rangle})$$
$$= -\left[2F^{\langle 1,1\rangle} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F^{\langle 1,1\rangle}}{\partial\theta}\right)\right]$$
$$+ 2F^{\langle 1,0\rangle}\left[F^{\langle 0,1\rangle} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F^{\langle 0,1\rangle}}{\partial\theta}\right)\right]$$
$$+ 2F^{\langle 0,1\rangle}\left[F^{\langle 1,0\rangle} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F^{\langle 1,0\rangle}}{\partial\theta}\right)\right] \qquad (r=1), \tag{4.18}$$

$$\int_0^{\pi}(2F^{\langle 1,0\rangle}F^{\langle 0,1\rangle} + F^{\langle 1,1\rangle})\sin\theta\, d\theta = 0. \tag{4.19}$$

$$\int_0^{\pi}(3F^{\langle 1,0\rangle}F^{\langle 0,1\rangle} + F^{\langle 1,1\rangle})\cos\theta\sin\theta\, d\theta = 0, \tag{4.20}$$

$$\int_0^{\pi}[V_r^{\langle 1,1\rangle} + 4F^{\langle 1,0\rangle}F^{\langle 0,1\rangle}V_{rr}^{\langle 0,0\rangle} + 2F^{\langle 1,0\rangle}F^{\langle 0,1\rangle}V_r^{\langle 0,0\rangle} + F^{\langle 1,0\rangle}F^{\langle 0,1\rangle}V_{rrr}^{\langle 0,0\rangle}$$
$$+ 2F^{\langle 1,0\rangle}V_r^{\langle 0,1\rangle} + F^{\langle 1,0\rangle}V_{rr}^{\langle 0,1\rangle} + 2F^{\langle 0,1\rangle}V_r^{\langle 1,0\rangle} + F^{\langle 0,1\rangle}V_{rr}^{\langle 1,0\rangle}$$
$$+ 2F^{\langle 1,1\rangle}V_r^{\langle 0,0\rangle} + F^{\langle 1,1\rangle}V_{rr}^{\langle 0,0\rangle} - F_\theta^{\langle 1,0\rangle}V_\theta^{\langle 0,1\rangle} - F_\theta^{\langle 0,1\rangle}V_\theta^{\langle 1,0\rangle}$$
$$- F^{\langle 1,0\rangle}F_\theta^{\langle 0,1\rangle}V_{r\theta}^{\langle 0,0\rangle} - F^{\langle 0,1\rangle}F_\theta^{\langle 1,0\rangle}V_{r\theta}^{\langle 0,0\rangle}]_{r=1}\sin\theta\, d\theta = 0, \tag{4.21}$$

$$\frac{\partial}{\partial\theta}[V^{\langle 1,1\rangle} + F^{\langle 1,0\rangle}F^{\langle 0,1\rangle}V_{rr}^{\langle 0,0\rangle} + F^{\langle 1,0\rangle}V_r^{\langle 0,1\rangle} + F^{\langle 0,1\rangle}V_r^{\langle 1,0\rangle} + F^{\langle 1,1\rangle}V_r^{\langle 0,0\rangle}] = 0$$
$$(r=1). \quad 4.22$$

By combining the kinematic condition (4.17) and normal stress condition (4.18), an equation for the coefficients of the shape function is obtained. The solvability condition for this equation leads to a determination of the first correction to the characteristic frequencies, which is of primary interest.

The kinematic condition (4.17) leads to

$$l\beta_l^{\langle 1,1\rangle} = \frac{\partial \alpha_l^{\langle 1,1\rangle}}{\partial T_{(0,0)}} + \frac{\partial \alpha_l^{\langle 0,1\rangle}}{\partial T_{(1,0)}}$$
$$- B_{-2}^{\langle 1,1\rangle}(l)\frac{\partial \alpha_{l-2}^{\langle 0,1\rangle}}{\partial T_{(0,0)}} - B_0^{\langle 1,1\rangle}(l)\frac{\partial \alpha_l^{\langle 0,1\rangle}}{\partial T_{(0,0)}} - B_{+2}^{\langle 1,1\rangle}(l)\frac{\partial \alpha_{l+2}^{\langle 0,1\rangle}}{\partial T_{(0,0)}}, \tag{4.23}$$

where

$$B_{-2}^{\langle 1,1\rangle}(l) = \frac{9}{8}(l-1)I_{+2}(l-2),$$
$$B_0^{\langle 1,1\rangle}(l) = \frac{9}{8}\left[(l+1)I_0(l) - \frac{(l-1)}{3} - \frac{2l}{(2l-1)}\right],$$
$$B_{+2}^{\langle 1,1\rangle}(l) = \frac{9}{8}\left[(l+3)I_{-2}(l+2) - \frac{2(l+1)}{(2l+3)}\right],$$

with

$$I_{-2}(l) = \frac{l(l-1)}{(2l-1)(2l+1)},$$

$$I_0(l) = \frac{2l^2+2l-1}{(2l-1)(2l+3)},$$

$$I_{+2}(l) = \frac{(l+1)(l+2)}{(2l+1)(2l+3)}.$$

The equation for the normal stress balance (4.18) yields

$$\begin{aligned}
-\Delta p_0^{\langle 1,1\rangle} &+ \sum_{l=1}^{\infty} \frac{\partial \beta_l^{\langle 1,1\rangle}}{\partial T_{(0,0)}} P_l + \sum_{l=0}^{\infty} (l-1)(l+2)\alpha_l^{\langle 1,1\rangle} P_l \\
&= -\sum_{l=2}^{\infty} \frac{1}{l} \frac{\partial^2 \alpha_l^{\langle 0,1\rangle}}{\partial T_{(0,0)} \partial T_{(1,0)}} P_l \\
&\quad + \sum_{l=0}^{\infty} [C_{-2}^{\langle 1,1\rangle}(l)\alpha_{l-2}^{\langle 0,1\rangle} + C_0^{\langle 1,1\rangle}(l)\alpha_l^{\langle 0,1\rangle} + C_{+2}^{\langle 1,1\rangle}(l)\alpha_{l+2}^{\langle 0,1\rangle}] P_l, \qquad (4.24)
\end{aligned}$$

where

$$C_{-2}^{\langle 1,1\rangle}(l) = \frac{9}{8}[2(l^2-3l+6) + 8(l-2) + l(l-2)(l-3)] I_{+2}(l-2),$$

$$C_0^{\langle 1,1\rangle}(l) = \frac{9}{8}\left\{[2(l^2+l+4) + l(l-1)(l+2)]\left[I_0(l) - \frac{1}{3}\right] + 8l\left[I_0(l) - \frac{2l}{(2l+1)(2l-1)}\right]\right\},$$

$$C_{+2}^{\langle 1,1\rangle}(l) = \frac{9}{8}[2(l^2+5l+10) + 8l + (l+1)(l+2)(l+4)] I_{-2}(l+2).$$

By eliminating $\beta_l^{\langle 1,1\rangle}$ from (4.23) and (4.24), an equation is obtained for the coefficients of the shape function

$$\begin{aligned}
\frac{\partial^2 \alpha_l^{\langle 1.1\rangle}}{\partial T_{(0,0)}^2} &+ l(l-1)(l+2)\alpha_l^{\langle 1,1\rangle} \\
&= -2\frac{\partial^2 \alpha_l^{\langle 0,1\rangle}}{\partial T_{(0,0)} \partial T_{(1,0)}} + A_{+2}^{\langle 1,1\rangle}(l)\alpha_{l+2}^{\langle 0,1\rangle} \\
&\quad + A_0^{\langle 1,1\rangle}(l)\alpha_l^{\langle 0,1\rangle} + A_{-2}^{\langle 1,1\rangle}(l)\alpha_{l-2}^{\langle 0,1\rangle}, \quad \text{for } l \neq 0, \qquad (4.25)
\end{aligned}$$

where

$$A_{-2}^{\langle 1,1\rangle}(l) = lC_{-2}^{\langle 1,1\rangle}(l) - l(l-2)(l-3)B_{-2}^{\langle 1,1\rangle}(l),$$

$$A_0^{\langle 1,1\rangle}(l) = lC_0^{\langle 1,1\rangle}(l) - l(l-1)(l+2)B_0^{\langle 1,1\rangle}(l),$$

$$A_{+2}^{\langle 1,1\rangle}(l) = lC_{+2}^{\langle 1,1\rangle}(l) - (l+1)(l+2)(l+4)B_{+2}^{\langle 1,1\rangle}(l).$$

The solvability condition for (4.25) requires that

$$-2\frac{\partial^2 \alpha_l^{\langle 0,1\rangle}}{\partial T_{(0,0)} \partial T_{(1,0)}} + A_0^{\langle 1,1\rangle}(l)\alpha_l^{\langle 0,1\rangle} = 0. \qquad (4.26)$$

As a result,

$$\alpha_l^{\langle 0,1\rangle} = c_l^{\langle 1,1\rangle} \exp\left[i\left(\omega_l T_{(0,0)} - \frac{A_0^{\langle 1,1\rangle}(l)}{2\omega_l} T_{(1,0)}\right)\right], \tag{4.27}$$

where $c_l^{\langle 1,1\rangle}$ could be functions of $T_{(0,1)}, T_{(0,2)}, T_{(1,1)}, \ldots$.

Thus a more accurate expression for the characteristic frequencies is found to be

$$\omega_l\left(1 - \frac{A_0^{\langle 1,1\rangle}(l)}{2\omega_l^2}\varepsilon_1\right) \qquad (l = 2, 3, 4, \ldots) \tag{4.28}$$

where ω_l is the linear normal mode frequency given by (4.15).

Moreover the coefficients $\alpha_0^{\langle 1,1\rangle}$, $\alpha_1^{\langle 1,1\rangle}$, and $\Delta p_0^{\langle 1,1\rangle}$ can be obtained form (4.19), (4.20), and (4.24):

$$\alpha_0^{\langle 1,1\rangle} = -\frac{3}{10}\alpha_2^{\langle 0,1\rangle}, \qquad \alpha_1^{\langle 1,1\rangle} = -\frac{81}{140}\alpha_3^{\langle 0,1\rangle}, \qquad \Delta p_0^{\langle 1,1\rangle} = -\frac{18}{5}\alpha_2^{\langle 0,1\rangle}; \tag{4.29}$$

and (4.25) yields

$$\alpha_l^{\langle 1,1\rangle} = -\frac{A_{+2}^{\langle 1,1\rangle}(l)}{2(l+2)(3l+2)}\alpha_{l+2}^{\langle 0,1\rangle} + \frac{A_{-2}^{\langle 1,1\rangle}(l)}{2l(3l-4)}\alpha_{l-2}^{\langle 0,1\rangle}. \tag{4.30}$$

As discussed in [1], the solution to the homogeneous part of (4.25) need not be included because it can always be absorbed in the linear modes (4.14).

The coefficients for the electric potential can be generally written as

$$\xi_l^{\langle 1,1\rangle} = H_{-3}^{\langle 1,1\rangle}(l)\alpha_{l-3}^{\langle 0,1\rangle} + H_{-1}^{\langle 1,1\rangle}(l)\alpha_{l-1}^{\langle 0,1\rangle} + H_{+1}^{\langle 1,1\rangle}(l)\alpha_{l+1}^{\langle 0,1\rangle} + H_{+3}^{\langle 1,1\rangle}(l)\alpha_{l+3}^{\langle 0,1\rangle},$$

where $H_{-3}^{\langle 1,1\rangle}(l)$, $H_{-1}^{\langle 1,1\rangle}(l)$, $H_{+1}^{\langle 1,1\rangle}(l)$, and $H_{+3}^{\langle 1,1\rangle}(l)$ can be obtained from (4.21) and (4.22) in a straightforward but tedious way.

5. Discussion. One interesting aspect of the results obtained in this paper is the lowering of the characteristic frequencies in the presence of an externally applied electric field, which may be regarded as the result of a net reduction in the restoring force on the drop surface. This effect is evident as a reduction in the uniform pressure difference across the interface as seen in (4.6), where the perturbation correction to Δp_0 is negative. Physical insight may be gained by examining the surface forces at some special locations. A reduction in the surface restoring force occurs at the equator where the curvature is diminished by electrostatic distortion in the relative absence of induced charge. At the poles the curvature effect tends to increase the restoring force, but the Coulomb repulsion due to the concentration of induced charge effectively softens the drop surface.

For convenience, each of the characteristic frequencies in (4.28) is normalized by the corresponding linear normal mode frequency ω_l and is ploted in Fig. 1 as a function of a normalized electric field strength $E_0 = \sqrt{\varepsilon_1}$ instead of the small parameter ε_1. The quadratic decrease in the lowest characteristic frequency with E_0 predicted here is generally in good agreement with the one obtained from the spheroidal model by Brazier-Smith et al. [5], except in the region where the drop deformation is no longer small. When the deformation increases, the drop will become unstable so the

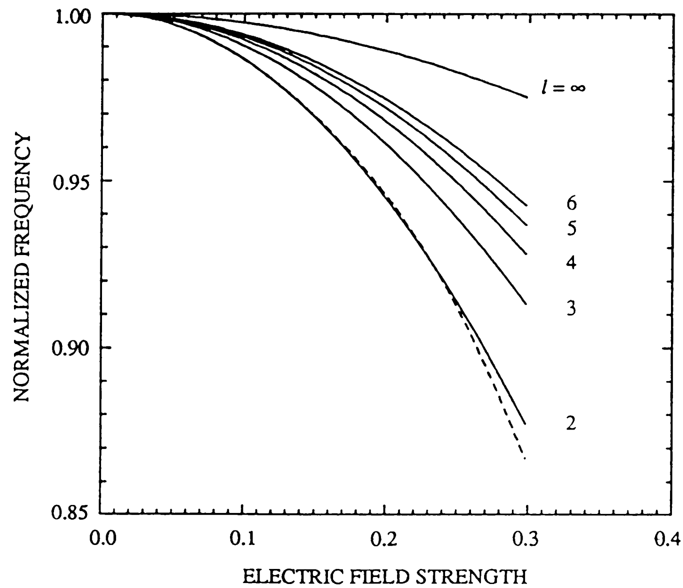

FIG. 1. The normalized characteristic frequency $1-(A_0^{(1,1)}(l)/2\omega_l^2)E_0^2$ of a conducting drop in an electric field as a function of the normalized electric field strength, $E_0 = \sqrt{\varepsilon_1}$. Results of the method of multiple-parameter perturbations (solid curves, numbered by values of l) and that of the spheroidal model of Brazier-Smith et al. (dashed curve) for the two-lobed mode are shown.

effects of higher-order terms in the perturbation expansion could be more significant. The calculations here, however, are based on low-order expansions which may be only valid for small deformations. The results obtained in this paper differ somewhat from those of Sample et al. [8] and Morrison et al. [9], because their failure to retain all terms of the same order is avoided in using the consistent method of multiple-parameter perturbations.

Acknowledgment. This work was supported by the National Science Foundation under grants ATM 8601549 and 8722688. The author is also grateful to Prof. Kenneth Beard for his active interest and wishes to thank Prof. Mankin Mak and Dr. Ming Cai for valuable discussions.

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