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IS QUENCHING IN INFINITE TIME POSSIBLE?

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Consider the problem

$$u_t - \Delta u = -u^{-\alpha} \quad \text{for } t > 0, \ x \in D, \tag{1}$$

$$u = 1 \quad \text{for } t > 0, \ x \in \partial D, \tag{2}$$

$$u(0, x) = u_0(x) > 0 \quad \text{for } x \in D, \tag{3}$$

where $D \subset \mathbb{R}^N$ is a bounded domain and $\alpha > 0$. It is well known that for suitable domains D and initial data u_0 the solution of (1) (2) (3) can approach zero in finite time ("it quenches"), while for other initial data or other domains it can remain globally bounded away from zero; see for instance [10]. A third possibility is conceivable, i.e.,

(P) The solution of (1) (2) (3) tends to zero in infinite time.

In this case we speak of quenching in infinite time. It was shown in [2, 3, 13] that (P) cannot occur for $N = 1$ and $u_0 \equiv 1$. Other results concerning the impossibility of (P) for $N = 1$ can be found in [11]. For $N \geq 2$ the question whether (P) can occur was stated as an open problem in [10, p. 279]. There is also a related problem for which it has been shown by Levine [9] that quenching in infinite time is impossible in one dimension, and by Levine and Lieberman [12] that (P) is possible in two dimensions. In terms of our notation they studied problems like

$$u_t - \Delta u = 0 \quad \text{for } t > 0, \ x \in D, \tag{4}$$

$$u = 1 \quad \text{for } t > 0, \ x \in \sigma, \tag{5}$$

$$\frac{\partial u}{\partial n} = -u^{-\alpha} \quad \text{for } t > 0, \ x \in \Sigma, \tag{6}$$

$$u(0, x) = u_0(x) > 0 \quad \text{for } x \in D, \tag{7}$$

in which ∂D is split in two subsets σ and Σ. So for problem (4)–(7) the answer to the question whether (P) can occur depends on the space dimension N. Our result reads as follows:

THEOREM. Let u be a decreasing solution ($u_t \leq 0$) to (1) (2) (3) and suppose that $D \subset \mathbb{R}^N$ is convex. Then (P) cannot occur for $N = 1, 2$ and $\alpha > 1$ or for $N = 3$ and $\alpha > 3$.

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The assumption that u is decreasing is satisfied for instance for initial data $u_0 \equiv 1$ or, more generally, for u_0 satisfying $\Delta u_0 - u_0^{-\alpha} \le 0$.

One ingredient in the proof is a transformation of the quenching problem into a blow up problem. The equivalence between quenching and blow up problems was pointed out in [1]. For blow up problems it is known that the occurence of blow up in infinite time depends on the space dimension, see [14, 5]. Our strategy is to apply the concavity method as in [8] to the new blow up problem. In order to do this we need to know that

$$\int_{\partial D} \frac{\partial u}{\partial n}\, dx \le c(u_0) < \infty \tag{8}$$

holds, where c is independent of t. Property (8) follows form the proof of Theorem 2.3 of Deng and Levine [4] (for a corresponding result in the context of blow up see [7, proof of Theorem 3.3]). Here the convexity of D was used. Finally we need to use Theorem 3.4 from [6], which states that

$$\lim_{t \to T^-} \int_D u^{-\lambda}(t, x)\, dx = +\infty \quad \text{for } \lambda > \frac{N}{2}(1+\alpha) \text{ and } \alpha \ge 1. \tag{9}$$

Here $T \le +\infty$ is the quenching time. We remark that $\int_D u^{-\lambda}(t, x)\, dx$ may stay bounded for $t \in [0, T)$ if $\lambda < \frac{N}{2}(1+\alpha)$, see [6].

Proof. We make the ansatz $v = u^{-\beta}$ with positive β. Later we shall restrict the range of β. Then v solves the problem

$$\begin{aligned} v_t - \Delta v = \beta v^{(\alpha+\beta+1)/\beta} - \frac{\beta+1}{\beta} v^{-1} |\nabla v|^2 \quad & \text{for } t > 0,\ x \in D, \\ v = 1 \quad & \text{for } t > 0,\ x \in \partial D, \\ v(0, x) = v_0(x) = u_0(x)^{-\beta} \quad & \text{for } x \in D. \end{aligned}$$

Assume that u quenches in infinite time and that u is decreasing. Then v blows up in infinite time and is increasing. It is easy to see that the functional

$$V(w) := \frac{1}{2}\int_D |\nabla w|^2\, dx - \frac{\beta^2}{\alpha + 2\beta + 1} \int_D w^{(\alpha+2\beta+1)/\beta}\, dx$$

decreases along v. In fact,

$$\begin{aligned} \frac{d}{dt} V(v) &= \int_D \{\nabla v \nabla v_t - \beta v^{(\alpha+\beta+1)/\beta} v_t\}\, dx \\ &= -\int_D v_t^2\, dx - \frac{\beta+1}{\beta} \int_D \frac{|\nabla v|^2}{v} v_t\, dx \le -\int_D v_t^2\, dx, \end{aligned}$$

where we used the monotonicity of v with respect to t. Therefore

$$\int_0^t \int_D v_t^2\, dx\, dt + V(v(t, x)) \le V(v_0(x)). \tag{10}$$

Denote $M(t) := \int_0^t \int_D v^2\, dx\, dt$, then using (8) and (10) and setting $\gamma := \beta - \frac{2\beta(2\beta+1)}{\alpha+2\beta+1}$

we obtain

$$\begin{aligned}
\frac{1}{2}M''(t) &= \int_D v v_t \, dx \\
&= -\int_D |\nabla v|^2 \, dx + \int_{\partial D} \frac{\partial v}{\partial n} \, dx + \beta \int_D v^{(\alpha+2\beta+1)/\beta} \, dx - \frac{\beta+1}{\beta} \int_D |\nabla v|^2 \, dx \\
&\geq -\frac{2(2\beta+1)}{\beta} V(v) - \beta c(u_0) + \gamma \int_D v^{(\alpha+2\beta+1)/\beta} \, dx \\
&\geq \frac{2(2\beta+1)}{\beta} \left(\int_0^t \int_D v_t^2 \, dx \, dt - V(v_0) \right) - \beta c(u_0) + \gamma \int_D v^{(\alpha+2\beta+1)/\beta} \, dx.
\end{aligned}$$

We want to have $\gamma > 0$. Therefore we choose $\beta < \frac{\alpha-1}{2}$. This explains the assumption $\alpha > 1$ in our theorem. Now we identify the last integral with

$$\int_D u^{-\lambda} dx, \quad \text{where } \lambda = \alpha + 2\beta + 1.$$

We use the fact that according to (9) this integral goes to $+\infty$ as $t \to \infty$, provided $\lambda > \frac{N}{2}(1+\alpha)$. This and $\beta < \frac{\alpha-1}{2}$ imply $N < (4-N)\alpha$, which explains our restrictions on N and α. Therefore $M''(t) \to \infty$ (and hence M and $M' \to \infty$) as $t \to \infty$. Now we calculate

$$\begin{aligned}
MM'' &- \frac{2\beta+1}{\beta}(M')^2 \\
&\geq \frac{4(2\beta+1)}{\beta} \left[\int_0^t \int_D v^2 \, dx \, dt \cdot \int_0^t \int_D v_t^2 \, dx \, dt - \left(\int_0^t \int_D v v_t \, dx \, dt \right)^2 \right] \\
&+ M \left\{ 2\gamma \int_D v^{(\alpha+2\beta+1)/\beta} \, dx - \frac{4(2\beta+1)}{\beta} V(v_0) - 2\beta c(u_0) \right\} \\
&- M' \left(\frac{2(2\beta+1)}{\beta} \int_D v_0^2 \, dx \right) + \frac{2\beta+1}{\beta} \left(\int_D v_0^2 \, dx \right)^2.
\end{aligned}$$

Notice that we have used the identity

$$M'(t) = \int_D v_0^2 \, dx + \int_0^t \int_D (v^2)_t \, dx \, dt.$$

The term in the square brackets is nonnegative because of Schwarz's inequality. The first term in the curly brackets can be estimated from below by $c_1(M')^\delta$ with some $\delta > 1$. The second and third terms in the curly brackets are equal to some constant $-c_2$.

Therefore we can rewrite our estimate as follows:

$$MM'' - \frac{2\beta+1}{\beta}(M')^2 \geq c_1 M (M')^\delta - c_2 M - c_3 M', \tag{11}$$

and for sufficiently large t the right-hand side of (11) is positive. Hence $(M^{-\mu})'' < 0$ for t large enough and for $\mu = \frac{\beta+1}{\beta}$. But $M^{-\mu}$ is decreasing and concave and must have a root $t_0 > 0$—a contradiction to the assumption that M is well defined for all $t > 0$. □

REMARK. Suppose it is known that the stationary version of problem (1) (2) has a minimal solution w which is classical and unstable from below (for $N = 1$ and D sufficiently small this is known, see [11]). Then by a result of Matano [15] there is a monotone decreasing solution u of (1) (2) defined on $(-\infty, 0] \times \overline{D}$ which, for $t \to -\infty$, tends to w in $C^2(\overline{D})$. According to our theorem the solution emanating from $u(0, \cdot)$ must quench in finite time. Hence by the maximum principle any solution of (1) (2) (3) with $u_0 \leq w$ but $u_0 \not\equiv w$ will quench in finite time. Our argument should be compared with the reasoning of Levine [11, proof of Theorem 3.1A]. We do not require a priori knowledge about nonexistence of weak stationary solutions.

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