QUARTERLY OF APPLIED MATHEMATICS

VOLUME XLVIII SEPTEMBER · 1990 NUMBER 3

SEPTEMBER 1990, PAGES 401-413

ENERGY BALANCE CRITERIA FOR VISCOELASTIC FRACTURE

By

J. M. GOLDEN [1] (*Environmental Research Unit, St. Martin's House, Dublin, Ireland*)

AND

G. A. C. GRAHAM [2] (*Dublin Institute for Advanced Studies, Dublin, Ireland*)

Abstract. An energy balance criterion of the Griffith type has been used to derive conditions that are valid, in the isothermal noninertial approximation, for the growth of cracks in viscoelastic bodies. These bodies are acted upon by general position and time-dependent load. The conditions have the same form as the instability conditions obtained for the corresponding problems in elasticity theory and, in particular, are independent of crack velocity. The analysis relies upon an exact calculation of the displacement and stress fields that is derived in the Appendix with the aid of extensions to viscoelasticity of the Kolosov-Muskhelishvili equations of elasticity theory.

1. Introduction. Energy balance theories for crack growth in elastic bodies have their origin in the work of Griffith [1, 2]. One notable feature of Griffith's result is that, in the noninertial approximation, it gives no information on the velocity of crack extension. This is not so if the surface energy, T, associated with crack extension is presumed to depend on the velocity of crack growth. Experimental evidence for this has been presented, for example, by Vincent and Gotham [3], Francis, Carlton, and Lindsey [4], and in papers referred to by Knauss [5], while Gurtin [6] has provided a theoretical foundation for the assumption. Willis [7], Kostrov and Nikitin [8], and Blackburn [9] also considered this possibility. Graham [10, 11] gave a derivation of the crack extension criterion for viscoelastic materials which indicated that it has a form similar to that for elastic materials, the instantaneous moduli of the viscoelastic body playing the role of elastic moduli. Kostrov and Nikitin [8] used an elaborate analysis of energy balance at the crack tip to arrive at the same conclusion. However, there is the work of Cherepanov [12] who, starting from unarguably valid energy relations, obtains a criterion of different form to that of Graham.

Nuismer [13] points out the contradictory nature of these results and derives a criterion which is in agreement with that of Graham. He also points to flaws in the work of Cherepanov and others which explain why they obtained different results.

Received August 24, 1987.

[1] Research Associate, Dublin Institute for Advanced Studies.

[2] On leave from: Department of Mathematics and Statistics, Simon Fraser University, Burnaby, B. C., Canada V5A 1S6.

More recent work (Christensen and McCartney [14] and references therein) supports the position of Graham and Nuismer if singular stresses are admitted.

On the other hand, several authors, on the basis of various phenomenological approaches, have given noninertial criteria which fix the velocity of crack extension. Contributions to this development have been made and surveyed by Knauss [5, 15], McCartney [16], Christensen and McCartney [14], Schapery [17, 18], Kaminskii [19], Kaminskii and Pestrikov [20], Coussy [21], Schovanec and Walton [22], Goleniewski [23], and Golden and Graham [24]; each of which contains further relevant bibliographical information.

In the present paper, a simple transparent derivation of the result of Graham and Nuismer is presented, which is based on a manifestly fundamental energy principle. In fact, the results obtained generalize the earlier treatments in that tangential and position-dependent applied stresses are included. It is hoped that this work will help to further clarify the issue. Since it includes position-dependent stresses on the crack, there is the implication that if one assigns structure to the crack tip simply by the inclusion of cohesive, or indeed shear forces, the energy balance criterion, allowing singular stresses, will not give a velocity-dependent criterion for crack growth. This is also true if singular stresses are eliminated by the standard Barenblatt [25] argument using a cohesive zone of negligible length (Golden and Graham [26]). If the length of the cohesive zone is not allowed to go to zero, then it would seem from the work of several authors, notably Schovanec and Walton [22], that a velocity-dependent criterion is obtained. Note that this procedure amounts to constructing a model of the velocity dependence of the surface energy.

The point of departure of this study, which is restricted to the linear isothermal noninertial case, is an energy balance equation given by Erdogan [27]. The development of the implication of this equation is believed to be novel, though somewhat similar ideas have recently been used in the context of elastic composites by McCartney [28].

2. Energy balance criteria for crack extension in linear viscoelasticity. Goodier [29] shows that for the case of fixed applied tension in the noninertial approximation, the strain energy of a linear elastic body increases if crack extension occurs, but only half the work done by the boundary forces goes to this increase. The other half is available for creating new surface.

In order to discuss the linear viscoelastic case we adopt a more formal approach, which incidentally also applies to the linear elastic case in the limit, and illustrates the observation of the previous paragraph. Let us write down an isothermal energy balance relation of the form

$$\int_B ds\sigma_{ij}n_j\dot{u}_i = \dot{S} + \dot{H} + 2T(\dot{b} - \dot{a}), \tag{1}$$

where ds is a surface element (becoming a line element in the plane strain case), $\dot{S}$ is the rate of increase of stored mechanical energy and $\dot{H}$ is the rate of dissipation of mechanical energy. Here, and throughout the paper, we have adhered to the standard conventions of cartesian tensor notation. The crack, which is allowed to grow with time, occupies the interval $[a(t), b(t)]$ of the x-axis (see Appendix), while T, which is assumed to be a constant, is the surface energy per unit length. Since the quantity on the left is the rate of work of the boundary forces this relation may be taken as self-evident (see Erdogan [27], for example). In the noninertial theory, $\dot{S}(t)+\dot{H}(t)$ can be replaced by the space integral of $\sigma_{ij}\dot{\varepsilon}_{ij}$ (see Christensen [30] and Golden and Graham [26]) and we have

$$\int_B ds\sigma_{ij}(\underline{r}, t)n_j(\underline{r})\dot{u}_i(\underline{r}, t) = \int_V dv\sigma_{ij}(\underline{r}, t)\dot{\varepsilon}_{ij}(\underline{r}, t) + 2T(\dot{b}(t)-\dot{a}(t)). \tag{2}$$

For an elastic material (e.g., see Sokolnikoff [31])

$$H(t)=0, \qquad S(t)=\frac{1}{2}\int_B ds\sigma_{ij}(\underline{r}, t)n_j(\underline{r})u_i(\underline{r}, t) \tag{3}$$

and, instead of (2), we may take

$$\int_B ds\sigma_{ij}(\underline{r}, t)n_j(\underline{r})\dot{u}_i(\underline{r}, t) = \frac{1}{2}\frac{d}{dt}\int_B ds\sigma_{ij}(\underline{r}, t)n_j(\underline{r})u_i(\underline{r}, t) + 2T(\dot{b}(t)-\dot{a}(t)). \tag{4}$$

A centrally important observation is that (2) is an identity, in the absence of crack extension. In the presence of an extending crack, however, it becomes nontrivial, as we shall see later. The former observation implies that all terms will cancel except those that involve in some essential manner a derivative of the crack length.

Let us now adapt our notation more explicitly to the crack problem considered in the Appendix for which the stresses vanish at infinity. The boundary B consists of the two crack faces on which the tractions are assumed to be equal with opposite sign. Thus

$$\begin{aligned}\int_B & ds\sigma_{ij}(\underline{r}, t)n_j(\underline{r})\dot{u}_i(\underline{r}, t) \\ &= (-)\int_{a(t)}^{b(t)} dx\{\sigma_{12}(x, 0, t)\dot{\Delta}_1(x, t)+\sigma_{22}(x, 0, t)\dot{\Delta}_2(x, t)\} \\ &= (-)\,\mathrm{Im}\left[\int_{a(t)}^{b(t)} dx\overline{\Sigma}(x, t)\dot{\Delta}(x, t)\right],\end{aligned} \tag{5}$$

where

$$\Delta_i(x, t) = u_i(x, 0^+, t) - u_i(x, 0^-, t); \qquad \Sigma(x, t) = (\sigma_{22}-i\sigma_{12})(x, t), \tag{6}$$

and

$$\Delta(x, t) = (\Delta_1 + i\Delta_2)(x, t) \tag{7}$$

is given by (A26). We decompose $\Delta(x, t)$ into an instantaneous portion and a remainder, to obtain

$$\Delta(x, t) = \Delta_0(x, t) + \Delta_r(x, t) \tag{8}$$

where

$$\begin{aligned}\Delta_0(x, t) &= \frac{2ik_0}{\pi}\int_{a(t)}^{b(t)} dy'\Sigma(y', t)M(x, y'; t), \\ \Delta_r(x, t) &= \frac{2i}{\pi}\int_{t_1(x)}^{t} dt'k_1(t-t')\int_{a(t')}^{b(t')} dy'\Sigma(y', t')M(x, y'; t'),\end{aligned} \tag{9}$$

it being presumed that (see (A8) and (A17))

$$k(t) = k_0\delta(t) + k_1(t)H(t), \tag{10}$$

where $k_1(t)$ is a smooth function. The function $M(x, y'; t)$ is given by (A28). Then

$$\dot{\Delta}(x, t) = \dot{\Delta}_0(x, t) + \dot{\Delta}_r(x, t), \tag{11}$$

where, by virtue of (A29),

$$\dot{\Delta}_0(x, t) = \frac{2ik_0}{\pi}\int_{a(t)}^{b(t)} dy'\{\dot{\Sigma}(y', t)M(x, y'; t) + \Sigma(y', t)\dot{M}(x, y'; t)\}, \tag{12a}$$

$$\begin{aligned}\dot{\Delta}_r(x, t) = {} & \frac{2ik_1(0)}{\pi}\int_{a(t)}^{b(t)} dy'\Sigma(y', t)M(x, y'; t) \\ & + \frac{2i}{\pi}\int_{t_1(x)}^{t} dt'\dot{k}_1(t-t')\int_{a(t')}^{b(t')} dy'\Sigma(y', t')M(x, y'; t').\end{aligned} \tag{12b}$$

We now consider the first term in the right-hand side of (2) and rewrite it as

$$\int_V dv\sigma_{ij}(\underline{r}, t)\frac{\partial}{\partial x_j}\dot{u}_i(\underline{r}, t). \tag{13}$$

In normal circumstances, and in particular for a crack that is not expanding, Green's theorem, and the fact that $\sigma_{ij,j}$ vanishes, gives that this is equal to the boundary term on the left of (2). However the integral contains divergent terms, as a result of crack extension, so that we must proceed with caution. The displacements off the crack can be written down from (A13 (iii)) and (A14) by transferring the hereditary integral to the right-hand side of (A13 (iii)). The explicit expression for $\phi(z, t)$ is given by (A20). All that is essential for our purposes is that it possess square root singularities at the crack ends. Similarly, the stresses at points off the crack face may be evaluated with the aid of (A13 (i), (ii)). These also have square root singularities at the crack tips. Let us write at a general point

$$u_i(\underline{r}, t) = u_{i0}(\underline{r}, t) + u_{ir}(\underline{r}, t) \tag{14}$$

where $u_{i0}(\underline{r}, t)$ is the instantaneous portion of the hereditary integral that gives the displacement and $u_{ir}(\underline{r}, t)$ is the remainder, consisting of an integral with a smooth kernel over the history of the real and imaginary parts of the right-hand side of (A13 (iii)). Now the quantity $(\partial/\partial x_j)\dot{u}_{ir}(\underline{r}, t)$ will possess at most a square root singularity. This is so because the time derivative will act only on the kernel as in (12b). This singularity, combined with that in the stresses gives rise to a linear singularity which is integrable since the integral in (11) is two-dimensional. Thus, Green's theorem can be applied and what is obtained is the noninstantaneous portion of (5). The noninstantaneous terms in fact cancel out of the equation. This observation is the crucial one, for our purposes. It means that specifically viscoelastic effects do not contribute. The above argument does not go through for the instantaneous terms because of the presence of higher-order singularities. For the same reason Eq. (2) is not suitable for dealing with these terms. However, the instantaneous response of a viscoelastic material is elastic (see Gurtin and Sternberg [32]). Therefore we can use (4), instead of (2), in deriving the conditions for fracture.

Recalling the manipulations that led up to (5) we find that for the crack geometry of this paper

$$S(t) = -\frac{1}{2}\,\mathrm{Im}\int_{a(t)}^{b(t)} dx\,\overline{\Sigma}(x, t)\Delta_0(x, t). \tag{15}$$

Remembering that the noninstantaneous portion of (5) has already been cancelled we finally write (4) in the form

$$\begin{aligned}(-)\,\mathrm{Im}&\left\{\frac{2ik_0}{\pi}\int_{a(t)}^{b(t)} dx\,\overline{\Sigma}(x, t)\int_{a(t)}^{b(t)} dy'[\dot{\Sigma}(y', t)M(x, y'; t) + \Sigma(y', t)\dot{M}(x, y'; t)]\right\}\\ &= (-)\frac{d}{dt}\,\mathrm{Im}\left\{\frac{ik_0}{\pi}\int_{a(t)}^{b(t)} dx\,\overline{\Sigma}(x, t)\int_{a(t)}^{b(t)} dy'\Sigma(y', t)M(x, y'; t)\right\}\\ &\quad + 2T(\dot{b}(t) - \dot{a}(t))\end{aligned} \tag{16}$$

with aid of (A26). Using the symmetry of M (see A28) together with (A29), (A30) it is found that the terms not involving $\dot{a}(t)$, $\dot{b}(t)$ cancel and we are left with

$$\frac{\pi k_0}{2}\{(K_1^2(b) + K_2^2(b))\dot{b}(t) - (K_1^2(a) + K_2^2(a))\dot{a}(t)\} = 2T(\dot{b}(t) - \dot{a}(t)), \tag{17}$$

in terms of the stress intensity factors K_i, $i = 1, 2$ defined by (A32), (A33). We conclude that the conditions for crack growth at the respective crack tips are

$$\pi k_0(K_1^2(b) + K_2^2(b)) = 4T; \qquad \pi k_0(K_1^2(a) + K_2^2(a)) = 4T. \tag{18}$$

If the middle point of the crack face is chosen as the origin and $\Sigma(x, t)$ is even in x, these conditions reduce to the single condition

$$\pi k_0(K_1^2(c) + K_2^2(c)) = 4T \tag{19}$$

where $c(t) = b(t) = -a(t)$ and K_1, K_2 are given in (A36). In particular if $\Sigma = \Sigma(t)$ is independent of x it follows from (A40) that (19) becomes

$$\pi k_0|\Sigma(t)|^2 c(t) = 4T \tag{20}$$

which, in the case of purely normal stresses $\Sigma(t) = -p(t)$ is

$$\pi k_0 p^2(t)c(t) = 4T. \tag{21}$$

Equation (20) may more simply be derived by using Eqs. (A38)–(A41). Equation (21) has been obtained by Graham [10, 11] and Nuismer [13] using different approaches.

Strictly (1) should be an inequality stating that the left-hand side is greater than or equal to the right, in which case conditions (18)–(21) become inequalities. These conditions have the same form as the Griffith criterion for crack extensions for an elastic body with k_0, which is an instantaneous inverse modulus, replacing the elastic inverse modulus defined by (see (A17))

$$k_0 = \frac{1 - \nu}{\mu_0}$$

for a material with a unique Poisson's ratio ν, where μ_0 is the instantaneous shear modulus.

Note that in (16) the first term on the right is half the corresponding term on the left. This is a manifestation of the theorem of Goodier [29] for elastic bodies mentioned at the beginning of the section.

It may be confirmed that superposition of a prescribed time-dependent stress field at infinity does not alter the form of conditions (18). Also, these results may be extended to other crack geometries, for example, that of a growing penny-shaped crack in an infinite viscoelastic body (see also Graham [10]).

What has been shown here is that energy considerations for a viscoelastic medium in the noninertial approximation give no more than a Griffith instability criterion similar to that for an elastic medium. One cannot, therefore, hope to obtain a condition determining crack velocity from a noninertial energy equation, if the surface energy T is constant.

Appendix. In this appendix we consider the problem of a single straight line crack occupying the interval $[a(t), b(t)]$ of the x-axis in an infinite linear viscoelastic body that is in a state of plane strain in the (x, y) cartesian coordinate plane. Rehealing of crack faces is not considered so that $a(t)$ and $b(t)$ are constant or monotone decreasing and increasing with time, respectively. A position and time-dependent distribution of normal and shear stress is considered to act on the surface of the crack, while all the stresses are assumed to vanish at infinity. It is easy to verify that the problem arising when a time-dependent stress field $\{\sigma_{11}, \sigma_{22}, \sigma_{12}\}(t)$ is applied at infinity may be reduced to the above problem by superposition of solutions.

If we write

$$\Sigma(\underline{r}, t) = (\sigma_{22} - i\sigma_{12})(\underline{r}, t); \qquad D(\underline{r}, t) = (u_1 + iu_2)(\underline{r}, t); \tag{A1}$$

then the boundary conditions on the crack surface may be written in the form

$$\Sigma(x, t) = (\sigma_{22} - i\sigma_{12})(x, t) = (-p + is)(x, t), \qquad a(t) \le x \le b(t). \tag{A2}$$

If, at time t, the crack is open at the point x then $p(x, t)$ and $s(x, t)$ must be prescribed and

$$u_2(x^+, t) - u_2(x^-, t) > 0. \tag{A3}$$

On the other hand if the crack is closed at x at time t then

$$u_2(x^+, t) - u_2(x^-, t) = 0 \quad \text{and} \quad p(x, t) > 0. \tag{A4}$$

By resorting to the extended Correspondence Principle (see Graham and Sabin [33] and the Kolosov-Mushkelishvili equations (see Green and Zerna [34]) we find that a solution to the field equations of the quasistatic theory of viscoelasticity, for a body containing a crack that may grow with time, may be written as follows $(z = x + iy)$

$$\begin{aligned}
(\sigma_{11} + \sigma_{22})(\underline{r}, t) &= 2[\phi(z, t) + \overline{\phi}(\bar{z}, t)] = 4\,\mathrm{Re}[\phi(z, t)], \\
\Sigma(\underline{r}, t) = (\sigma_{22} - i\sigma_{12})(\underline{r}, t) &= \phi(z, t) + \overline{\phi}(\bar{z}, t) + z\overline{\phi}'(\bar{z}, t) + \overline{\psi}(\bar{z}, t), \\
2\int_{-\infty}^{t} dt'\mu(t - t')D'(\underline{r}, t') &= \int_{-\infty}^{t} dt'\kappa(t - t')\phi(z, t') \\
&\quad - \overline{\phi}(\bar{z}, t) - z\overline{\phi'}(\bar{z}, t) - \overline{\psi}(\bar{z}, t),
\end{aligned} \tag{A5}$$

where

$$D'(\underline{r}, t) = \frac{\partial}{\partial x}(u_1 + iu_2)(\underline{r}, t) \tag{A6}$$

and $\phi(z, t)$, $\psi(z, t)$ are analytic complex functions at all points outside $[a(t), b(t)]$. For the problem in hand, since the stresses and rotations vanish at infinity and the resultant of all the forces acting on the crack face cancel to zero, it may be assumed that (see Green and Zerna [34])

$$\phi(z, t) \sim O\left(\frac{1}{z^2}\right), \qquad \psi(z, t) \sim O\left(\frac{1}{z^2}\right) \tag{A7}$$

at large z.

The functions $\mu(t)$ and $\kappa(t)$ are both zero for negative time t and are related to the relaxation functions of the material for shear and dilation $G_1(t)$, $G_2(t)$ (see Golden and Graham [26]) in the following manner

$$\begin{aligned} 2\mu(t) &= \frac{d}{dt}(H(t)G_1(t)) = \delta(t)G_1(0) + H(t)\dot{G}_1(t); \\ 3\lambda(t) &= \delta(t)(G_2(0) - G_1(0) + H(t)(\dot{G}_2(t) - \dot{G}_1(t)); \\ \hat{\nu}(\omega) &= \frac{\hat{\lambda}(\omega)}{2[\hat{\lambda}(\omega) + \hat{\mu}(\omega)]}; \qquad \hat{\kappa}(\omega) = 3 - 4\hat{\nu}(\omega); \end{aligned} \tag{A8}$$

where, for example,

$$\hat{\mu}(\omega) = \int_{-\infty}^{\infty} dt e^{-i\omega t}\mu(t) \tag{A9}$$

is the Fourier transform of μ; $\delta(t)$ refers to the Dirac delta function and $H(t)$ is the Heaviside unit step function. The functions $\mu(t)$ and $\lambda(t)$ are clearly zero for $t < 0$.

Since, for the problem under consideration, the stresses are continuous at every point on the x-axis we find from (A5 (ii)) that

$$\begin{aligned} &\phi^+(x, t) + \overline{\phi^-}(x, t) + x\overline{\phi^{-\prime}}(x, t) + \overline{\psi^-}(x, t) \\ &\quad = \phi^-(x, t) + \overline{\phi^+}(x, t) + x\overline{\phi^{+\prime}}(x, t) + \overline{\psi^+}(x, t) \end{aligned} \tag{A10}$$

where $\phi^{\pm}(x, t)$ are the limits of $\phi(z, t)$ from above and below the real axis. Write (A10) as

$$\begin{aligned} &\phi^+(x, t) - \overline{\phi^+}(x, t) - x\overline{\phi^{+\prime}}(x, t) - \overline{\psi^+}(x, t) \\ &\quad = \phi^-(x, t) - \overline{\phi^-}(x, t) - x\overline{\phi^{-\prime}}(x, t) - \overline{\psi^-}(x, t) \end{aligned} \tag{A11}$$

and it follows that the function $\{\phi(z, t) - \overline{\phi}(z, t) - z\overline{\phi'}(z, t) - \overline{\psi}(z, t)\}$ is analytic everywhere. Since, by (A7), this function is zero at infinity it must, by Liouville's theorem (see Ahlfors [35]), be zero everywhere. It follows that

$$\overline{\psi}(\bar{z}, t) = \phi(\bar{z}, t) - \overline{\phi}(\bar{z}, t) - \bar{z}\overline{\phi'}(\bar{z}, t) \tag{A12}$$

and we write (A5) as

$$\begin{aligned}
(\sigma_{11}+\sigma_{22})(\underline{r}, t) &= 2(\overline{\phi}(\bar{z}, t)+\phi(z, t)) = 4\operatorname{Re}[\phi(z, t)], \\
\Sigma(\underline{r}, t) = (\sigma_{22}-i\sigma_{12})(\underline{r}, t) &= \phi(z, t)+\phi(\bar{z}, t)+(z-\bar{z})\overline{\phi'}(\bar{z}, t), \\
2d(\underline{r}, t) &= \int_{-\infty}^{t} dt'\kappa(t-t')\phi(z, t') - \phi(\bar{z}, t) - (z-\bar{z})\overline{\phi'}(\bar{z}, t),
\end{aligned} \tag{A13}$$

where

$$d(\underline{r}, t) = \int_{-\infty}^{t} dt'\mu(t-t')D'(\underline{r}, t'). \tag{A14}$$

Approaching the x-axis from above and below in (A13 (iii)), and subtracting gives

$$2(d(x, 0^+, t) - d(x, 0^-, t)) = \int_{-\infty}^{t} dt'(\delta+\kappa)(t-t')[\phi^+(x, t') - \phi^-(x, t')] \tag{A15}$$

or

$$\frac{1}{2}\int_{-\infty}^{t} dt' l(t-t')\Delta'(x, t') = \phi^+(x, t) - \phi^-(x, t), \tag{A16}$$

where, by virtue of the Faltung theorem (see Sneddon [36]), l is defined by the fact that its Fourier transform is given by

$$\hat{l}(\omega) = \frac{4\hat{\mu}(\omega)}{1+\hat{\kappa}(\omega)} = \frac{\hat{\mu}(\omega)}{1-\hat{\nu}(\omega)} = \frac{1}{\hat{k}(\omega)}. \tag{A17}$$

The quantity $k(t)$, defined by (A17), is used extensively later and in the main body of the paper. Also

$$\Delta'(x, t) = D'(x, 0^+, t) - D'(x, 0^-, t). \tag{A18}$$

If the left-hand side of (16) is zero at a given x then $\phi(z, t)$ is continuous at that point. This will be true if $\Delta'(x, t')$ is zero for all $t' \le t$ and in particular at points on the x-axis outside $[a(t), b(t)]$. From (A13 (ii)) we have

$$\phi^+(x, t) + \phi^-(x, t) = \Sigma(x, t) = (-p+is)(x, t), \qquad a(t) \le x \le b(t). \tag{A19}$$

This equation and the continuity of $\phi(x, t)$ across the x-axis outside of the crack specify a Hilbert problem whose solution is given by Muskhelishvili [37]

$$\begin{aligned}
\phi(z, t) &= \frac{X(z, t)}{2\pi i}\int_{a(t)}^{b(t)} dx' \frac{\Sigma(x', t)}{X^+(x', t)(x'-z)}, \\
X(z, t) &= \{(z-a(t))(z-b(t))\}^{-1/2},
\end{aligned} \tag{A20}$$

where it is appropriate to choose the branch of $\{(z-a(t)(z-b(t)\}^{-1/2}$ which is such that $zX(z, t) \to 1$ as $|z| \to \infty$.

On the real axis, for $x \in [a(t), b(t)]$

$$\begin{aligned}
X^+(x, t) &= -X^-(x, t) = 1/[im(x, t)], \\
m(x, t) &= \{(x-a(t))(b(t)-x)\}^{1/2},
\end{aligned} \tag{A21}$$

and for $x \notin [a(t), b(t)]$

$$\begin{aligned}
X(x, t) &= \pm 1/n(x, t), \\
n(x, t) &= |(x-a(t))(x-b(t))|^{1/2}.
\end{aligned} \tag{A22}$$

It is positive for $x > b(t)$ and negative for $x < a(t)$, as a consequence of choosing the branch of $X(z, t)$ as specified above. We write

$$\phi(z, t) = \frac{X(z, t)}{2\pi} \int_{a(t)}^{b(t)} dx' \frac{\Sigma(x', t) m(x', t)}{x' - z}. \tag{A23}$$

Note that it is singular at the end points of the cracks which implies that the stresses will also be singular there.

Consider now the expression for the derivative of the gap. From (A21), (A23) and the Plemelj formula (see Muskhelishvili [37]) we find that

$$\phi^{+}(x, t) - \phi^{-}(x, t) = \frac{1}{\pi i m(x, t)} \int_{a(t)}^{b(t)} dx' \frac{\Sigma(x', t) m(x', t)}{x' - x} \tag{A24}$$

which is a principal value integral. Using (A16), we obtain

$$\Delta'(x, t) = \frac{2}{\pi i} \int_{t_1(x)}^{t} dt' k(t - t') \frac{1}{m(x, t')} \int_{a(t')}^{b(t')} dx' \frac{\Sigma(x', t') m(x', t')}{x' - x}, \tag{A25}$$

where $k(t)$ is defined by (A17) and $t_1(x)$ is the time that x crosses the crack tip. Integrating (A25) from either crack tip and making a change in integration orders we find, using the fact that the Hilbert transform of $1/m(x, t)$ is zero on $[a(t), b(t)]$ (see Erdelyi [38]), that

$$\Delta(x, t) = \frac{2i}{\pi} \int_{t_1(x)}^{t} dt' k(t - t') \int_{a(t')}^{b(t')} dy' \Sigma(y', t') M(x, y'; t'), \tag{A26}$$

where

$$M(x, y'; t') = m(y', t') \int_{x}^{b(t')} \frac{dy}{(y' - y) m(y, t')}. \tag{A27}$$

The integral in (A27) may be evaluated to yield

$$\begin{aligned} M(x, y'; t') &= \log_e \left| \frac{\sqrt{y' - a(t')}\sqrt{b(t') - x} - \sqrt{b(t') - y'}\sqrt{x - a(t')}}{\sqrt{y' - a(t')}\sqrt{b(t') - x} + \sqrt{b(t') - y'}\sqrt{x - a(t')}} \right| \\ &= M(y', x; t'). \end{aligned} \tag{A28}$$

Equations (A27), (A28) imply that

$$\begin{aligned} M(a(t'), x; t') = M(b(t'), x; t') = M(x, a(t'); t') &= M(x, b(t'); t') \\ &= 0. \end{aligned} \tag{A29}$$

Also, by differentiating (A28) we find that

$$\dot{M}(x, y'; t) = \frac{1}{(b - a)} \left\{ \dot{a} \sqrt{\frac{b - x}{x - a}} \sqrt{\frac{b - y'}{y' - a}} - \dot{b} \sqrt{\frac{x - a}{b - x}} \sqrt{\frac{y' - a}{b - y'}} \right\}. \tag{A30}$$

Equations (A29), (A30) are used in the main part of the paper.

The complex stress $\Sigma(\underline{r}, t)$ off the crack surface is given by (A13). On the real axis, this has the form $\phi^+(x, t) + \phi^-(x, t)$. From (A22), (A23), and the Plemelj formula (see Muskhelishvili [37]) we obtain

$$\Sigma(x, t) = \pm\frac{1}{\pi n(x, t)}\int_{a(t)}^{b(t)} dx'\frac{\Sigma(x', t)m(x', t)}{x' - x}, \qquad x \notin [a(t), b(t)], \tag{A31}$$

the upper sign referring to $x > b(t)$. This is the same as in elastic theory. Note that it is independent of material parameters. It is easily checked that this is true everywhere, which, it will be perceived, is a consequence of Michell's theorem [39]. The complex stress intensities are given by

$$\begin{aligned} K_1(b) - iK_2(b) &= \mathrm{Lim}_{x\to b}\{[2(x-b)]^{1/2}\Sigma(x, t)\} \\ &= (-)\frac{1}{\pi\sqrt{c(t)}}\int_{a(t)}^{b(t)} dx'\Sigma(x', t)\left\{\frac{x' - a(t)}{b(t) - x'}\right\}^{1/2} \end{aligned} \tag{A32}$$

and

$$K_1(a) - iK_2(a) = (-)\frac{1}{\pi\sqrt{c(t)}}\int_{a(t)}^{b(t)} dx'\Sigma(x', t)\left\{\frac{b(t) - x'}{x' - a(t)}\right\}^{1/2} \tag{A33}$$

where

$$c(t) = \frac{b(t) - a(t)}{2}. \tag{A34}$$

It is easy to verify that if the crack is growing at $x = a(t)$ at time t then Eq. (A25) may be integrated to give

$$\Delta_2(x, t) \approx_{x\to a(t)} \frac{2k_0K_1(a(t))}{\sqrt{c(t)}}m(x, t). \tag{A35}$$

Therefore, in view of (A3), (A4), we may conclude that in these circumstances $K_1(a(t)) \geq 0$, depending on whether the crack is open or closed. It is noteworthy that if the crack is stationary for a period before and including t then the right-hand side of (A35) is replaced by an hereditary integral, with the consequence that $K_1(a(t))$ may take negative values. The remarks of this paragraph, which apply equally well at $x = b(t)$, are due to Graham and Sabin [40, 41] who derive them for the case of position-independent normal stress, by means of more explicit methods.

It is interesting to note the fundamental qualitative differences between extending and stationary cracks. These are traceable to the fact that the dominant singular term for an extending crack comes from the delta function part of the hereditary integral, while this is not so for a stationary crack. Note that this "instantaneous" property of singular terms, in the case of extending cracks leads to properties similar to those found in the elastic case, while stationary viscoelastic cracks behave quite differently to the elastic case. In the main part of this paper, dealing with propagation criteria, the interesting similarity between elastic and extending viscoelastic cracks is further manifested.

If the middle point of the crack face is chosen as the origin and $\Sigma(x, t)$ is even in x over the crack face then (A32), (A33) give the same result, namely, choosing

$b(t) = -a(t) = c(t)$

$$K_1(c) - iK_2(c) = (-)\frac{\sqrt{c(t)}}{\pi}\int_{-c(t)}^{c(t)} dx' \frac{\Sigma(x', t)}{(c^2(t) - x'^2)^{1/2}}. \tag{A36}$$

The most interesting special case is where $\Sigma(x, t)$ is independent of x on the crack face. We write it as $\Sigma(t)$. Then, (A25) gives, with the aid of standard integral (see Erdelyi [38])

$$\Delta'(x, t) = -2i\int_{t_1(x)}^{t} dt' \frac{k(t-t')}{m(x, t')}\left\{\frac{b(t') + a(t')}{2} - x\right\}\Sigma(t'), \tag{A37}$$

which can be integrated to give

$$\Delta(x, t) = -2i\int_{t_1(x)}^{t} dt' k(t-t')\Sigma(t')m(x, t'). \tag{A38}$$

The complex stress $\Sigma(x, t)$, given by (A31), becomes

$$\Sigma(x, t) = \Sigma(t)\left\{1 \pm \frac{(b(t) + a(t))/2 - x}{n(x, t)}\right\}, \qquad x \notin [a(t), b(t)] \tag{A39}$$

and the complex stress intensities are then given by

$$K_1(b) - iK_2(b) = K_1(a) - iK_2(a) = -c^{1/2}(t)\Sigma(t). \tag{A40}$$

In particular, for purely normal stresses acting on the crack face, $\Sigma(t) = -p(t)$ and

$$K_1(a) = K_1(b) = c^{1/2}(t)p(t). \tag{A41}$$

The results presented in this appendix were first given in a special case by Kachanov [42] and more generally by Kaminskii and Rushchitskii [43]; and by Graham [10, 11] using an alternative form of the elastic results, namely that described by Sneddon and Lowengrub [44].

Acknowledgment. This research was supported by funds provided by the Natural Sciences and Engineering Research Council of Canada. The paper was improved by the incorporation of reviewers' suggestions.

References

[1] A. A. Griffith, *The phenomenon of rupture and flow in solids*, Phil. Trans. Roy. Soc. A **221**, 163–198 (1921)

[2] A. A. Griffith, *The theory of rupture*, Proc. 1st Intern. Congr. Appl. Mech., Delft, 1924, p. 55–63

[3] P. I. Vincent and K. V. Gotham, *Effect of crack propagation velocity on the fracture energy of poly(methylmethacrylate)*, Nature **210**, 1254 (1966)

[4] E. C. Francis, C. H. Carlton, and G. H. Lindsey, *Viscoelastic fracture of solid propellants in pressurization loading conditions*, J. Spacecraft **11**, 691–696 (1974)

[5] W. G. Knauss, *Fracture of solids possessing deformation rate sensitive material properties*, The Mechanics of Fracture, AMD-Vol. 19, F. Erdogan (Editor), The American Society of Mechanical Engineers, New York, 1976, p. 69–103

[6] M. E. Gurtin, *Thermodynamics and the cohesive zone in fracture*, J. Appl. Math. Phys. (ZAMP) **30**, 991–1003 (1979)

[7] J. R. Willis, *Crack propagation is viscoelastic media*, J. Mech. Phys. Solids **15**, 229–240 (1967)

[8] B. V. Kostrov and L. V. Nikitin, *Some general problems of mechanics of brittle fracture*, Arch. Mech. Stosow. **6**, 749–776 (1970)

[9] W. S. Blackburn, *Steady crack growth in a linear viscoelastic material*, Int. J. Fract. Mech. **7**, 354–356 (1971)

[10] G. A. C. Graham, *Two extending crack problems in linear viscoelasticity*, Quart. Appl. Math. **27**, 497–507 (1970)

[11] G. A. C. Graham, *Quasi-static crack growth in linear viscoelastic bodies that are acted upon by alternating tensile and compressive loads*, Proc. Roy. Irish Acad. **75A**, 263–269 (1975)

[12] G. P. Cherepanov, *Crack propagation in continuous media*, J. Appl. Math. Mech. **31**, 503–512 (1967)

[13] R. J. Nuismer, *On the governing equation for quasi-static crack growth in linearly viscoelastic materials*, J. Appl. Mech. **41**, 631–634 (1974); **42**, 521 (1975)

[14] R. M. Christensen and L. N. McCartney, *Viscoelastic crack growth*, Int. J. Fract. **23**, R11–R13 (1983)

[15] W. G. Knauss, *The mechanics of polymer fracture*, Appl. Mech. Rev. **26**, 1–17 (1973)

[16] L. N. McCartney, *Crack growth laws for a variety of viscoelastic solids using energy and COD fracture criteria*, Int. J. Fract. **15**, 31–40 (1979); **16**, R27–R30, R109–R110 (1980)

[17] R. A. Schapery, *On the analysis of crack initiation and growth in nonhomogeneous viscoelastic media*, Fracture Mechanics (R. Burridge, Ed.), American Mathematical Society, Providence, 1979, pp. 137–152

[18] R. A. Schapery, *Time-dependent fracture: continuum aspects of crack growth*, Encyclopedia of Materials Science and Engineering, Vol. 7, M. B. Bever (Editor-in-Chief), Pergamon Press, Oxford, 1986, pp. 5043–5053

[19] A. A. Kaminskii, *Investigations in the field of the mechanics of the fracture of viscoelastic bodies*, Soviet Applied Mechanics **16**, 741–759 (1980); *Mechanics of fracture of viscoelastic bodies* (in Russian), Naukova Dumka, Kiev, 1980

[20] A. A. Kaminskii and V. M. Pestrikov, *Kinetics of crack growth in aging viscoelastic bodies*, Soviet Applied Mechanics **22**, 790–797 (1986)

[21] O. Coussy, *Un modèle de viscoélasticité confinée en mécanique de la rupture. Propagation subcritique*, C. R. Acad. Sci. Paris Série II **302**, 53–56 (1986)

[22] L. Schovanec and J. R. Walton, *The energy release rate for a quasi-static mode I crack in a nonhomogeneous linearly viscoelastic body*, Engineering Fracture Mechanics **28**, 445–454 (1987)

[23] G. Goleniewski, *Dynamic crack growth in a viscoelastic material*, Int. J. Fract. **37**, R39–R44 (1988)

[24] J. M. Golden and G. A. C. Graham, *Fatigue crack growth in viscoelastic materials*, Arabian J. Sci. Engrg. **9**, 77–85 (1984)

[25] G. I. Barenblatt, *The mathematical theory of equilibrium cracks in brittle fracture*, Advances in Applied Mechanics, Vol. 7, H. L. Dryden and Th. von Karman (Editors), Academic Press, New York and London, 1962

[26] J. M. Golden and G. A. C. Graham, *Boundary volume problems in linear viscoelasticity*, Springer-Verlag, Berlin, Heidelberg, 1988

[27] F. Erdogan, *Crack-propagation theories*, Fracture, An Advanced Treatise, Vol. II, Mathematical Fundamentals, H. Liebowitz (Editor), Academic Press, New York, 1968, pp. 497–590

[28] L. N. McCartney, *Mechanics of matrix-cracking in brittle-matrix fibre-reinforced composites*, Proc. Roy. Soc. Lond. **A409**, 329–350 (1987)

[29] J. N. Goodier, *Mathematical theory of equilibrium cracks*, Fracture, An Advanced Treatise, Vol II, Mathematical Fundamentals, H. Liebowitz (Editor), Academic Press, New York, 1968, pp. 1–66

[30] R. M. Christensen, *Theory of Viscoelasticity: An Introduction*, Academic Press, New York, 1982

[31] I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, McGraw-Hill, New York, 1956

[32] M. E. Gurtin and E. Sternberg, *On the linear theory of viscoelasticity*, Arch. Rational Mech. Anal. **11**, 291–356 (1962)

[33] G. A. C. Graham and G. C. W. Sabin, *The correspondence principle of linear viscoelasticity for problems that involve time-dependent regions*, Internat. J. Engrg. Sci. **11**, 123–140 (1973)

[34] A. E. Green and W. Zerna, *Theoretical Elasticity*, Clarendon Press, Oxford, 1968

[35] L. V. Ahlfors, *Complex Analysis*, McGraw Hill, New York, 1966

[36] I. N. Sneddon, *Fourier Transforms*, McGraw-Hill, New York, 1951

[37] N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Groningen, 1963

[38] A. Erdelyi, *Tables of Integral Transforms*, Vol. II, McGraw Hill, New York, 1954

[39] J. H. Michell, *On the direct determination of stress in an elastic solid, with application to the theory of plates*, Proc. Lond. Math. Soc. **31**, 100–124 (1899)

[40] G. A. C. Graham and G. C. W. Sabin, *The opening and closing of a growing crack in a linear viscoelastic body that is subject to alternating tensile and compressive loads*, Int. J. Fract. **14**, 639–649 (1978)

[41] G. A. C. Graham and G. C. W. Sabin, *Steady state solutions for a cracked standard linear viscoelastic body*, Mech. Res. Commun. **8**, 361–368 (1981)

[42] L. M. Kachanov, *On the kinetics of crack growth*, J. Appl. Math. Mech. **25**, 739–745 (1961)

[43] A. A. Kaminskii and Ya. Ya. Rushchitskii, *Applicability of the Volterra principle in the analysis of crack propagation in hereditary elastic media*, Soviet Applied Mechanics **5**, 102–108 (1969)

[44] I. N. Sneddon and M. Lowengrub, *Crack Problems in the Classical Theory of Elasticity*, Wiley, New York, 1969