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RESONANCE IN UNDAMPED SECOND-ORDER NONLINEAR EQUATIONS WITH PERIODIC FORCING

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We consider the scalar equation

$$x'' + g(x) = p(t) \tag{1}$$

where g and p are real valued functions continuous on the reals R, $p(t+2\pi) = p(t)$, and the solutions of (1) are uniquely determined by their initial conditions. If g is nonlinear, the question of whether all solutions of (1) are bounded on R has long been recognized as nontrivial and challenging. For the special case of $g(x) = 2x^3$ Morris [1] was able to show that this question has an affirmative answer. Later Dieckerhoff and Zehnder [2] were able to prove it for the case $g(x) = x^{2n+1} + p_{2n}(t)x^{2n} + \cdots + p_1(t)x$ where the $p_k(t)$ are 2π-periodic and sufficiently smooth. A more recent and considerably more general result is due to Ding [3] and asserts that if g is continuously differentiable and $g(x)/x \to \infty$ as $|x| \to \infty$, the answer is in the affirmative. On the other hand, Littlewood [4] has given an example of an equation like (1) with $g(x)/x \to \infty$ as $|x| \to \infty$ where $g(x)$ and $p(t)$ are not continuous which has an unbounded solution on R. As Morris has pointed out in [1], it is easy to modify the $g(x)$ in Littlewood's example so that it is continuously differentiable, in fact, C^∞, and still have an unbounded solution for (1); whether the function $p(t)$, which in Littlewood's example is piecewise constant-valued, can be modified to be continuous seems not entirely obvious. However, in Ding's proof in [3], only the fact that the Poincaré map associated with (1) is an area-preserving twist homeomorphism is used, and since this is also true if $g(x)$ is continuously differentiable and $p(t)$ only piecewise continuous, a question arises as to the validity of Ding's proof or Littlewood's example.

In this note we consider the problem of giving conditions on g for which all solutions of (1) are unbounded on R; specifically on $[0, \infty)$, primarily for cases where g is nonlinear. Our main result states that if for some integer n, $|g(x) - n^2 x|$ is bounded on R, and if a suitable Fourier coefficient of $p(t)$ has sufficiently large absolute value, then all solutions of (1) are unbounded on $[0, \infty)$.

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Our result is based on the following two results, which we state without proofs. First we define for $f(t)$ any function continuous on $[0, 2\pi]$ the notation

$$M_f(n, \delta) = \int_0^{2\pi} f(t) \cos(nt + \delta)\, dt.$$

LEMMA 1. If $f(t)$ is continuous and 2π-periodic, and there exists a δ such that $M_f(n, \delta) \neq 0$, then all solutions of $x'' + n^2 x = f(t)$ are unbounded on $[0, \infty)$.

LEMMA 2. If all solutions of (1) exist for $t \geq 0$ and there exists a solution which together with its derivative is bounded on $[0, \infty)$, then there exists a 2π-periodic solution of (1).

Lemma 1 is well known and easy to prove; in fact it also follows easily that if $M_f(n, \delta) = 0$ for all δ, $0 \leq \delta < 2\pi$, then all solutions of $x'' + n^2 x = f(t)$ are bounded on R. Lemma 2 is a special case of a theorem due to Massera [5] and its proof uses a nontrivial fixed point theorem in R^2.

THEOREM 1. Suppose there exists a $g_0 > 0$, and an integer $n \geq 0$, such that

$$\left| g(x) - n^2 x \right| \leq g_0 \quad \text{for } x \text{ in } R. \tag{1.1}$$

Then if $|M_p(\delta_0, n)| > 2\pi g_0$, where $\delta_0 = -\tan^{-1}(s_0/c_0)$, $c_0 = \int_0^{2\pi} p(t)\cos nt\, dt$, $s_0 = \int_0^{2\pi} p(t) \sin nt\, dt$, all solutions of (1) are unbounded on $[0, \infty)$.

Proof. Using standard continuation theorems, one can easily show that under the hypotheses on g, all solutions of (1) can be continued to $t = \infty$; we omit the details.

We next show that if $x(t)$ is a solution bounded on $[0, \infty)$ then $x'(t)$ is also. Suppose not; then there exists $t_0 > 0$ such that

$$\left(x'(t_0)\right)^2 > 4M_0 B_0 \tag{1.2}$$

where $B_0 = \sup\{|x(t)| : t \geq 0\}$,

and $M_0 = \sup\left\{|g(x) - p(t)| : |x| \leq B_0,\ t \text{ in } R\right\}$.

Suppose $x'(t_0) > 0$; the case $x'(t_0) < 0$ can be dealt with in the same way. Since $x(t)$ increases without bound if $x'(t) > 0$ and is bounded away from zero for $t \geq 0$, it follows that there exists an increasing sequence $\{t_k\}$, $k = 1, 2, \ldots$, such that $t_1 > t_0$, $x'(t_k) \to 0$ as $k \to \infty$, and $x'(t) > 0$ for $t_0 \leq t < \bar{t}$ where $\bar{t} = \lim_{k\to\infty} t_k$. Put $x_k = x(t_k)$, $k = 0, 1, 2, \ldots$, and define $y(x) = x'(t(x))$, where $t(x)$ is the inverse of $x(t)$ on $[t_0, \bar{t})$. It is easy to show that for any integer $k \geq 1$, $d(y^2(x))/dx = 2(p(t(x)) - g(x))$, $x_0 < x < x_k$.

From this and the mean value theorem we get

$$\left| y^2(x_k) - y^2(x_0) \right| \leq 2M_0 (x_k - x_0) \leq 4M_0 B_0 \tag{2}$$

and since $y(x_0) = x'(t_0)$ and $y(x_k) = x'(t_k) \to 0$ as $k \to \infty$, we obtain from (2) that $(x'(t_0))^2 \leq 4M_0 B_0$, which contradicts (1.2). This proves the assertion that $x'(t)$ is alsobounded on $[0, \infty)$.

Thus if (1) has a solution bounded on $[0, \infty)$, by Lemma 2 it has a 2π-periodic solution, say $\overline{x}(t)$. So the equation

$$x'' + n^2 x = n^2 \overline{x}(t) - g(\overline{x}(t)) + p(t) \tag{3}$$

has a 2π-periodic solution $\overline{x}(t)$. Using Lemma 1 with $f(t) = n^2\overline{x}(t) - g(\overline{x}(t)) + p(t)$ we find that for all δ,

$$M_p(\delta, n) = -\int_0^{2\pi} \left(n^2 \overline{x}(t) - g(\overline{x}(t)\right) \cos(nt + \delta)\, dt.$$

So $|M_p(\delta, n)| \le 2\pi g_0$ for all δ. But by elementary calculus, it follows easily that

$$\max\left\{M_p(\delta, n) : \delta \text{ real}\right\} = \left|M_p(\delta_0, n)\right|,$$

and we arrive at a contradiction since by hypotheses $|M_p(\delta_0, n)| > 2\pi g_0$. This proves the theorem.

We state a result for a more general equation which can be proved using the method in the proof of Theorem 1.

THEOREM 2. Let $f(x, t)$ be real valued and continuous of $R \times R$ and $f(x, t+2\pi) = f(x, t)$ there. Suppose there exists an integer $n > 0$ and a δ such that

$$\int_0^{2\pi} \left(f(\overline{x}(t), t) + n^2 \overline{x}(t)\right) \cos(nt + \delta)\, dt \ne 0$$

for all continuously differentiable 2π-periodic functions $\overline{x}(t)$. Suppose all solutions of

$$x'' = f(t, x) \tag{4}$$

are uniquely determined by these initial conditions and extend to $[0, \infty)$. Then all solutions of (4) are unbounded on $[0, \infty)$.

We conclude by applying Theorem 1 to some special cases of (1), first to obtain a lower bound on the suprema of the absolute values of certain bounded solutions of $x'' + x^3 = p(t)$.

Let n_0 and m_0 be positive integers, and define

$$\begin{aligned} g(x) &= x^3, \qquad 0 \le x \le m_0, \\ &= n_0^2(x - m_0) + m_0^3, \; x_0 > m_0 \end{aligned}$$

and $g(x) = -g(-x)$ for $x < 0$. We use Theorem 1 with $n = n_0$. If

$$g_0 = \max\left\{m_0\left|m_0^2 - n_0^2\right|, \; 2n_0^3/3\sqrt{3}\right\},$$

it follows by straightforward calculations that

$$\left|g(x) - n_0^2 x\right| \le g_0, \; x \quad \text{in } R.$$

Thus if $M_p(n_0, \delta_0) > 2\pi g_0$, with δ_0 as defined in Theorem 1 with $n = n_0$, all solutions of (1) for this g are unbounded on $[0, \infty)$; i.e., for any solution $x(t)$ of (1) there exist $t_k \to \infty$ as $k \to \infty$ such that $|x(t_k)| > m_0$, $n = 1, 2, \ldots$.

This shows also that if $x(t)$ solves $x'' + x^3 = p(t)$ and $|x(t)| \le m_0$ for t in R, then $M_p(n_0, \delta_0) \le 2\pi g_0$.

If we take $m_0 = n_0$, clearly $g_0 = 2n_0^3/3\sqrt{3}$, and we conclude that if

$$\int_0^{2\pi} p(t)\cos(n_0 t + \delta_0)\, dt > 4\pi n_0^3/3\sqrt{3},$$

then there exist no solutions $x(t)$ of $x'' + x^3 = p(t)$ such that $|x(t)| \le n_0$ for t in R.

Next consider the piecewise linear $g(x)$ given by

$$\begin{aligned} g(x) &= x, & 0 \le x \le 1, \\ &= 4x - 3, & x > 1 \end{aligned}$$

with $g(x) = -g(-x)$, $x < 0$.

A simple application of Theorem 1 shows that if δ_0 is defined as in Theorem 1 with $n = 2$, and

$$\left|M_p(2, \delta_0)\right| > 6\pi, \tag{5}$$

then all solutions of (1) are unbounded on $[0, \infty)$. An interesting and apparently open question is that if $p(t)$ does not satisfy (5), are these solutions bounded on $[0, \infty)$; i.e., are there 2π-periodic solutions? It is easy to show that if for some δ,

$$M_p(1, \delta) \ne 0, \tag{6}$$

then for every solution $x(t)$ there exists a sequence $t_k \to \infty$ as $k \to \infty$ such that $|x(t_k)| > 1$, $k = 1, 2, \ldots$. If not, there exists a t_0 such that $|x(t)| \le 1$ for $t \ge t_0$, and so $x(t)$ satisfies $x'' + x = p(t)$ for all such t, and using (6) we easily get a contradiction.

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