QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLVIII, NUMBER 2
JUNE 1990, PAGES 341-359

EXISTENCE, UNIQUENESS, AND DECAY ESTIMATES FOR SOLUTIONS IN THE NONLINEAR THEORY OF ELASTIC, EDGE-LOADED, CIRCULAR TUBES

By

C. O. HORGAN (*University of Virginia, Charlottesville, Virginia*)

L. E. PAYNE (*Cornell University, Ithaca, New York*)

AND

J. G. SIMMONDS (*University of Virginia, Charlottesville, Virginia*)

Abstract. Solutions to the two, coupled, ordinary nonlinear differential equations for a semi-infinite circular elastic tube subjected to edge loads and undergoing small axisymmetric strains, but arbitrarily large axisymmetric rotations—the simplified Reissner equations—are analyzed. First, with the aid of a Green's function, the differential equations and boundary conditions are transformed to a complex-valued integral equation. From this equation existence, uniqueness, boundedness, and rate of decay are extracted for a dimensionless stress function, f, and an angle of rotation, β, for sufficiently small edge data. It is shown that these unique solutions must decay at least as fast as the linear solution. Second, it is shown that any solution that decays to zero must, at a sufficiently large distance from the edge, decay at the linear rate. Third, rates of decay are established for any solution for which the L_2 or the sup norm of β has certain bounds. Finally, an energy (or Lyapunov) function E, defined on solutions of the differential equations, is constructed and under certain *a priori* restrictions on the angle of rotation, three different upper bounds on E are obtained. These also provide exponential decay estimates for solutions. The energy approach is examined with a view to more general shells where a Green's function may not be readily available.

1. Introduction. We shall be concerned with a homogeneous, elastically isotropic, semi-infinite circular tube (cylindrical shell) of constant thickness h and midsurface radius R under axisymmetric edge loads. In classical, (first-approximation) theory, the *linear* field equations can be reduced to the following coupled system of second-order ordinary differential equations [1]:

$$f'' - \varepsilon^2 f + \beta = 0, \qquad \beta'' - \varepsilon^2 \beta - f = 0, \qquad 0 < x < \infty. \tag{1.1}$$

Here, f is a dimensionless stress function, β is the angle of rotation of a generator

Received June 26, 1989.

of the midsurface of the tube, εRx is distance along a generator from the end of the tube, and

$$\varepsilon^2 = \frac{h}{R\sqrt{12(1-\nu^2)}} \tag{1.2}$$

is a small parameter, where ν is Poisson's ratio.

The decaying solutions of (1.1) can be conveniently expressed in the complex form

$$f + i\beta \equiv \phi = \phi(0)e^{-px}, \qquad p = (i+\varepsilon^2)^{1/2} \equiv a + ib = \frac{1+i}{\sqrt{2}} + O(\varepsilon^2), \tag{1.3}$$

where i is the imaginary unit and $\phi(0)$ depends on the boundary conditions at $x = 0$. From (1.3), $|\phi(x)| = |\phi(0)|e^{-ax}$. Because $a = O(1)$ and $\varepsilon Rx = O(\sqrt{hR}x)$ is distance from the edge, the linear solution exhibits a (well-known) boundary layer whose width is the geometric mean of the thickness of the tube times its radius.

Our aim herein is to characterize, as far as possible, the analogous solution for a *nonlinear* version of (1.1), namely, the *simplified Reissner equations* [1,2],

$$f'' - \varepsilon^2 f + \sin\beta = 0, \qquad \beta'' - \varepsilon^2\sin\beta - f\cos\beta = 0, \qquad 0 < x < \infty, \tag{1.4}$$

which assume that, while the elastic strains are negligible compared to one (as in the linear theory), the rotations may be arbitrarily large. The classical boundary conditions at the end of the tube associated with (1.4) prescribe the (dimensionless) radial displacement or the horizontal stress resultant, i.e.,

$$f'(0) + \nu f(0)\sin\beta(0) \quad \text{or} \quad f(0), \tag{1.5}$$

and the moment or the rotation at the end of the tube, i.e.,

$$\beta'(0) - 2\nu\sin^2[\beta(0)/2] \quad \text{or} \quad \beta(0). \tag{1.6}$$

To concentrate on essentials, we shall assume that

$$\phi(0) = f(0) + i\beta(0) \quad \text{is prescribed} \tag{1.7}$$

and that

$$\int_0^\infty e^{-ax}|\phi(x)|dx < \infty \quad \text{and} \quad e^{-ax}|\phi'(x)| \to 0 \quad \text{as } x \to \infty. \tag{1.8}$$

Note that $(1.8)_1$ implies that

$$e^{-ax}|\phi(x)| \to 0 \quad \text{as } x \to \infty. \tag{1.9}$$

As we shall see in the next section, (1.8) are sufficient conditions for us to convert the differential equations (1.4) with edge conditions (1.7) into an integral equation.

We note that the nonlinear terms in (1.4) allow for nonunique solutions. For example, if the end loads are zero, a semi-infinite tube may be in the stress-free state, $f = \beta = 0$, or in an everted state in which $f \approx 0$ and $\beta \approx \pi$, except in a narrow edge zone where f and β change rapidly to meet the stress-free boundary conditions. We also note that (1.4), g4.1 (1.7), and (1.8) admit an even simpler, nondecaying *constant* solution, namely

$$f = \sqrt{1-\varepsilon^8}/\varepsilon^2, \qquad \beta = \pi/2 + \sin^{-1}\varepsilon^4. \tag{1.10}$$

(Of course, in a semi-infinite tube, this solution would violate the assumption of small strain on which the simplified Reissner equations are based.) Thus, to guarantee uniqueness and decay, it will not seem unreasonable if we impose either bounds on the end data or an *a priori* bound on β.

2. Existence and uniqueness. To prove existence and uniqueness, we first convert (1.4) and the associated boundary conditions to a complex-valued integral equation. We then show that we have a contraction for $|\phi(0)|$ sufficiently small. To this end, we first add $(1.4)_2$ multiplied by i to $(1.4)_1$ and move nonlinear terms to the right of the resulting equation to obtain

$$\mathscr{L}\phi = -iN(\phi), \tag{2.1}$$

where

$$\mathscr{L} = d^2/dx^2 - p^2 \tag{2.2}$$

and

$$N = 2f \sin^2 \beta/2 + p^2(\beta - \sin\beta), \tag{2.3}$$

with ϕ and p defined in (1.3).

The Green's function associated with the linear operator $\mathscr{L}$ and the boundary conditions

$$G(0, \xi) = 0, \qquad \lim_{x\to\infty} G(x, \xi) = 0 \tag{2.4}$$

is given by

$$pG(x, \xi) = \begin{cases} -e^{-p\xi}\,\mathrm{sh}\,px, & x < \xi \\ -e^{-px}\,\mathrm{sh}\,p\xi, & \xi < x \end{cases}. \tag{2.5}$$

Thus, (2.1), (2.4), and (2.5) together with the edge and decay conditions (1.7) and (1.8) imply that

$$\phi = \phi(0)e^{-px} - i\mathscr{G}\phi(x) \equiv \mathscr{N}\phi(x), \tag{2.6}$$

where

$$\mathscr{G}\phi = \int_0^\infty G(\xi, x)N(\phi(\xi))\,d\xi. \tag{2.7}$$

The existence of the integral on the right of (2.7) follows from $(1.8)_1$, (2.5), and Eq. (A.4) of Appendix A.

To establish existence and uniqueness, it is convenient to multiply both sides of (2.6) by e^{px}. We thus obtain a new integral equation of the form

$$e^{px}\phi(x) \equiv \theta = \phi(0) - i\mathscr{K}\theta(x) \equiv \mathscr{P}\theta(x), \tag{2.8}$$

where

$$\mathscr{K}\theta = \int_0^\infty K(\xi, x)P(\theta(\xi))\,d\xi, \tag{2.9}$$

$$pK(x, \xi) = e^{p\xi}pG(x, \xi)e^{-3px} = \begin{cases} -e^{-3px}\,\mathrm{sh}\,px, & x < \xi \\ -e^{p(\xi-4x)}\,\mathrm{sh}\,p\xi, & \xi < x \end{cases}, \tag{2.10}$$

and

$$P = e^{3px}N = e^{3px}[2f\sin^2\beta/2 + p^2(\beta - \sin\beta)]. \tag{2.11}$$

The analysis of the complex-valued integral equation (2.8) will be our main concern in the remainder of this section.

The set of all bounded, continuous, complex-valued functions on $[0, \infty)$, with norm

$$\|\cdot\| = \sup_{0 \le x < \infty} |\cdot|, \tag{2.12}$$

is a Banach space, B. If $\psi \in B$, then, from (1.3), (2.8)–(2.11), (A.5), and (A.14), we have

$$\begin{aligned} |\mathscr{P}\psi(x)| &\le |\phi(0)| + \int_0^\infty |K(\xi, x)||P(\psi(\xi))|\, d\xi \\ &\le |\phi(0)| + M\|\psi\|^3, \end{aligned} \tag{2.13}$$

where

$$M = \frac{\sqrt{9 + |p|^4}}{16a} = \frac{\sqrt{5}}{8} + O(\varepsilon^2). \tag{2.14}$$

Hence, $\mathscr{P}$ maps elements of B into bounded functions.

To show that $\mathscr{P}$ maps B into B, we note by (2.8), (2.9), (2.12), (A.5), and (A.15) that

$$\begin{aligned} |\mathscr{P}\psi(x) - \mathscr{P}\psi(y)| &\le \int_0^\infty |K(\xi, x) - K(\xi, y)||P(\psi(\xi))|\, d\xi \\ &\le (M/3)\|\psi\|^3 e^{-2ay}[3e^{2a(y-x)} - 2 - e^{-2a(y-x)} + |1 - e^{-2p(y-x)}|]. \end{aligned} \tag{2.15}$$

Because the right side of (2.15) approaches zero (uniformly) as $y \to x$, $\mathscr{P}\psi(x)$ is continuous on $[0, \infty)$.

Finally, to show that $\mathscr{P}$ is a contraction for sufficiently small values of $|\phi(0)|$, we consider the iteration

$$\theta_{n+1} = \mathscr{P}\theta_n(x), \qquad n = 0, 1, \dots, \tag{2.16}$$

where permissible values of θ_0 will be specified presently. It follows from (2.8), (2.9), (2.12), (A.13), and (A.14) that if

$$\|\theta_n\|^2 \le (1 - \delta)L_*^2, \qquad n = 0, 1, \dots, \tag{2.17}$$

where δ is any positive number and

$$L_*^2 = \frac{16a|p|}{\sqrt{7 + 2\sqrt{10}}\sqrt{9 + |p|^4}} = 0.980\cdots + O(\varepsilon^2), \tag{2.18}$$

then

$$\begin{aligned} |\mathscr{P}\theta_m(x) - \mathscr{P}\theta_n(x)| &\le \int_0^\infty |K(\xi, x)||P(\theta_m(\xi)) - P(\theta_n(\xi))|\, d\xi \\ &< (1 - \delta)\|\theta_m - \theta_n\|. \end{aligned} \tag{2.19}$$

Taking the supremum on the left, we have a contraction.

To show that we can guarantee (2.17), note from (2.8), (2.13), and (2.16) that

$$\|\theta_{n+1}\| \le |\phi(0)| + M\|\theta_n\|^3. \tag{2.20}$$

If $0 < |\phi(0)| \le 2/(3\sqrt{3M}) = 0.728\cdots + O(\varepsilon^2)$, Fig. 1 shows that the sequence of real numbers generated by the iteration

$$\Theta_{n+1} = |\phi(0)| + M\Theta_n^3, \; \Theta_0 = \|\theta_0\| < L_2 \tag{2.21}$$

has a positive limit L_1 satisfying

$$L_1(1 - ML_1^2) = |\phi(0)|, \tag{2.22}$$

where

$$0 < L_1 \le 1/\sqrt{3M} \le L_2 < 1/\sqrt{M}. \tag{2.23}$$

(See Fig. 2.) Fig. 1. also shows that if $0 \le \Theta_0 \le L_1$, then $\Theta_n \uparrow L_1$. Because

$$\|\theta_1\| \le |\phi(0)| + M\|\theta_0\|^3 = \Theta_1, \tag{2.24}$$

it follows by induction that $\|\theta_n\| \le \Theta_n \le L_1$. Choosing $|\phi(0)|$ so that $L_1 < L_*$, we

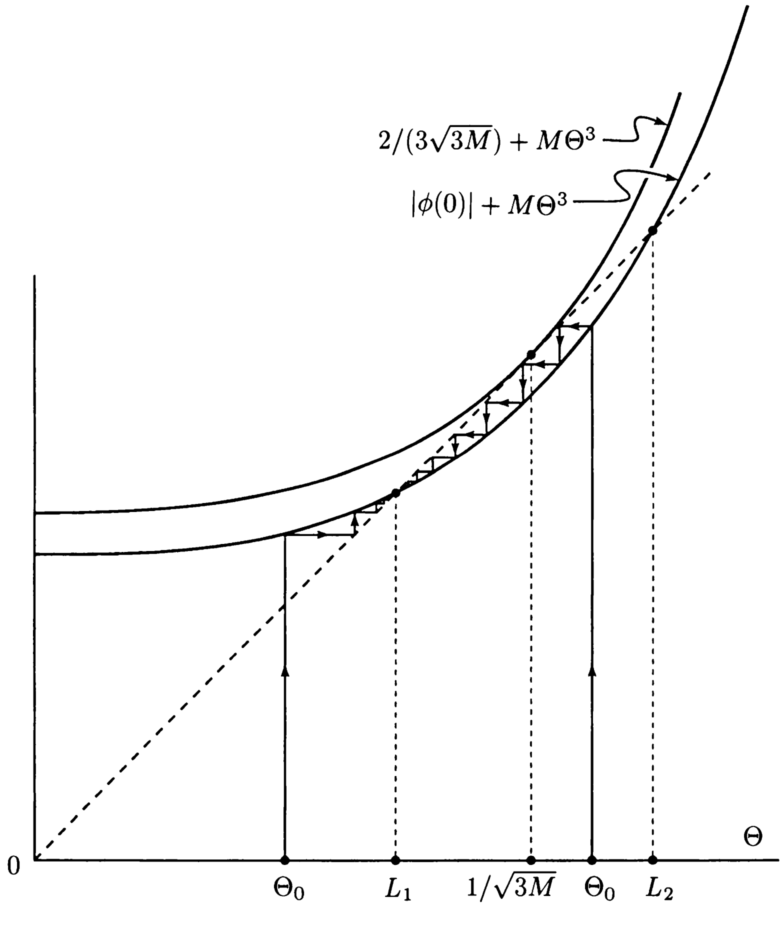

FIG. 1. The iteration $\Theta_{n+1} = |\phi(0)| + M\Theta_n^3$.

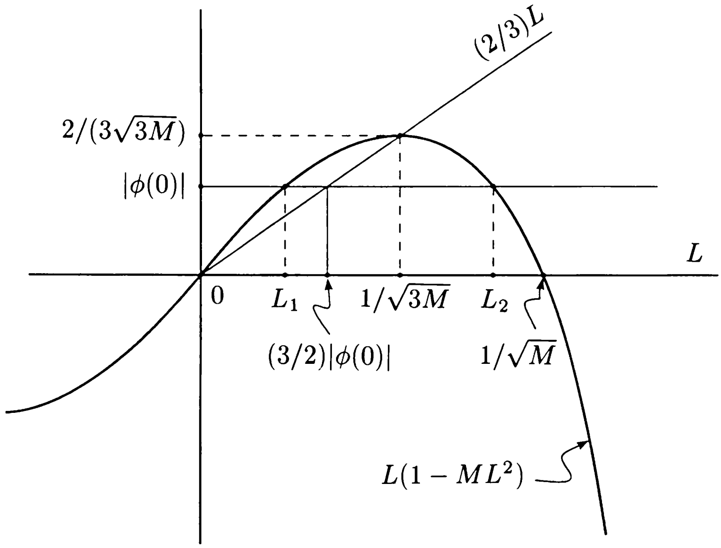

FIG. 2. Roots of and other constants associated with $L(1 - ML^2) = |\phi(0)|$.

arrive at the

THEOREM. If $\theta_0 = \phi(0)$ and

$$|\phi(0)| < L_*(1 - ML_*^2) = 0.718\cdots + O(\varepsilon^2), \tag{2.25}$$

where L_* is given in (2.18), then the integral equation (2.8) has the unique solution

$$\theta = \lim_{n\to\infty} \theta_n(x). \tag{2.26}$$

3. Exponential decay estimates. The integral equation approach just described provides existence and uniqueness of the differential equations (1.4), subject to the auxiliary conditions (1.7) and (1.8), if the edge data satisfy (2.25). The exponential decay estimate,

$$|\phi| = e^{-ax}|\theta(x)| < L_* e^{-ax}, \qquad 0 \le x < \infty, \tag{3.1}$$

then follows immediately from (2.8), (2.17), and (2.26), where a is given in (1.3). Comparing (3.1) with (1.3), we see that solutions to the nonlinear differential equations, (1.4), subject to the boundary conditions (1.7), and (1.8), and the restriction (2.25), *decay exponentially at least as fast* as do solutions of the linear differential equations (1.1), subject to the same boundary conditions.

In fact, we can say more: if $\phi(x)$ is *any* solution of the integral equation (2.6) that decays to zero, then as $x \to \infty$, *such a solution decays neither slower nor faster than the linear solution.* To see why, note that if $\phi(x)$ decays to zero, there must be an $x_0 > 0$ such that

$$|\phi(x_0)| < L_*(1 - ML_*^2). \tag{3.2}$$

Now let

$$\tilde{x} \equiv x - x_0, \qquad \tilde{\phi}(\tilde{x}) \equiv \phi(\tilde{x} + x_0). \tag{3.3}$$

Since the differential equations (1.4) are invariant under the change of variable, $(3.3)_1$, it follows that all of the preceding results concerning solutions of the integral equations (2.6) or (2.8) continue to hold with x, ϕ, and θ replaced by $\tilde{x}$, $\tilde{\phi}$, and $\tilde{\theta} \equiv e^{p\tilde{x}}\tilde{\phi}(\tilde{x})$, respectively. Thus, because (3.2) is the same condition as (2.25) except that $\phi(0)$ is replaced by $\tilde{\phi}(0)$, (3.1) holds in the form

$$|\tilde{\phi}| = e^{-a\tilde{x}}|\tilde{\theta}(\tilde{x})| < L_* e^{-a\tilde{x}}, \qquad 0 \le \tilde{x} < \infty, \tag{3.4}$$

and the first part of our assertion is established.

To establish the second part, we note that (2.8), (2.9), (2.12), (A.5), and (A.14) imply that

$$|\tilde{\theta}(\tilde{x})| \ge |\tilde{\phi}(0)| - M\|\tilde{\theta}\|^3. \tag{3.5}$$

Further, since $\tilde{\theta} = \lim_{n\to\infty} \tilde{\theta}_n$ and $\|\tilde{\theta}_n\| \le L_1$, we have, with the aid of (2.22),

$$|\tilde{\theta}(\tilde{x})| \ge |\tilde{\phi}(0)| - ML_1^3 = 2|\tilde{\phi}(0)| - L_1. \tag{3.6}$$

But Fig. 2 shows that $L_1 \le \frac{3}{2}|\tilde{\phi}(0)|$. Thus,

$$|\tilde{\phi}(\tilde{x})| = |\tilde{\theta}(\tilde{x})|e^{-a\tilde{x}} \ge \tfrac{1}{2}|\tilde{\phi}(0)|e^{-a\tilde{x}}, \qquad 0 \le \tilde{x} < \infty, \tag{3.7}$$

as claimed.

4. Exponential decay estimates with a priori bounds on norms of β. If the initial data violate (2.25), we can still deduce exponential decay estimates if we assume that a solution of the integral equation (2.6) exists and if we assume that *either* (1) the L_2-norm of β or (2) the sup norm of β is suitably bounded.

(1) *L_2-norm of β suitably bounded.* From (1.3), (2.5)–(2.8), and (A.4) we have

$$\begin{aligned}
|p||\theta| &\le |p|\left[|\phi(0)| + \left|e^{px}\int_0^\infty G(\xi, x)N(\phi(\xi))\,d\xi\right|\right] \\
&\le |p||\phi(0)| + \left|\int_0^x \operatorname{sh} p\xi N\,d\xi\right| + \left|e^{px}\operatorname{sh} px\int_x^\infty e^{-p\xi}N\,d\xi\right| \\
&\le |p||\phi(0)| + \tfrac{1}{6}\sqrt{9+|p|^4}\left(\int_0^x |\operatorname{sh} p\xi|\beta^2|\phi|d\xi + |e^{px}\operatorname{sh} px|\int_x^\infty e^{-a\xi}\beta^2|\phi|d\xi\right) \\
&\le |p||\phi(0)| + \tfrac{1}{12}\sqrt{9+|p|^4}\|\theta\|\left(\int_0^x |1-e^{-2p\xi}|\beta^2\,d\xi + |e^{2px}-1|\int_x^\infty e^{-2a\xi}\beta^2\,d\xi\right) \\
&\le |p||\phi(0)| + \tfrac{1}{6}\sqrt{9+|p|^4}\|\theta\|\int_0^\infty \beta^2\,d\xi.
\end{aligned} \tag{4.1}$$

Taking the supremum of the left side of this inequality, and recalling that $\|\theta\| \equiv \sup_{[0,\infty)}|\phi(x)|e^{ax}$, we conclude that if

$$\int_0^\infty \beta^2\,d\xi \le \frac{6|p|(1-\delta)}{\sqrt{9+|p|^4}} < 1.897\cdots + O(\varepsilon^2), \qquad \delta > 0, \tag{4.2}$$

then

$$|\phi(x)| \le \|\theta\|e^{-ax} \le (1/\delta)|\phi(0)|e^{-ax}, \qquad 0 \le x < \infty. \tag{4.3}$$

That is, no solution of the integral equation (2.6) can decay slower than the linear solution, so long as (4.2) holds.

(2) *Sup norm of β suitably bounded.* For some real number $\gamma \in (0, a)$, we multiply both sides of (2.6) by $e^{\gamma x}$, so that

$$\psi \equiv \phi e^{\gamma x} = \phi(0) e^{-(p-\gamma)x} - i e^{\gamma x} \mathcal{G} \phi(x). \tag{4.4}$$

Then, with the aid of (A.4),

$$\begin{aligned} |p||\psi(x)| &\le |p||\phi(0)| + \left| e^{-(p-\gamma)x} \int_0^x \operatorname{sh} p\xi N \, d\xi \right| + \left| e^{\gamma x} \operatorname{sh} px \int_x^\infty e^{-p\xi} N \, d\xi \right| \\ &\le |p||\phi(0)| + \tfrac{1}{6}\sqrt{9 + |p|^4} \|\beta\|^2 \|\psi\| \\ &\quad \times \left(|e^{-(p-\gamma)x}| \int_0^x e^{-\gamma\xi} |\operatorname{sh} p\xi| \, d\xi + e^{\gamma x} |\operatorname{sh} px| \int_x^\infty e^{-(a+\gamma)\xi} \, d\xi \right) \\ &\le |p||\phi(0)| + \tfrac{1}{12}\sqrt{9 + |p|^4} \|\beta\|^2 \|\psi\| \\ &\quad \times \left(\frac{1 - e^{-(a-\gamma)x}}{a - \gamma} + \frac{1 + e^{-(a-\gamma)x}}{a + \gamma} \right) \\ &\le |p||\phi(0)| + \frac{a\sqrt{9 + |p|^4}}{6(a^2 - \gamma^2)} \|\beta\|^2 \|\psi\|. \end{aligned} \tag{4.5}$$

Taking the supremum of the left side of this inequality, we conclude that if

$$\|\beta\|^2 \le \frac{6(1-\delta)|p|(a^2 - \gamma^2)}{a\sqrt{9 + |p|^4}} < 1.341 \cdots + O(\varepsilon^2), \qquad \delta > 0, \tag{4.6}$$

then

$$|\phi(x)| \le (1/\delta)|\phi(0)| e^{-\gamma x}, \qquad 0 \le x < \infty. \tag{4.7}$$

From (4.6) we see that as we try to make the upper bound on $\|\beta\|$ bigger by making δ or γ smaller, we either increase the amplitude or lower the rate of decay, respectively, on the right side of (4.7).

5. Decay estimates using an energy approach. Differential inequalities for "energy" (or Lyapunov) functions have been widely used to obtain exponential decay estimates for solutions of linear and nonlinear partial differential equations on semi-infinite strips and cylinders. Such estimates arise in the analysis of Saint-Venant's Principle in elasticity theory (see, e.g., [3,4] for a review) and in establishing theorems of Phragmén-Lindelöf type. In this section, we present three different energy arguments which lead to exponential decay estimates for solutions of (1.4) satisfying the edge data (1.7). Our assumptions on asymptotic behavior as $x \to \infty$ will be stronger than (1.8). The results obtained are different from (3.1) both in structure and in the magnitude of the estimated decay rate, and are established under different *a priori* assumptions. We pursue the energy approach because we think it holds promise for the analysis of more general shells where an integral equation formulation via Green's functions may be difficult.

We begin by constructing the energy function[1]

$$E(x) = f^2 + g(\beta), \qquad 0 \le x < \infty \tag{5.1}$$

on the solutions of the coupled system of ordinary differential equations (1.4). Here, g is, as yet, an unknown function. To guarantee that E vanishes when both f and β vanish, we shall require that $g(0) = 0$.

From (5.1), we find that

$$E' = 2ff' + (dg/d\beta)\beta'. \tag{5.2}$$

A further differentiation of (5.1) together with (1.4) yields

$$\begin{aligned} E'' = 2(f')^2 + (d^2g/d\beta^2)(\beta')^2 + 2\varepsilon^2 f^2 + \varepsilon^2(dg/d\beta)\sin\beta \\ + f[(dg/d\beta)\cos\beta - 2\sin\beta]. \end{aligned} \tag{5.3}$$

Since the last line in (5.3) is of indefinite sign, we choose g so that the term in brackets vanishes. That is, we set

$$dg/d\beta = 2\tan\beta, \tag{5.4}$$

so that, since $g(0) = 0$, we have

$$g = \ln\sec^2\beta. \tag{5.5}$$

Thus, (5.1)–(5.3) reduce to

$$E = f^2 + \ln\sec^2\beta \tag{5.6}$$

$$E' = 2ff' + 2\tan\beta\,\beta' \tag{5.7}$$

$$E'' = 2[(f')^2 + (\beta')^2\sec^2\beta + \varepsilon^2(f^2 + \sin\beta\tan\beta)], \tag{5.8}$$

for $0 \le x < \infty$.

Result 1. We establish our first exponential decay estimate for $E(x)$ under the *hypothesis* that

$$-\pi/2 < \beta(x) < \pi/2, \qquad 0 \le x < \infty. \tag{5.9}$$

This restriction is motivated by the existence of the nondecaying solution (1.10). If (5.9) holds, it follows that

$$\sin\beta\tan\beta \ge \ln\sec^2\beta \ge \sin^2\beta, \qquad 0 \le x < \infty. \tag{5.10}$$

Using (5.6)–(5.8), we find that

$$\begin{aligned} EE'' - \tfrac{1}{2}(E')^2 = 2\varepsilon^2(f^2 + \ln\sec^2\beta)(f^2 + \sin\beta\tan\beta) \\ + 2\{(f^2 + \ln\sec^2\beta)[(f')^2 + (\beta')^2\sec^2\beta] - (ff' + \beta'\tan\beta)^2\}. \end{aligned} \tag{5.11}$$

Employing both inequalities of (5.10) on the right in (5.11), we obtain

$$\begin{aligned} EE'' - \tfrac{1}{2}(E')^2 &\ge 2\{\varepsilon^2E^2 + (f^2 + \sin^2\beta)[(f')^2 + (\beta')^2\sec^2\beta] - (ff' + \beta'\tan\beta)^2\} \\ &\ge 2\varepsilon^2E^2, \end{aligned} \tag{5.12}$$

[1]The *strain energy density* for the elastic tube associated with the simplified Reissner equations is proportional to $(f')^2 - 2\varepsilon\nu f' f\sin\beta + \varepsilon^2 f^2\sin^2\beta + (\beta')^2 - 4\nu\varepsilon\beta'\sin^2\beta/2 + 4\varepsilon^2\sin^4\beta/2$.

where Cauchy's inequality has been used to obtain the last line. The second-order differential inequality (5.12) can be written as

$$(E^{1/2})'' \geq \varepsilon^2 E^{1/2}, \qquad 0 \leq x < \infty, \tag{5.13}$$

and integrated once to yield the first-order differential inequality

$$(E^{1/2})' + \varepsilon E^{1/2} \leq 0, \qquad 0 \leq x < \infty, \tag{5.14}$$

if

$$\lim_{x \to \infty} e^{-\varepsilon x}(E'/2E^{1/2} + \varepsilon E^{1/2}) = 0. \tag{5.15}$$

The inequality (5.14) implies

$$E'(x) \leq 0, \qquad 0 \leq x < \infty, \tag{5.16}$$

so that $E(x)$ is a monotone decreasing function of x for $E(x) > 0$. On further integration of (5.14), we obtain the decay estimate

$$E(x) \leq E(0)e^{-2\varepsilon x}, \qquad 0 \leq x < \infty, \tag{5.17}$$

where

$$E(0) = f^2(0) + \ln \sec^2 \beta(0) \tag{5.18}$$

is an explicit function of the edge data.

The decay estimates (3.1) and (5.17) have been established under very different hypotheses. The asymptotic behavior as $x \to \infty$ assumed in (5.15) is more restrictive than (1.8). Also, assumption (2.25) restricts the magnitude of the edge data while (5.9) assumes that $|\beta(x)| < \pi/2$ for *all* $x > 0$. It would be desirable to establish conditions on the edge data that would *guarantee* that (5.9) holds, but such a result has not yet been found. The major disadvantage of (5.17) compared to (3.1) is that the decay rate in the former is much slower than in the latter. Thus, we now consider an alternative argument which provides a larger decay rate than does (5.17).

Result 2. The second decay estimate for $E(x)$ is established by hypothesizing that

$$\cos\beta > \gamma > \varepsilon^2/2, \qquad 0 \leq x < \infty, \tag{5.19}$$

where γ is a constant. It is clear from (5.17) that if $|\beta| < \pi/2$, then, in fact, $\cos\beta \geq \exp[-E(0)/2]$. Thus, if ε is sufficiently small and $|\beta| < \pi/2$, the inequality (5.19) will *automatically* be satisfied for some suitable γ. Since (5.19) ensures that (5.9) holds, we may use (5.14) to deduce that both $E(x)$ and $\int_x^\infty E(s)\,ds$ are bounded for $x \geq 0$. We shall show that

$$E(x) \leq E(0)e^{-k_1 x}, \qquad 0 \leq x < \infty, \tag{5.20}$$

where the estimated decay rate is given by

$$k_1 = 2^{-1/4}\gamma^{1/2} + O(\varepsilon^2). \tag{5.21}$$

The proof of (5.21) makes use of (5.10), the inequalities

$$\beta^2 \leq \ln \sec^2 \beta \leq \beta \tan \beta, \tag{5.22}$$

and the identity

$$(f\beta' - \beta f')' = -\varepsilon^2 f(\beta - \sin\beta) + \beta\sin\beta + f^2\cos\beta\,, \tag{5.23}$$

which follows from the differential equations (1.4). On integrating (5.23) and assuming

$$\lim_{x\to\infty}(f\beta' - \beta f') = 0\,, \tag{5.24}$$

we obtain the identity

$$\int_x^\infty (f^2 + \beta\tan\beta)\cos\beta\, ds = \varepsilon^2\int_x^\infty f(\beta - \sin\beta)\, ds + \beta f' - f\beta'. \tag{5.25}$$

Using the arithmetic-geometric mean inequality in the first term on the right in (5.25) and the same inequality with weight α, i.e.,

$$2ab \le \alpha a^2 + b^2/\alpha\,, \qquad \alpha > 0\,, \tag{5.26}$$

in the second term, we get

$$\begin{aligned}\int_x^\infty (f^2 + \beta\tan\beta)\cos\beta\, ds &\le (\varepsilon^2/2)\int_x^\infty [f^2 + (\beta - \sin\beta)^2]\, ds \\ &+ (\alpha/2)[(f')^2 + (\beta')^2] + (1/2\alpha)(f^2 + \beta^2).\end{aligned} \tag{5.27}$$

Using the inequalities (5.19) and (5.22) in (5.27), we find that

$$\begin{aligned}2(\gamma - \varepsilon^2/2)\int_x^\infty (f^2 + \ln\sec^2\beta)\, ds &\le \alpha[(f')^2 + (\beta')^2] + (1/\alpha)(f^2 + \ln\sec^2\beta) \\ &\le \alpha[(f')^2 + (\beta')^2\sec^2\beta] + (1/\alpha)(f^2 + \ln\sec^2\beta)\,,\end{aligned} \tag{5.28}$$

where $\alpha > 0$ is an arbitrary constant.

Since (5.17) implies that

$$\lim_{x\to\infty} E(x) = 0\,, \tag{5.29}$$

we now write

$$E(x) = -\int_x^\infty E'(s)\, ds\,, \tag{5.30}$$

and use (5.7) to obtain

$$E = -2\int_x^\infty (ff' + \beta'\tan\beta)\, ds. \tag{5.31}$$

Applying first Cauchy's inequality and then (5.26) with weight $\delta > 0$, we find

$$E \le 2\int_x^\infty (f^2 + \sin^2\beta)^{1/2}[(f')^2 + (\beta')^2\sec^2\beta]^{1/2}\, ds \tag{5.32}$$

$$\le \delta\int_x^\infty (f^2 + \sin^2\beta)\, ds + \frac{1}{\delta}\int_x^\infty [(f')^2 + (\beta')^2\sec^2\beta]\, ds. \tag{5.33}$$

Adding and subtracting

$$\varepsilon^2(f^2 + \sin\beta\tan\beta) \tag{5.34}$$

in the integrand in the second integral on the right in (5.33), and using (5.10), we obtain

$$E(x) \le (\delta - \varepsilon^2\delta)\int_x^\infty (f^2 + \ln\sec^2\beta)\,ds + \frac{1}{\delta}\int_x^\infty [(f')^2 + (\beta')^2\sec^2\beta + \varepsilon^2(f^2 + \sin\beta\tan\beta)]\,ds, \tag{5.35}$$

where $\delta > 0$ is an arbitrary constant. If δ is chosen so that

$$\delta - \varepsilon^2/\delta > 0, \tag{5.36}$$

we may, in view of the right-hand inequality of (5.19), use the inequality (5.28) in (5.35) to deduce that

$$E \le \frac{(\delta - \varepsilon^2/\delta)\alpha}{2(\gamma - \varepsilon^2/2)}[(f')^2 + (\beta')^2\sec^2\beta] + \frac{(\delta - \varepsilon^2/\delta)E}{2\alpha(\gamma - \varepsilon^2/2)} + \frac{1}{2\delta}\int_x^\infty E''(s)\,ds. \tag{5.37}$$

Rewriting (5.37) in the form

$$\left[1 - \frac{(\delta - \varepsilon^2/\delta)}{2\alpha(\gamma - \varepsilon^2/2)}\right]E \le \frac{(\delta - \varepsilon^2/\delta)\alpha}{2(\gamma - \varepsilon^2/2)}[(f')^2 + (\beta')^2\sec^2\beta + \varepsilon^2(f^2 + \sin\beta\tan\beta)] - \frac{\alpha\varepsilon^2(\delta - \varepsilon^2/\delta)(f^2 + \sin\beta\tan\beta)}{2(\gamma - \varepsilon^2/2)} + \frac{1}{2\delta}\int_x^\infty E''(s)\,ds, \tag{5.38}$$

using the left-hand inequality of (5.10), and recalling (5.8), we deduce from (5.38) that

$$\left[1 - \frac{(\delta - \varepsilon^2/\delta)(1/\alpha - \varepsilon^2\alpha)}{2(\gamma - \varepsilon^2/2)}\right]E \le \frac{(\delta - \varepsilon^2/\delta)\alpha}{4(\gamma - \varepsilon^2/2)}E'' + \frac{1}{2\delta}\int_x^\infty E''(s)\,ds. \tag{5.39}$$

If

$$\lim_{x\to\infty} E'(x) = 0, \tag{5.40}$$

then the inequality (5.39) can be written as

$$E'' - AE' - BE \ge 0, \qquad 0 \le x < \infty, \tag{5.41}$$

where the constants A and B are given by

$$A = \frac{2(\gamma - \varepsilon^2/2)}{\alpha\delta(\delta - \varepsilon^2/\delta)} \tag{5.42}$$

$$B = \frac{4(\gamma - \varepsilon^2/2)}{\alpha(\delta - \varepsilon^2/\delta)}\left[1 - \frac{(\delta - \varepsilon^2/\delta)(1 - \varepsilon^2\alpha^2)}{2\alpha(\gamma - \varepsilon^2/2)}\right]. \tag{5.43}$$

The second-order differential inequality (5.41) may be integrated using (5.29) and (5.40) to yield the estimate

$$E(x) \le E(0)e^{-kx}, \qquad 0 \le x < \infty, \tag{5.44}$$

where

$$k = \tfrac{1}{2}\left(\sqrt{A^2 + 4B} - A\right). \tag{5.45}$$

If B is positive, then the constant k is positive, and so (5.44) is a *decay* estimate. We see from the right-hand inequality of (5.19), (5.36), and (5.43) that this will be true if

$$1 - (\delta - \varepsilon^2/\delta)(1 - \varepsilon^2\alpha^2)/2\alpha(\gamma - \varepsilon^2/2) > 0. \tag{5.46}$$

We have established (5.44) with the decay rate (5.45) under the assumption that γ satisfies (5.19) and that the asymptotic limits in (5.24), (5.29), and (5.40) hold. Choosing the arbitrary positive constants α and δ to *maximize* k, subject to the constraints (5.36) and (5.46), can be shown to lead to the second decay estimate, (5.20) and (5.21). We note that (5.19) implies that (5.9) holds which in turn yields (5.29), and so we see that

$$f(x) \quad \text{and} \quad \beta(x) \to 0 \quad \text{as } x \to \infty. \tag{5.47}$$

Thus, (5.24), (5.29), and (5.40) will be satisfied if we impose the requirement that

$$\lim_{x\to\infty} e^{-2\varepsilon x}\{[f'(x)]^2 + [\beta'(x)]^2\} = 0. \tag{5.48}$$

Result 3. The third decay estimate for $E(x)$ is also established under the hypothesis (5.19). We shall show that

$$E(x) \le (2k_2)^{-1}Q(0)e^{-k_2 x}, \qquad 0 \le x < \infty, \tag{5.49}$$

where

$$k_2 = 2^{1/4}\gamma^{1/2} + O(\varepsilon^2), \tag{5.50}$$

and $Q(0)$ can be explicitly bounded in terms of the edge data, $f(0)$ and $\beta(0)$. (See (B.17) in Appendix B.) Comparing (5.21) with (5.50), we see that the decay rate in (5.49) is *larger* than in (5.20), although the constant multiplying the exponential in (5.49) is more complicated than the analogous factor in (5.20).

The proof of (5.49) and (5.50) is similar to the proof of (5.20) and (5.21) and uses the inequality (5.28), established under the assumption (5.24). In contrast to the *second-order* differential inequality (5.41), we here construct a *third-order* differential inequality for the function

$$F(x) \equiv \int_x^\infty (f^2 + \ln\sec^2\beta)\,ds = \int_x^\infty E(s)\,ds. \tag{5.51}$$

From (5.51) and (5.6)–(5.8), we have

$$F' = -(f^2 + \ln\sec^2\beta) = -E \tag{5.52}$$

$$F'' = -2(ff' + \beta'\tan\beta) = -E' \tag{5.53}$$

$$F''' = -2[(f')^2 + (\beta')^2\sec^2\beta] - 2\varepsilon^2(f^2 + \sin\beta\tan\beta) = -E''. \tag{5.54}$$

The inequality (5.28) can be written as

$$(\gamma - \varepsilon^2/2)F \le -\{(\alpha/4)[F''' + 2\varepsilon^2(f^2 + \sin\beta\tan\beta)] + (\tfrac{1}{2}\alpha)F'\} \tag{5.55}$$

$$\le -[(\alpha/4)F''' + \tfrac{1}{2}(1/\alpha - \varepsilon^2\alpha)F'], \tag{5.56}$$

where the left side of (5.10) has been used to obtain (5.56) from (5.55). We write (5.56) as

$$F''' + (2/\alpha)(1/\alpha - \varepsilon^2\alpha)F' + (4/\alpha)(\gamma - \varepsilon^2/2)F \le 0, \qquad 0 \le x < \infty, \tag{5.57}$$

which is the desired third-order differential inequality for $F(x)$.

To integrate (5.57), we rewrite it as

$$Q' + mQ \leq 0, \qquad 0 \leq x < \infty, \tag{5.58}$$

where

$$Q \equiv F'' - mF' + [m^2 + (2/\alpha)(1/\alpha - \varepsilon^2\alpha)]F \tag{5.59}$$

and m is a root of the cubic equation

$$(\alpha/4)m^3 + \tfrac{1}{2}(1/\alpha - \varepsilon^2\alpha)m - (\gamma - \varepsilon^2/2) = 0. \tag{5.60}$$

On integrating (5.58) and assuming

$$\lim_{x \to \infty} Q(x) = 0, \tag{5.61}$$

we obtain

$$Q(x) \leq Q(0)e^{-mx}, \qquad 0 \leq x < \infty. \tag{5.62}$$

It is easily seen that the assumption (5.48) is a sufficient condition to guarantee that (5.61) holds.

To ensure that (5.62) is, indeed, a *decay* estimate and to find the *largest* possible decay rate, we *maximize* $m_1(\alpha)$, the positive root of (5.60). Differentiating (5.60) implicitly with respect to α, we obtain

$$\tfrac{1}{4}m_1^3 - \tfrac{1}{2}(1/\alpha_1^2 + \varepsilon^2)m_1 = 0, \tag{5.63}$$

where α_1 is the value of α that maximizes $m_1(\alpha)$. Thus,

$$\alpha_1^2 = 2/(m_1^2 - 2\varepsilon^2). \tag{5.64}$$

Insertion of this value of α_1 into (5.63) yields

$$m_1 = 2^{1/4}\gamma^{1/2} + O(\varepsilon^2). \tag{5.65}$$

Thus, (5.62) has been established with the estimated decay rate m_1 given above.

To obtain the final result (5.49), we observe that with α and m given by (5.64) and (5.65), $Q(x)$ in (5.59) may be written

$$Q = F'' - m_1 F' + 2(m_1^2 - 2\varepsilon^2)F. \tag{5.66}$$

But, from (5.51), $F(x) \geq 0$, and so (5.66) yields

$$Q \geq F'' - m_1 F'. \tag{5.67}$$

By virtue of (5.62), the inequality (5.67) may be written as

$$(F'e^{-m_1 x})' \leq Q(0)E^{-2m_1 x}. \tag{5.68}$$

On integrating (5.68) and recalling (5.52), we arrive at the desired decay estimate (5.49) with decay rate $m_1 = k_2$, given by (5.50). We show in Appendix B how to bound $Q(0)$ explicitly in terms of the edge data $f(0)$ and $\beta(0)$.

Conclusions. We have obtained decay estimates for solutions of the coupled nonlinear differential equations (1.4) subject to the boundary conditions (1.7) and various decay conditions as $x \to \infty$ including (1.8) which allowed the boundary value problem for the differential equations to be replaced by a complex-valued integral equation. For sufficiently small initial data satisfying (2.25), we found that the integral equation had a unique solution that had to decay as fast as the linear solution. Moreover, we showed that *any* decaying solution of the integral equation could decay no slower or faster than the linear solution as $x \to \infty$. Alternatively, instead of assuming decaying solutions of the integral equation, we found that merely assuming the existence of a solution such that (1) the L_2-norm of the angle of rotation satisfied a certain bound guaranteed decay at a rate at least as fast as that of the linear solution whereas (2) if the sup norm of the angle of rotation satisfied another bound, then decay, but at a slower rate than the linear solution, was guaranteed. Finally, by working with an "energy" (or, perhaps, more properly, a Lyapunov) function, defined on assumed solutions of the nonlinear *differential equations*, (1.4), we obtained three other decay estimates. The energy methods were developed with an eye to obtaining decay estimates for more general shells for which an integral equation formulation might not be easy to obtain.

There remains the open question raised in Sec. 5: If we start with edge data such that $|\beta(0)| < \pi/2$, will $|\beta(x)|$ remain less than $\pi/2$ for all x?

Appendix A.

Bounds on the nonlinear function N. Note that $|\sin\beta| \le |\beta|$ implies

$$|f\sin^2\beta/2| \le |f|\min(1, \beta^2/4). \tag{A.1}$$

Further,

$$|\beta - \sin\beta| \le \left(1 - \frac{\sin\beta}{\beta}\right)|\beta| \le k|\beta|, \tag{A.2}$$

where $k = 2\sin^2\beta_*/2 = 1.46\cdots$, β_* being the smallest positive root of $\beta = \tan\beta$. Also,

$$|\beta - \sin\beta| = \left|\int_0^\beta (1-\cos\gamma)\,d\gamma\right| = 2\left|\int_0^\beta \sin^2\frac{\gamma}{2}\,d\gamma\right| \le \frac{1}{2}\left|\int_0^\beta \gamma^2\,d\gamma\right| \le |\beta|^3/6. \tag{A.3}$$

Since $|\phi| = \sqrt{f^2+\beta^2}$, it follows from (2.3), (A.1)–(A.3), and Cauchy's inequality that

$$|N| \le |\phi|\min\left[\sqrt{4+k|p|^4},\ \tfrac{1}{6}\sqrt{9+|p|^4}\beta^2\right],\quad 0 \le x < \infty. \tag{A.4}$$

Moreover, because $\theta = e^{px}\phi(x)$ and $P = e^{3px}N$,

$$|P| \le \tfrac{1}{6}\sqrt{9+|p|^4}|\theta|^3, \qquad 0 \le x < \infty. \tag{A.5}$$

Bounds on $|P(\theta_m) - P(\theta_n)|$. Using elementary inequalities, we have

$$\begin{aligned}|f_m\sin^2\beta_m/2 - f_n\sin^2\beta_n/2| &= |f_m(\sin\beta_m/2 - \sin\beta_n/2)(\sin\beta_m/2 + \sin\beta_n/2)\\ &\quad + (f_m - f_n)\sin^2\beta_n/2| \\ &\le \tfrac{1}{4}[|f_m||\beta_m - \beta_n|(|\beta_m| + |\beta_n|) + |f_m - f_n|\beta_n^2].\end{aligned} \tag{A.6}$$

Introducing the shorthand notation

$$\hat{f} = \tfrac{1}{2}(|f_m| + |f_n|), \qquad \hat{\beta} = \tfrac{1}{2}(|\beta_m| + |\beta_n|), \qquad \Delta f = |f_m - f_n|, \qquad \Delta\beta = |\beta_m - \beta_n|, \tag{A.7}$$

interchanging m and n in the last line of (A.6), and adding (to obtain an inequality symmetric in m and n), we obtain

$$|f_m \sin^2 \beta_m/2 - f_n \sin^2 \beta_n/2| \le \tfrac{1}{4}[2\hat{f}\hat{\beta}\Delta\beta + \tfrac{1}{2}(\beta_m^2 + \beta_n^2)\Delta f]. \tag{A.8}$$

Furthermore, using an obvious modification of (A.2), we find that

$$\begin{aligned} |(\beta_m - \sin\beta_m) - (\beta_n - \sin\beta_n)| &= \left| \int_{\beta_n}^{\beta_m} (1 - \cos\gamma)\, d\gamma \right| \\ &\le \tfrac{1}{6}|\beta_m^3 - \beta_n^3| \\ &\le \tfrac{1}{6}|\beta_m^2 + \beta_m\beta_n + \beta_n^2|\Delta\beta. \end{aligned} \tag{A.9}$$

Thus, by an appeal to the Cauchy and arithmetic-geometric mean inequalities, we have

$$\begin{aligned} |N(\phi_m) - N(\phi_n)| &\le \tfrac{1}{6}\sqrt{9 + |p|^4}\sqrt{\lambda(\Delta f)^2 + 2\mu\Delta f\Delta\beta + \nu(\Delta\beta)^2} \\ &\le (S/6)\sqrt{9 + |p|^4}|\phi_m - \phi_n|, \end{aligned} \tag{A.10}$$

where

$$\begin{gathered} \lambda = \tfrac{1}{4}(\beta_m^2 + \beta_n^2)^2, \qquad \mu = \hat{f}\hat{\beta}(\beta_m^2 + \beta_n^2), \\ \nu = 4\hat{f}^2\hat{\beta}^2 + |\beta_m^2 + \beta_m\beta_n + \beta_n^2|^2, \end{gathered} \tag{A.11}$$

and

$$S = \sqrt{\tfrac{1}{2}(\nu + \lambda) + \sqrt{\tfrac{1}{4}(\nu - \lambda)^2 + \mu^2}}. \tag{A.12}$$

Finally, setting $\theta = e^{px}\phi$, $P = e^{3px}N$, and supposing that there exists a positive constant L_* such that $|\theta_n| < L_*$, we have

$$|P(\theta_m) - P(\theta_n)| < \tfrac{1}{6}\sqrt{7 + 2\sqrt{10}}\sqrt{9 + |p|^4}L_*^2|\theta_m - \theta_n|. \tag{A.13}$$

Bounds on integrals of K. From (1.3) and (2.10),

$$\begin{aligned} |p|\int_0^\infty |K(\xi, x)|\, d\xi &= \int_0^x |e^{-3p\xi} \operatorname{sh} p\xi|\, d\xi + |e^{px} \operatorname{sh} px| \int_x^\infty e^{-4a\xi}\, d\xi \\ &\le \tfrac{1}{2}\left[\int_0^x (e^{-2a\xi} + e^{-4a\xi})\, d\xi + e^{ax}|\operatorname{sh} px| \int_x^\infty e^{-4a\xi}\, d\xi\right] \\ &= (1/8a)(3 - 2e^{-2ax} - e^{-4ax} + e^{-3ax}|\operatorname{sh} px|) \\ &\le (3/8a), \qquad 0 \le x < \infty. \end{aligned} \tag{A.14}$$

If $x < y$, then from (2.10)

$$\begin{aligned} |p|\int_0^\infty |K(\xi, x) - K(\xi, y)|\, d\xi &= \int_x^y |e^{-3p\xi}\operatorname{sh} p\xi - e^{p(x-4\xi)}\operatorname{sh} px|\, d\xi \\ &\quad + \tfrac{1}{2}|e^{2py} - e^{2px}| \int_y^\infty |e^{-4p\xi}|\, d\xi \\ &\le (e^{-2ay}/8a)[3e^{2a(y-x)} - 2 - e^{-2a(y-x)} \\ &\quad + |1 - e^{-2p(y-x)}|], \qquad y > x. \end{aligned} \tag{A.15}$$

If $y < x$, (A.15) holds with x and y interchanged.

Appendix B.

An upper bound on $Q(0)$. From (5.52), (5.53), and (5.66), we have

$$Q(0) = F''(0) - m_1 F'(0) + 2(m_1^2 - 2\varepsilon^2)F(0) \tag{B.1}$$

$$= -2(f_0 f_0' + \beta_0' \tan\beta_0) + m_1(f_0^2 + \ln\sec^2\beta_0) + 2(m_1^2 - 2\varepsilon^2)F(0), \tag{B.2}$$

where

$$f_0 \equiv f(0), \qquad \beta_0 = \beta(0), \qquad f_0' \equiv f'(0), \qquad \beta_0' = \beta'(0). \tag{B.3}$$

On using the arithmetic-geometric mean inequality (5.26) with weight $\sigma > 0$, we find that

$$\begin{aligned} Q(0) &\le \sigma[(f_0')^2 + (\beta_0')^2] + (1/\sigma)(f_0^2 + \tan^2\beta_0) \\ &\quad + m_1(f_0^2 + \ln\sec^2\beta_0) + 2(m_1^2 - 2\varepsilon^2)F(0). \end{aligned} \tag{B.4}$$

We now find an upper bound for the first term on the right in (B.4) in terms of f_0, β_0, and $F(0)$. To this end, we multiply $(1.4)_1$ by f', $(1.4)_2$ by β', add the resulting equations and integrate. Using (5.47) and assuming also that f' and β' approach zero as $x \to \infty$, we get

$$(f_0')^2 + (\beta_0')^2 = \varepsilon^2(f_0^2 + 4\sin^2\beta_0/2) + 2\int_0^\infty (f'\sin\beta - f\beta'\cos\beta)\, dx. \tag{B.5}$$

Using the arithmetic-geometric mean inequality (5.26) with weight ω, we find

$$\begin{aligned} (f_0')^2 + (\beta_0')^2 &\le \varepsilon^2(f_0^2 + 4\sin^2\beta_0/2) + \omega\int_0^\infty [(f')^2 + (\beta')^2\cos^2\beta]\, dx \\ &\quad + \frac{1}{\omega}\int_0^\infty (f^2 + \sin^2\beta)\, dx, \end{aligned} \tag{B.6}$$

for any $\omega > 0$. On adding to and subtracting from the integrand in the second term on the right in (B.6) the quantity $\varepsilon^2(f^2 + \sin\beta\tan\beta)$, and using the inequality (5.10), we get

$$\begin{aligned} (f_0')^2 + (\beta_0')^2 &\le \varepsilon^2(f_0^2 + 4\sin^2\beta_0/2) \\ &\quad + \omega\int_0^\infty [(f')^2 + (\beta')^2\sec^2\beta + \varepsilon^2(f^2 + \sin\beta\tan\beta)]\, dx \\ &\quad + (1/\omega - \varepsilon^2\omega)F(0). \end{aligned} \tag{B.7}$$

On recalling (5.54), we see that the integral on the right is equal to

$$-\frac{1}{2}\int_0^\infty F'''(x)\,dx = \frac{1}{2}F''(0), \tag{B.8}$$

and so using (5.53), we deduce from (B.7) that

$$(f_0')^2 + (\beta_0')^2 \le \varepsilon^2(f_0^2 + 4\sin^2\beta_0/2) - \omega(f_0 f_0' + \beta_0'\tan\beta_0) + (1/\omega - \varepsilon^2\omega)F(0). \tag{B.9}$$

Using the arithmetic-geometric inequality (5.26) with weight τ in the second term on the right in (B.9), we obtain

$$\begin{aligned}(1-\omega\tau/2)[(f_0')^2 + (\beta_0')^2] &\le \varepsilon^2(f_0^2 + 4\sin^2\beta_0/2) \\ &\quad + (\omega/2\tau)(f_0^2 + \tan^2\beta_0) + (1/\omega - \varepsilon^2\omega)F(0),\end{aligned} \tag{B.10}$$

for arbitrary positive constants ω and τ. On recalling the definition of $F(x)$ in (5.51), we obtain from (5.28), with $x = 0$, the inequality

$$F(0) \le \tfrac{1}{2}(\gamma - \varepsilon^2/2)^{-1}\{\alpha[(f_0')^2 + (\beta_0')^2] + (1/\alpha)(f_0^2 + \ln\sec^2\beta_0)\}, \tag{B.11}$$

where $\alpha > 0$ is an arbitrary constant. On substituting (B.11) into (B.10) and assuming that $1/\omega - \varepsilon^2\omega > 0$, we obtain

$$\begin{aligned}[1 - \omega\tau/2 - \alpha/2(\gamma - \varepsilon^2/2)][(f_0')^2 + (\beta_0')^2] &\le \varepsilon^2(f_0^2 + 4\sin^2\beta_0/2) \\ &\quad + (\omega/2\tau)(f_0^2 + \tan^2\beta_0) \\ &\quad + \frac{(1/\omega - \varepsilon^2\omega)}{2\alpha(\gamma - \varepsilon^2/2)}(f_0^2 + \ln\sec^2\beta_0).\end{aligned} \tag{B.12}$$

For simplicity, we now choose

$$\omega = 1, \qquad \tau = \tfrac{1}{2}, \qquad \alpha = \tfrac{1}{2}(\gamma - \varepsilon^2/2), \tag{B.13}$$

so that (B.12) reduces to

$$\begin{aligned}\tfrac{1}{2}[(f_0')^2 + (\beta_0')^2] &\le \varepsilon^2(f_0^2 + 4\sin^2\beta_0/2) + f_0^2 + \tan^2\beta_0 \\ &\quad + \frac{(1-\varepsilon^2)}{(\gamma - \varepsilon^2/2)^2}(f_0^2 + \ln\sec^2\beta_0).\end{aligned} \tag{B.14}$$

With α chosen as in (B.13), (B.11) reads

$$F(0) \le \tfrac{1}{4}[(f_0')^2 + (\beta_0')^2] + (\gamma - \varepsilon^2/2)^{-2}(f_0^2 + \ln\sec^2\beta_0), \tag{B.15}$$

which, by virtue of (B.14), can be written as

$$2F(0) \le \varepsilon^2(f_0^2 + 4\sin^2\beta_0/2) + f_0^2 + \tan^2\beta_0 + \frac{(3-\varepsilon^2)}{(\gamma - \varepsilon^2/2)^2}(f_0^2 + \ln\sec^2\beta_0). \tag{B.16}$$

Substituting from (B.14) and (B.16) into the right side of (B.4) and choosing $\sigma = 1$ for simplicity, we obtain the bound we sought:

$$\begin{aligned}Q(0) &\le (3 + m_1^2 - 2\varepsilon^2)(f_0^2 + \tan^2\beta_0) + \varepsilon^2(2 + m_1^2 - 2\varepsilon^2)(f_0^2 + 4\sin^2\beta_0/2) \\ &\quad + \left[m_1 + \frac{2 - 8\varepsilon^2 + m_1^2(3-\varepsilon^2) + 2\varepsilon^4}{(\gamma - \varepsilon^2/2)^2}\right](f_0^2 + \ln\sec^2\beta_0),\end{aligned} \tag{B.17}$$

where m_1 is given by (5.65).

Acknowledgments. The work of C. O. H. was supported by the National Science Foundation (NSF) under Grant No. MSM-89-04719 and by the Center for Advanced Studies, University of Virginia. The work of L. E. P. was carried out while he held an appointment as Visiting Professor, Department of Applied Mathematics, University of Virginia, Spring, 1989, and was partially supported by the NSF under Grant No. DMS-86-00250. The work of J. G. S. was supported by NSF under Grant No. MSM-8618657.

References

[1] A. Libai and J. G. Simmonds, *The Nonlinear Theory of Elastic Shells: One Spatial Dimension*, Academic Press, San Diego, 1988

[2] J. G. Simmonds and A. Libai, *A simplified version of Reissner's non-linear equations for a first-approximation theory of shells of revolution*, Computational Mechanics **2**, 99–103 (1987)

[3] C. O. Horgan and J. K. Knowles, *Recent developments concerning Saint-Venant's principle*, Advances in Applied Mechanics (eds. T. Y. Wu and J. W. Hutchinson), Vol. 23, Academic Press, San Diego, 1983, pp. 179–269

[4] C. O. Horgan, *Recent developments concerning Saint-Venant's principle—an update*, Applied Mechanics Reviews **42**, 295–303 (1989)