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ON THE STABILITY OF VARLEY-DAY SOLUTIONS FOR HARMONIC MATERIALS

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1. Introduction. Varley and Day have considered the problem of determining deformations that can occur in a homogeneous, isotropic elastic body at constant pressure on the boundary and null Cauchy deviatoric stress and have found certain solutions to this problem [1] which possess constant principal stretches. Ericksen [2–3] has conjectured that the solutions obtained in [1] should be at best neutrally stable. This conjecture was proved for plane deformations by Adeleke [4].

In a recent paper [5] nonconformal plane deformations characterized by the requirement that the ratio of the principal stretches of the deformation is constant were shown to be Varley-Day type solutions (in the absence of body forces) for certain unconstrained isotropic elastic solids. By employing the stability criterion due to Beatty [6] the solutions in this class were also shown in [5] to be at best neutrally stable. The solutions found in [5] however are not the only plane nonconformal[1] Varley-Day solutions with nonconstant principal stretches. Such solutions may occur whenever the Baker-Ericksen (B-E) inequality [7, Sec. 51] is violated but the weakened B-E inequality is not [5].

One of the implications of the requirement that a Harmonic material [9] admits a regular state of uniaxial tension in plane strain is that the B-E inequality cannot be satisfied at all states of deformation [10]. Specifically, at deformations characterized by the condition that the sum of their principal stretches is a certain material-dependent constant the B-E inequality fails but the weakened B-E inequality does not [9, 10]. As it will become clear in the following this condition characterizes the class of plane Varley-Day solutions for Harmonic materials.[2,3]

In this paper we show by means of examples that among the Varley-Day solutions for Harmonic materials which admit a regular state of uniaxial tension there are solutions with nonconstant principal stretches. Then, by using Beatty's stability

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[1] In the absence of body forces all conformal equilibrium solutions for unconstrained homogeneous isotropic elastic solids that satisfy the classical Pressure-Compression inequality [7, Sec. 51] must necessarily be homogeneous [8]. As such, they possess principal stretches that are both constant and equal.

[2] Throughout this paper the body forces are taken to be zero.

[3] There is an abuse of language here; the conformal (Varley-Day) solutions at which the B-E inequality is satisfied are not considered in this paper.

criterion for the infinitesimal stability of equilibrium configurations of unconstrained elastic solids subject to uniform hydrostatic loading everywhere on the boundary [6], we show that the Varley-Day solutions can be at best neutrally stable and, in particular, that the Varley-Day solutions with constant principal stretches are unstable. An illustrative example is considered in the last section of the paper.

2. Preliminaries. Consider the plane deformations described by a suitable smooth and invertible transformation

$$\mathbf{x} = \mathbf{x}(\mathbf{X}) \tag{2.1}$$

where $\mathbf{X}$ and $\mathbf{x}$ are points that belong, respectively, to the domains D and $\overline{D}$ of the two-dimensional Euclidean Space R^2. Associated with such deformations is the deformation gradient, given by

$$F = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \tag{2.2}$$

for which

$$\det \mathbf{F} > 0, \tag{2.3}$$

where $\det(\cdot)$ stands for determinant. The deformation gradient can be uniquely represented in the form

$$\mathbf{F} = \mathbf{VQ}, \tag{2.4}$$

where $\mathbf{Q}$ is a proper orthogonal tensor and $\mathbf{V}$ is a symmetric, positive-definite tensor often referred to as the left-stretch tensor. By the spectral theorem there is an orthogonal basis $\{\mathbf{e}^1, \mathbf{e}^2\}$ such that

$$\mathbf{V} = \lambda_1 \mathbf{e}^1 \otimes \mathbf{e}^1 + \lambda_2 \mathbf{e}^2 \otimes \mathbf{e}^2, \tag{2.5}$$

where $\otimes$ denotes the tensor product of two vectors. The scalars λ_1, λ_2 are called principal stretches while the vectors $\mathbf{e}^1$, $\mathbf{e}^2$ are the principal axes of strain. One set of invariants of $\mathbf{V}$ is related to the principal stretches by the relations

$$\operatorname{tr} \mathbf{V}^2 \equiv I = \lambda_1^2 + \lambda_2^2, \qquad \det \mathbf{V} \equiv J = \lambda_1 \lambda_2, \tag{2.6}$$

where tr denotes the trace operator.

The plane part of the stress-deformation relation for a plane deformation of an isotropic homogeneous elastic solid can be written in the form

$$\mathbf{T} = \mathbf{t}(\mathbf{V}) = \varphi_0 \mathbf{1} + \varphi_1 \mathbf{V}^2. \tag{2.7}$$

In Eq. (2.7) $\mathbf{1}$ is the identity tensor, φ_0 and φ_1 are scalar-valued functions of the invariants of $\mathbf{V}$, and $\mathbf{T}$ is the Cauchy stress tensor. When a strain-energy function W exists, then

$$W = W(I, J) \equiv \widehat{W}(\lambda_1, \lambda_2), \tag{2.8}$$

and

$$\varphi_0 = \frac{\partial W}{\partial J}, \qquad \varphi_1 = \frac{2}{J}\frac{\partial W}{\partial I}. \tag{2.9}$$

If we combine (2.5), (2.7), and (2.9) we find that

$$\mathbf{T} = t_1 \mathbf{e}^1 \otimes \mathbf{e}^1 + t_2 \mathbf{e}^2 \otimes \mathbf{e}^2, \tag{2.10}$$

where the principal stresses t_1, t_2 are given by

$$t_i = \frac{\partial W}{\partial J} + \frac{2\lambda_i^2}{J}\frac{\partial W}{\partial I}, \qquad i = 1, 2. \tag{2.11}$$

The equilibrium condition, in the absence of body forces, is

$$\operatorname{div} \mathbf{T} = \mathbf{0}, \quad \text{on } \overline{D}, \tag{2.12}$$

where $\operatorname{div}(\cdot)$ denotes the divergence operator with respect to $\mathbf{x}$.

Harmonic materials are unconstrained elastic solids with a strain-energy density function in plane strain given by [9]

$$W(I, J) = 2\mu[H(Q) - J], \qquad Q \equiv (I + 2J)^{1/2} = \lambda_1 + \lambda_2, \tag{2.13}$$

where μ is a constant and $H(\cdot) \in C^2(0, \infty)$. Combining (2.7), (2.9), and (2.13) we find

$$\mathbf{T} = 2\mu\left\{\left[\frac{H'(Q)}{Q} - 1\right]\mathbf{1} + \frac{H'(Q)}{QJ}\mathbf{V}^2\right\}. \tag{2.14}$$

The requirement that both the strain-energy and the stress vanish in the undeformed configuration (i.e., for $\lambda_1 = \lambda_2 = 1$) yields

$$H(2) = H'(2) = 1. \tag{2.15}$$

A material is strongly-elliptic at $\mathbf{F}$ if [7, Sec. 44][4]

$$A_{\alpha\beta\gamma\sigma} a_\alpha a_\gamma b_\beta b_\sigma > 0, \qquad \alpha, \beta, \gamma, \sigma = 1, 2, \tag{2.16}$$

for every pair of nonzero vectors $\mathbf{a}(a_\alpha)$, $\mathbf{b}(b_\alpha)$. Here $A_{\alpha\beta\gamma\sigma}$ denote the components of the elasticity tensor $\mathbf{A}$ which is the Fréchet derivative of the Piola stress[5]

$$\mathbf{S}(\mathbf{F}) = J(\varphi_0 \mathbf{F}^{-T} + \varphi_1 \mathbf{F}). \tag{2.17}$$

If Cartesian coordinates are assigned to D and $\overline{D}$ the components of $\mathbf{A}$ are given by [11]

$$\begin{aligned} A_{\alpha\beta\gamma\sigma} = {} & 4\frac{\partial^2 W}{\partial I^2} F_{\alpha\beta} F_{\gamma\sigma} + 2J\frac{\partial^2 W}{\partial I \partial J}[F_{\beta\alpha}^{-1} F_{\gamma\sigma} + F_{\alpha\beta} F_{\sigma\gamma}^{-1}] + J^2 F_{\beta\alpha}^{-1} F_{\sigma\gamma}^{-1}\frac{\partial^2 W}{\partial J^2} \\ & + 2\frac{\partial W}{\partial I}\delta_{\alpha\gamma}\delta_{\beta\sigma} + J\frac{\partial W}{\partial J}[F_{\beta\alpha}^{-1} F_{\sigma\gamma}^{-1} - F_{\sigma\alpha}^{-1} F_{\beta\gamma}^{-1}], \end{aligned} \tag{2.18}$$

where $F_{\alpha\beta}$ denote the components of $\mathbf{F}$, $F_{\alpha\beta}^{-1}$ the components of $\mathbf{F}^{-1}$, and $\delta_{\alpha\beta}$ the two-dimensional Kronecker symbol.

We shall suppose that strong ellipticity holds for infinitesimal deformations of Harmonic materials. As shown in [11], this is the case if and only if

$$\mu > 0, \qquad H''(2) > 0. \tag{2.19}$$

[4] The summation convention over repeated indices is employed throughout the text.

[5] $\mathbf{F}^T$ is the transpose of $\mathbf{F}$ and $\mathbf{F}^{-T}$ is the inverse of the transpose of $\mathbf{F}$.

It then follows [10] that the strain-energy density (2.13) is positive, except in the undeformed state, if and only if

$$H(Q) > \frac{Q^2}{4}, \qquad Q \in (0, \infty) - \{2\}. \tag{2.20}$$

From (2.6), (2.7), and (2.10) we find that at conformal deformations (with $\lambda_1 = \lambda_2 \equiv \lambda$)

$$t_1 = t_2 = \varphi_0(2J, J) + J\varphi_1(2J, J) \equiv t = \hat{t}(\lambda). \tag{2.21}$$

The classical Pressure-Compression inequality is the requirement that t be a strictly increasing function of λ [7, Sec. 51]. Slightly stronger than this is the requirement that

$$\frac{d\hat{t}}{d\lambda} > 0, \qquad \lambda \in (0, \infty). \tag{2.22}$$

For Harmonic materials this condition becomes

$$QH''(Q) - H'(Q) > 0, \qquad Q \in (0, \infty). \tag{2.23}$$

At a deformation with $\lambda_1 \neq \lambda_2$ the stress-deformation relation (2.7) is said to satisfy the B-E inequality if [7, Sec. 51]

$$\varphi_1 > 0 \tag{2.24}$$

and the weakened B-E inequality if

$$\varphi_1 \geq 0. \tag{2.25}$$

At conformal deformations B-E inequality reduces to (2.25) [7, Sec. 51].

The inequality (2.24) is satisfied by the constitutive equation of Harmonic materials (2.14) if and only if

$$H'(Q) > 0, \qquad Q \in (0, \infty). \tag{2.26}$$

However, we shall suppose that the Harmonic materials admit a regular state of uniaxial tension in plane strain [10]. As shown in [10], for this condition to be satisfied by the Harmonic materials which obey (2.15), (2.19), and (2.23) it is necessary and sufficient that there exists a number $Q_0 \in (1, 2)$ such that

$$H'(Q_0) = 0, \qquad H'(Q)/Q \to 1 \quad \text{as } Q \to \infty \quad \text{and} \quad H''(Q) > 1 \quad \text{for } Q \in (Q_0, \infty). \tag{2.27}$$

A simple continuity argument shows that $(2.27)_3$ implies

$$H''(Q_0) \geq 1.^6 \tag{2.28}$$

An inspection of (2.14) and (2.12) now shows that the Varley-Day solutions for the class of Harmonic materials under consideration are characterized by the condition

$$\lambda_1 + \lambda_2 = Q_0. \tag{2.29}$$

[6] Note that at deformations with $\lambda_1 + \lambda_2 = Q_0$ the Coleman-Noll (C-N) convexity condition [7, Sec. 87] merely requires $H''(Q_0) > \frac{1}{2}$ [10].

3. Nonconformal Varley-Day solutions with nonconstant principal stretches. We start with the deformation

$$r = Rf(\Xi), \qquad \theta = g(R, \Xi), \tag{3.1}$$

where (r, θ) and (R, Ξ) are spatial and referential polar coordinates, respectively. The functions $f(\cdot)$ and $g(\cdot, \cdot)$ are to be determined such that the conditions (2.3) and (2.29) are satisfied.

The physical components of the deformation gradient corresponding to (3.1) are given by

$$\mathbf{F} = \begin{bmatrix} f & f_{,\Xi} \\ Rfg_{,R} & fg_{,\Xi} \end{bmatrix} \tag{3.2}$$

where, $(\cdot)$ stands for differentiation with respect to $(\cdot)$ (whether it is partial or total). From (2.4), (2.6), (3.2), and (2.29) we find the condition

$$I + 2J = f^2 + f^2_{,\Xi} + R^2 f^2 g^2_{,R} + f^2 g^2_{,\Xi} + 2(f^2 g_{,\Xi} - Rff_{,\Xi}g_{,R}) = Q_0^2. \tag{3.3}$$

To obtain our first example we assume that $g_{,R} = 0$ so that $(3.3)_2$ reduces to

$$f^2(1 + g_{,\Xi})^2 + f^2_{,\Xi} = Q_0^2. \tag{3.4}$$

Taking $f_{,\Xi} = k$, $k = \text{const.}$, $k \in (0, Q_0)$, we obtain from (3.4)

$$f = k\Xi + k_1, \qquad g = k^{-1}\sqrt{Q_0^2 - k^2}\ln(k\Xi + k_1) - \Xi + k_2, \tag{3.5}$$

where $k_1 > 0$ and k_2 are constants of integration. Using (2.4), (2.6), (3.2), and (3.5) it is easy to show that

$$I - 2J = 4(k\Xi + k_1)^2 - 4\sqrt{Q_0^2 - k^2}(k\Xi + k_1) + Q_0^2 \neq \text{const}. \tag{3.6}$$

Since there exists a number $\hat{k}$, independent on Q_0, such that, for $k_1 \in (0, \hat{k})$, the condition (2.3) is satisfied it follows that (3.1) and (3.5) define a deformation which meets our requirements.

Next we suppose that, in addition to (3.3), the functions f and g satisfy

$$f_{,\Xi} = -Rfg_{,R}. \tag{3.7}$$

Combining (3.7) and (3.3) and taking

$$f = \alpha_0 \exp(\alpha_1 \Xi), \qquad \alpha_0, \alpha_1 = \text{const.}, \tag{3.8}$$

in the resulting equation yields

$$g_{,\Xi} = \alpha_0^{-1}\sqrt{Q_0^2 \exp(-2\alpha_1\Xi) - 4\alpha_0^2\alpha_1^2} - 1, \tag{3.9}$$

where we assumed

$$4\alpha_0^2\alpha_1^2 \le Q_0^2 \exp(-4\alpha_1\pi). \tag{3.10}$$

From (3.7), (3.8), and (3.9) we deduce

$$\begin{aligned} g(R, \Xi) = {} & -\alpha_0^{-1}\alpha_1^{-1}\sqrt{Q_0^2 \exp(-2\alpha_1\Xi) - 4\alpha_0^2\alpha_1^2} - \Xi - \alpha_1 \ln R \\ & + 2\arctan\left[2^{-1}\alpha_0^{-1}\alpha_1^{-1}\sqrt{Q_0^2 \exp(-2\alpha_1\Xi) - 4\alpha_0^2\alpha_1^2}\right] + c, \\ & c = \text{const}. \end{aligned} \tag{3.11}$$

As condition (2.3) is satisfied if and only if $g_{,\Xi} > 0$, we choose α_0, α_1 so that

$$0 < \alpha_0 < 1\,, \qquad 4\alpha_0^2\alpha_1^2 + \alpha_0 \le \exp(-4\alpha_1\pi)\,. \tag{3.12}$$

Noticing that

$$I - 2J = \alpha_0^2(1 - g_{,\Xi})^2 \exp(2\alpha_1\Xi) \neq \text{const.}\,, \tag{3.13}$$

we conclude that (3.8), (3.11), (3.1), and (3.12) define another deformation of the type sought here.

4. Stability. A criterion for the infinitesimal stability of equilibrium configurations of unconstrained homogeneous elastic solids subject to uniform hydrostatic loading p everywhere on the boundary has been obtained by Beatty in [6]. The specialization of this criterion to plane deformations of isotropic bodies takes the form

$$K \equiv \int_{\overline{D}} \{\mathrm{tr}[\mathbf{H}\mathbf{t}(\mathbf{V})\mathbf{H}^T] + \mathbf{H}\cdot\mathbf{C}(\mathbf{F})[\mathbf{H}] - p[2\det\mathbf{E} + \mathrm{tr}(\mathbf{R}\mathbf{R}^T)]\}\, d\overline{D} > 0 \tag{4.1}$$

for all nonzero plane infinitesimal displacements $\mathbf{u}$. Here $\mathbf{H}(H_{\alpha\beta})$ is the gradient of $\mathbf{u}$, $\mathbf{E}$ and $\mathbf{R}$ the symmetric and skew-symmetric parts of $\mathbf{H}$, respectively, and

$$\mathbf{H}\cdot\mathbf{C}(\mathbf{F})[\mathbf{H}] \equiv \left(\frac{1}{J}F_{\beta\nu}F_{\sigma\xi}A_{\gamma\xi\alpha\nu} - \delta_{\alpha\gamma}t_{\beta\sigma}\right) H_{\alpha\beta}H_{\gamma\sigma}\,, \tag{4.2}$$

where $t_{\beta\sigma}$ denote the components of $\mathbf{t}(\mathbf{V})$ and $A_{\gamma\xi\alpha\nu}$ are given by (2.18). If K becomes negative for some $\mathbf{u}$ then the equilibrium configuration is unstable while if $K = 0$ for some $\mathbf{u} \neq \mathbf{0}$ but nonnegative for all $\mathbf{u}$ the equilibrium configuration is neutrally stable [6].

If $\mathbf{F} = \mathbf{V}\mathbf{Q}$ is the deformation gradient corresponding to a Varley-Day solution for a Harmonic material it follows from (2.7) and (2.14) that

$$\mathbf{d}t(\mathbf{V}) = p\mathbf{1} = -2\mu\mathbf{1}\,. \tag{4.3}$$

A straightforward but lengthy computation based on (2.18), (2.13), (4.2), (4.1), (4.3), and

$$\begin{aligned}\frac{\partial^2\widehat{W}}{\partial\lambda_\alpha\partial\lambda_\beta} = 4\lambda_\alpha\lambda_\beta\frac{\partial^2 W}{\partial I^2} + 2J\left(\frac{\lambda_\alpha}{\lambda_\beta} + \frac{\lambda_\beta}{\lambda_\alpha}\right)\frac{\partial^2 W}{\partial I\partial J} \\ + \frac{J^2}{\lambda_\alpha\lambda_\beta}\frac{\partial^2 W}{\partial J^2} + 2\delta_{\alpha\beta}\frac{\partial W}{\partial I} + (1 - \delta_{\alpha\beta})\frac{\partial W}{\partial J}\end{aligned} \tag{4.4}$$

leads to[7]

$$K = 2\mu\int_{\overline{D}}\left\{\left[\frac{\lambda_1}{\lambda_2}H''(Q_0) - 1\right]E_{11}^2 + 2H''(Q_0)E_{11}E_{22} + \left[\frac{\lambda_2}{\lambda_1}H''(Q_0) - 1\right]E_{22}^2\right\}\, d\overline{D}\,. \tag{4.5}$$

Since there exists a nonzero displacement vector $\mathbf{u}$ such that

$$E_{11} = E_{22} = 0 \tag{4.6}$$

the deformation with deformation gradient $\mathbf{F}$ can be at best neutrally stable.

[7] $F_{\alpha\beta}$ are the components of $\mathbf{F}$ with respect to the basis $\{\mathbf{e}^1, \mathbf{e}^2\}$.

From (4.5) we deduce that a Varley-Day deformation with constant principal stretches is neutrally stable if and only if

$$[H''(Q_0)]^{-1} \leq \frac{\lambda_1}{\lambda_2} \leq H''(Q_0) \tag{4.7}$$

and

$$H''(Q_0)\left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right) \leq 1\,. \tag{4.8}$$

Since[8]

$$\{(\lambda_1, \lambda_2) \mid \lambda_1 + \lambda_2 = Q_0,\ H''(Q_0)(\lambda_1\lambda_2^{-1} + \lambda_2\lambda_1^{-1}) \leq 1,\ H''(Q_0) \geq 1\} = \varnothing, \tag{4.9}$$

the condition (4.8) is violated in any Varley-Day deformation. In the case of Varley-Day deformations with constant principal stretches this fact precludes neutral stability as clearly, given any such deformation, an infinitesimal displacement **u** which renders $K < 0$ must always exist.

5. An example. The Harmonic material with

$$H(Q) = \frac{1}{2}Q^2 + \frac{Q}{m-1}\left(\frac{2}{Q}\right)^m + \frac{1+m}{1-m}, \qquad m \geq 0,\ m \neq 1\,, \tag{5.1}$$

which was recently considered in [12], satisfies the restrictions (2.15), (2.19), (2.20), (2.23), and (2.27) with

$$Q_0 = 2^{m/(m+1)} \tag{5.2}$$

and

$$H''(Q_0) = 1 + m\,. \tag{5.3}$$

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[8] See the remark following (2.27) and footnote 6.