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AN INVERSE PROBLEM FOR A GENERAL CONVEX DOMAIN WITH IMPEDANCE BOUNDARY CONDITIONS

By

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Abstract. The spectral function $\theta(t) = \sum_{n=1}^{\infty} \exp(-t\lambda_n)$, where $\{\lambda_n\}_{n=1}^{\infty}$ are the eigenvalues of the Laplace operator $\Delta = \sum_{i=1}^{2}(\partial/\partial x^i)^2$ in the x^1x^2-plane, is studied for a general convex domain $\Omega \subseteq R^2$ with a smooth boundary $\partial\Omega$ together with a finite number of piecewise smooth impedance boundary conditions on the parts $\Gamma_1, \ldots, \Gamma_m$ of $\partial\Omega$ such that $\partial\Omega = \bigcup_{j=1}^{m} \Gamma_j$.

1. Introduction. Let $\Omega \subseteq R^2$ be a simply connected bounded domain with a smooth boundary $\partial\Omega$. Consider the impedance problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \tag{1.1}$$

$$\left(\frac{\partial}{\partial n} + \gamma\right) u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\partial/\partial n$ denotes differentiation along the inward pointing normal to $\partial\Omega$, γ is a positive constant, and $u \in C^2(\Omega) \bigcap C(\overline{\Omega})$.

Denote the eigenvalues of problem (1.1), (1.2) counted according to multiplicity by an increasing sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty \quad \text{as } n \to \infty. \tag{1.3}$$

Sleeman and Zayed [4] have recently discussed the problem of determining the geometry of Ω as well as the impedance γ from the asymptotic expansion of the trace function

$$\theta(t) = \operatorname{tr}[\exp(-t\Delta)] = \sum_{n=1}^{\infty} \exp(-t\lambda_n) \quad \text{as } t \to 0. \tag{1.4}$$

Problem (1.1), (1.2) has been investigated by many authors (see, for example [1-3, 5, 6]) in the following special cases.

Case 1.1. $\gamma = 0$ (Neumann problem).

$$\theta(t) = \frac{|\Omega|}{4\pi t} + \frac{|\partial\Omega|}{8(\pi t)^{1/2}} + a_0 + \frac{7}{256}(t/\pi)^{1/2} \int_{\partial\Omega} k^2(\sigma)\, d\sigma + O(t) \quad \text{as } t \to 0. \tag{1.5}$$

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Case 1.2. $\gamma \to \infty$ (Dirichlet problem).

$$\theta(t) = \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8(\pi t)^{1/2}} + a_0 + \frac{1}{256}(t/\pi)^{1/2} \int_{\partial\Omega} k^2(\sigma)\, d\sigma + O(t) \quad \text{as } t \to 0. \tag{1.6}$$

In these formulae, $|\Omega|$ is the area of Ω, $|\partial\Omega|$ is the total length of $\partial\Omega$ and $k(\sigma)$ is the curvature of $\partial\Omega$. The constant term a_0 has geometric significance, e.g., if Ω is smooth and convex then $a_0 = \frac{1}{6}$ and if Ω is permitted to have a finite number H of smooth convex holes, then $a_0 = (1 - H)\frac{1}{6}$.

The object of this paper is to discuss the following problem:

Suppose that the eigenvalues (1.3) are known exactly for Eq. (1.1) together with the impedance boundary conditions

$$\left(\frac{\partial}{\partial n_j} + \gamma_j\right) u = 0 \quad \text{on } \Gamma_j,\ j = 1, \ldots, m, \tag{1.7}$$

where the boundary $\partial\Omega$ of the domain Ω consists of a finite number of parts $\Gamma_1, \ldots, \Gamma_m$ such that $\partial\Omega = \bigcup_{j=1}^m \Gamma_j$, while $\partial/\partial n_j$ denote differentiations along the inward pointing normals to Γ_j, and γ_j are positive constants.

The basic problem is that of determining the geometry of Ω as well as the impedances $\gamma_1, \ldots, \gamma_m$ from the asymptotic form of the spectral function $\theta(t)$ for small positive t.

2. Statement of results. Suppose that the parts $\Gamma_1, \ldots, \Gamma_m$ of the boundary $\partial\Omega$ are given locally by the equations $x^n = y^n(\sigma_j)$, $n = 1, 2$, $j = 1, \ldots, m$ in which σ_j are the arc lengths of the counterclockwise oriented boundary and $y^n(\sigma_j) \in C^\infty(\Gamma_j)$. Let $L_1, \ldots, L_m$ be the lengths of the parts $\Gamma_1, \ldots, \Gamma_m$, respectively, and let $k_1(\sigma_1), \ldots, k_m(\sigma_m)$ be the curvatures of $\Gamma_1, \ldots, \Gamma_m$, respectively. Then, the results of our problem (1.1), (1.7) can be summarized in the following cases.

Case 2.1. $(0 < \gamma_j \ll 1,\ j = 1, \ldots, k$ and $\gamma_j \gg 1,\ j = k+1, \ldots, m)$.

$$\begin{aligned}\theta(t) = \frac{|\Omega|}{4\pi t} + \frac{1}{8(\pi t)^{1/2}} &\left\{ \sum_{j=1}^{k} L_j - \sum_{j=k+1}^{m} \left[L_j + \gamma_j^{-1} \int_{\Gamma_j} k_j(\sigma_j)\, d\sigma_j \right] \right\} \\ &+ \left(1 - \frac{3}{\pi} \sum_{j=1}^{k} \gamma_j L_j \right) \frac{1}{6} + O(t^{1/2}) \quad \text{as } t \to 0.\end{aligned} \tag{2.1}$$

Case 2.2. $(\gamma_j \gg 1,\ j = 1, \ldots, k$ and $0 < \gamma_j \ll 1,\ j = k+1, \ldots, m)$.

In this case the asymptotic expansion of $\theta(t)$ as $t \to 0$ follows from (2.1) with the interchanges Γ_j, $j = 1, \ldots, k \leftrightarrow \Gamma_j$, $j = k+1, \ldots, m$, L_j, $j = 1, \ldots, k \leftrightarrow L_j$, $j = k+1, \ldots, m$ and γ_j, $j = 1, \ldots, k \leftrightarrow \gamma_j$, $j = k+1, \ldots, m$.

Case 2.3. $(\gamma_j \gg 1,\ j = 1, \ldots, m)$.

$$\theta(t) = \frac{|\Omega|}{4\pi t} - \frac{1}{8(\pi t)^{1/2}} \sum_{j=1}^{m} \left\{ L_j + \gamma_j^{-1} \int_{\Gamma_j} k_j(\sigma_j)\, d\sigma_j \right\} + \frac{1}{6} + O(t^{1/2}) \quad \text{as } t \to 0. \tag{2.2}$$

Case 2.4. $(0 < \gamma_j \ll 1,\ j = 1, \ldots, m)$.

$$\theta(t) = \frac{|\Omega|}{4\pi t} + \left(\sum_{j=1}^{m} L_j\right) / 8(\pi t)^{1/2} + \left(1 - \frac{3}{\pi} \sum_{j=1}^{m} \gamma_j L_j\right) \frac{1}{6} + O(t^{1/2}) \quad \text{as } t \to 0. \tag{2.3}$$

With reference to formulae (1.5), (1.6) the asymptotic expansions (2.1)–(2.3) may be interpreted as follows:

(i) Ω is a convex domain and we have the impedance boundary conditions (1.7) with small/large impedances $\gamma_1, \dots, \gamma_m$ as indicated in the specifications of the four respective cases.

(ii) For the first three terms, Ω is a convex domain of area $|\Omega|$.

In Case 2.1, it has $H = 3/\pi \sum_{j=1}^{k} \gamma_j L_j$ holes, the parts Γ_j, $j = 1, \dots, k$ of lengths $\sum_{j=1}^{k} L_j$ with Neumann boundary conditions and the other parts Γ_j, $j = k+1, \dots, m$ of lengths

$$\sum_{j=k+1}^{m} \left[L_j + \gamma_j^{-1} \int_{\Gamma_j} k_j(\sigma_j)\, d\sigma_j \right]$$

with Dirichlet boundary conditions, provided H is an integer.

In Case 2.3, it has no holes (i.e., $H = 0$), the parts Γ_j, $j = 1, \dots, m$ of lengths

$$\sum_{j=1}^{m} \left\{ L_j + \gamma_j^{-1} \int_{\Gamma_j} k_j(\sigma_j)\, d\sigma_j \right\}$$

together with Dirichlet boundary conditions.

In Case 2.4, it has $H = 3/\pi \sum_{j=1}^{m} \gamma_j L_j$ holes, the parts Γ_j, $j = 1, \dots, m$ of lengths $\sum_{j=1}^{m} L_j$ with Neumann boundary conditions, provided H is an integer.

We close this section with the remark that, the author [7] has recently discussed problem (1.1), (1.7) in its special case when $m = 2$ and has obtained results which are in agreement with the above results (2.1)–(2.3).

3. Formulation of the mathematical problem. Following the method of Kac [1] and following closely the procedure of Sec. 3 in Zayed [7], it is easy to show that $\theta(t)$ is given by

$$\theta(t) = \iint_{\Omega} G(\mathbf{x}, \mathbf{x}; t)\, d\mathbf{x} \tag{3.1}$$

where $G(\mathbf{x}_1, \mathbf{x}_2; t)$ is Green's function for the heat equation

$$\left(\Delta - \frac{\partial}{\partial t} \right) u = 0 \tag{3.2}$$

subject to the impedance boundary conditions (1.7) and the initial condition

$$\lim_{t \to 0} G(\mathbf{x}_1, \mathbf{x}_2; t) = \delta(\mathbf{x}_1 - \mathbf{x}_2), \tag{3.3}$$

where $\delta(\mathbf{x}_1 - \mathbf{x}_2)$ is the Dirac delta function located as the source point $\mathbf{x}_2$. Let us write

$$G(\mathbf{x}_1, \mathbf{x}_2; t) = G_0(\mathbf{x}_1, \mathbf{x}_2; t) + \chi(\mathbf{x}_1, \mathbf{x}_2; t), \tag{3.4}$$

where

$$G_0(\mathbf{x}_1, \mathbf{x}_2; t) = (4\pi t)^{-1} \exp\left\{ -\frac{|\mathbf{x}_1 - \mathbf{x}_2|^2}{4t} \right\}, \tag{3.5}$$

is the "fundamental solution" of the heat equation (3.2) while $\chi(\mathbf{x}_1, \mathbf{x}_2; t)$ is the "regular solution" chosen so that $G(\mathbf{x}_1, \mathbf{x}_2; t)$ satisfies the impedance boundary conditions (1.7).

On setting $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}$ we find that

$$\theta(t) = \frac{|\Omega|}{4\pi t} + K(t), \tag{3.6}$$

where

$$K(t) = \iint_\Omega \chi(\mathbf{x}, \mathbf{x}; t)\, d\mathbf{x}. \tag{3.7}$$

In what follows we shall use Laplace transform with respect to t and use s^2 as the Laplace transform parameter; thus we define

$$\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2) = \int_0^\infty e^{-s^2 t} G(\mathbf{x}_1, \mathbf{x}_2; t)\, dt. \tag{3.8}$$

An application of the Laplace transform to the heat equation (3.2) shows that $\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2)$ satisfies the membrane equation

$$(\Delta - s^2)\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2) = -\delta(\mathbf{x}_1 - \mathbf{x}_2) \quad \text{in } \Omega, \tag{3.9}$$

together with the impedance boundary conditions (1.7).

The asymptotic expansion of $K(t)$ as $t \to 0$ may then be deduced directly from the asymptotic expansion of $\overline{K}(s^2)$ as $s \to \infty$, where

$$\overline{K}(s^2) = \iint_\Omega \overline{\chi}(\mathbf{x}, \mathbf{x}; s^2)\, d\mathbf{x}. \tag{3.10}$$

4. Construction of Green's function. It is well known [6] that Eq. (3.9) has the fundamental solution

$$\overline{G}_0(\mathbf{x}_1, \mathbf{x}_2; s^2) = \frac{1}{2\pi} K_0(s r_{\mathbf{x}_1\mathbf{x}_2}),$$

where

$$r_{\mathbf{x}_1\mathbf{x}_2} = |\mathbf{x}_1 - \mathbf{x}_2|$$

is the distance between the points $\mathbf{x}_1 = (x_1^1, x_1^2)$, $\mathbf{x}_2 = (x_2^1, x_2^2)$ of the domain Ω and K_0 is the modified Bessel function of the second kind and of zero order. The existence of this solution enables us to construct integral equations for $\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2)$ satisfying the impedance boundary conditions (1.7) for small/large impedances $\gamma_1, \dots, \gamma_m$. Therefore, Green's theorem gives

Case 4.1. ($0 < \gamma_j \ll 1$, $j = 1, \dots, k$ and $\gamma_j \gg 1$, $j = k+1, \dots, m$).

$$\begin{aligned}\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2) = \frac{1}{2\pi} K_0(s r_{\mathbf{x}_1\mathbf{x}_2}) &- \frac{1}{\pi}\sum_{j=1}^{k} \int_{\Gamma_j} \overline{G}(\mathbf{x}_1, \mathbf{y}; s^2) \left\{ \frac{\partial}{\partial n_{j\mathbf{y}}} K_0(s r_{\mathbf{y}\mathbf{x}_2}) + \gamma_j K_0(s r_{\mathbf{y}\mathbf{x}_2}) \right\} d\mathbf{y} \\ &+ \frac{1}{\pi} \sum_{j=k+1}^{m} \int_{\Gamma_j} \frac{\partial}{\partial n_{j\mathbf{y}}} \overline{G}(\mathbf{x}_1, \mathbf{y}; s^2) \left\{ K_0(s r_{\mathbf{y}\mathbf{x}_2}) + \gamma_j^{-1} \frac{\partial}{\partial n_{j\mathbf{y}}} K_0(s r_{\mathbf{y}\mathbf{x}_2}) \right\} d\mathbf{y}.\end{aligned} \tag{4.1}$$

Case 4.2. ($\gamma_j \gg 1$, $j = 1, \dots, k$ and $0 < \gamma_j \ll 1$, $j = k+1, \dots, m$).

In this case $\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2)$ has the same form (4.1) with the interchanges Γ_j, $j = 1, \dots, k \leftrightarrow \Gamma_j$, $j = k+1, \dots, m$ and γ_j, $j = 1, \dots, k \leftrightarrow \gamma_j$, $j = k+1, \dots, m$.

Case 4.3. ($\gamma_j \gg 1$, $j = 1, \ldots, m$).

$$\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2) = \frac{1}{2\pi} K_0(s r_{\mathbf{x}_1 \mathbf{x}_2}) + \frac{1}{\pi} \sum_{j=1}^{m} \int_{\Gamma_j} \frac{\partial}{\partial n_{j\mathbf{y}}} \overline{G}(\mathbf{x}_1, \mathbf{y}; s^2) \left\{ K_0(s r_{\mathbf{y}\mathbf{x}_2}) + \gamma_j^{-1} \frac{\partial}{\partial n_{j\mathbf{y}}} K_0(s r_{\mathbf{y}\mathbf{x}_2}) \right\} d\mathbf{y}. \quad (4.2)$$

Case 4.4. ($0 < \gamma_j \ll 1$, $j = 1, \ldots, m$).

$$\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2) = \frac{1}{2\pi} K_0(s r_{\mathbf{x}_1 \mathbf{x}_2}) - \frac{1}{\pi} \sum_{j=1}^{m} \int_{\Gamma_j} \overline{G}(\mathbf{x}_1, \mathbf{y}; s^2) \left\{ \frac{\partial}{\partial n_{j\mathbf{y}}} K_0(s r_{\mathbf{y}\mathbf{x}_2}) + \gamma_j K_0(s r_{\mathbf{y}\mathbf{x}_2}) \right\} d\mathbf{y}. \quad (4.3)$$

On applying the iteration method (see [4, 7]) to the integral equation (4.1), we obtain the Green's function $\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2)$ which has the regular part

$$\begin{aligned}
&\overline{\chi}(\mathbf{x}_1, \mathbf{x}_2; s^2) \\
&= -\frac{1}{2\pi^2} \sum_{j=1}^{k} \int_{\Gamma_j} K_0(s r_{\mathbf{x}_1 \mathbf{y}}) \left\{ \frac{\partial}{\partial n_{j\mathbf{y}}} K_0(s r_{\mathbf{y}\mathbf{x}_2}) + \gamma_j K_0(s r_{\mathbf{y}\mathbf{x}_2}) \right\} d\mathbf{y} \\
&+ \frac{1}{2\pi^2} \sum_{j=k+1}^{m} \int_{\Gamma_j} \frac{\partial}{\partial n_{j\mathbf{y}}} K_0(s r_{\mathbf{x}_1 \mathbf{y}}) \left\{ K_0(s r_{\mathbf{y}\mathbf{x}_2}) + \gamma_j^{-1} \frac{\partial}{\partial n_{j\mathbf{y}}} K_0(s r_{\mathbf{y}\mathbf{x}_2}) \right\} d\mathbf{y} \\
&+ \frac{1}{2\pi^2} \sum_{j=1}^{k} \int_{\Gamma_j} \int_{\Gamma_j} K_0(s r_{\mathbf{x}_1 \mathbf{y}}) M_{\gamma_j}(\mathbf{y}, \mathbf{y}') \left\{ \frac{\partial}{\partial n_{j\mathbf{y}'}} K_0(s r_{\mathbf{y}'\mathbf{x}_2}) + \gamma_j K_0(s r_{\mathbf{y}'\mathbf{x}_2}) \right\} d\mathbf{y}\, d\mathbf{y}' \\
&+ \frac{1}{2\pi^2} \sum_{j=k+1}^{m} \int_{\Gamma_j} \int_{\Gamma_j} \frac{\partial}{\partial n_{j\mathbf{y}}} K_0(s r_{\mathbf{x}_1 \mathbf{y}}) M_{\gamma_j^{-1}}(\mathbf{y}, \mathbf{y}') \\
&\qquad \times \left\{ K_0(s r_{\mathbf{y}'\mathbf{x}_2}) + \gamma_j^{-1} \frac{\partial}{\partial n_{j\mathbf{y}'}} K_0(s r_{\mathbf{y}'\mathbf{x}_2}) \right\} d\mathbf{y}\, d\mathbf{y}' \\
&- \frac{1}{2\pi^2} \sum_{j=k+1}^{m} \int_{\Gamma_j} \left\{ \sum_{j=1}^{k} \int_{\Gamma_j} K_0(s r_{\mathbf{x}_1 \mathbf{y}}) M^*_{\gamma_j}(\mathbf{y}, \mathbf{y}')\, d\mathbf{y} \right\} \\
&\qquad \times \left\{ K_0(s r_{\mathbf{y}'\mathbf{x}_2}) + \gamma_j^{-1} \frac{\partial}{\partial n_{j\mathbf{y}'}} K_0(s r_{\mathbf{y}'\mathbf{x}_2}) \right\} d\mathbf{y}' \\
&- \frac{1}{2\pi^2} \sum_{j=1}^{k} \int_{\Gamma_j} \left\{ \sum_{j=k+1}^{m} \int_{\Gamma_j} \frac{\partial}{\partial n_{j\mathbf{y}}} K_0(s r_{\mathbf{x}_1 \mathbf{y}}) M^*_{\gamma_j^{-1}}(\mathbf{y}, \mathbf{y}')\, d\mathbf{y} \right\} \\
&\qquad \times \left\{ \frac{\partial}{\partial n_{j\mathbf{y}'}} K_0(s r_{\mathbf{y}'\mathbf{x}_2}) + \gamma_j K_0(s r_{\mathbf{y}'\mathbf{x}_2}) \right\} d\mathbf{y}'
\end{aligned} \quad (4.4)$$

where

$$M_{\gamma_j}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty}(-1)^{\nu}K_{\gamma_j}^{(\nu)}(\mathbf{y}',\mathbf{y}), \tag{4.5}$$

$$M_{\gamma_j^{-1}}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty}K_{\gamma_j^{-1}}^{(\nu)}(\mathbf{y}',\mathbf{y}), \tag{4.6}$$

$$M_{\gamma_j}^{*}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty}(-1)^{\nu}\,{}^{*}K_{\gamma_j}^{(\nu)}(\mathbf{y}',\mathbf{y}), \tag{4.7}$$

$$M_{\gamma_j^{-1}}^{*}(\mathbf{y},\mathbf{y}') = \sum_{\nu=0}^{\infty}{}^{*}K_{\gamma_j^{-1}}^{(\nu)}(\mathbf{y}',\mathbf{y}), \tag{4.8}$$

$$K_{\gamma_j}(\mathbf{y}',\mathbf{y}) = \frac{1}{\pi}\left\{\frac{\partial}{\partial n_{j\mathbf{y}}}K_0(sr_{\mathbf{y}\mathbf{y}'}) + \gamma_j K_0(sr_{\mathbf{y}\mathbf{y}'})\right\}, \tag{4.9}$$

$$K_{\gamma_j^{-1}}(\mathbf{y}',\mathbf{y}) = \frac{1}{\pi}\left\{\frac{\partial}{\partial n_{j\mathbf{y}}}K_0(sr_{\mathbf{y}\mathbf{y}'}) + \gamma_j^{-1}\frac{\partial^2}{\partial n_{j\mathbf{y}}\partial n_{j\mathbf{y}'}}K_0(sr_{\mathbf{y}\mathbf{y}'})\right\}, \tag{4.10}$$

$${}^{*}K_{\gamma_j}(\mathbf{y}',\mathbf{y}) = \frac{1}{\pi}\left\{\frac{\partial^2}{\partial n_{j\mathbf{y}}\partial n_{j\mathbf{y}'}}K_0(sr_{\mathbf{y}\mathbf{y}'}) + \gamma_j\frac{\partial}{\partial n_{j\mathbf{y}}}K_0(sr_{\mathbf{y}\mathbf{y}'})\right\}, \tag{4.11}$$

and

$${}^{*}K_{\gamma_j^{-1}}(\mathbf{y},\mathbf{y}') = \frac{1}{\pi}\left\{K_0(sr_{\mathbf{y}\mathbf{y}'}) + \gamma_j^{-1}\frac{\partial}{\partial n_{j\mathbf{y}}}K_0(sr_{\mathbf{y}\mathbf{y}'})\right\}. \tag{4.12}$$

Similarly, we can find $\overline{\chi}(\mathbf{x}_1,\mathbf{x}_2;s^2)$ for the other three cases.

On the basis of (4.4) the function $\overline{\chi}(\mathbf{x}_1,\mathbf{x}_2;s^2)$ will be estimated for large values of s together with small/large impedances $\gamma_1,\dots,\gamma_m$. The case when $\mathbf{x}_1$ and $\mathbf{x}_2$ lie in the neighbourhood of the parts $\Gamma_1,\dots,\Gamma_m$ of $\partial\Omega$ is particularly interesting. To this end we shall use coordinates similar to those obtained in [4, 7] as follows:

5. Differential geometry of the boundary. Let n_j, $j=1,\dots,m$ be the minimum distances from a point $\mathbf{x}=(x^1,x^2)$ of the domain Ω to the parts Γ_j, $j=1,\dots,m$ of $\partial\Omega$, respectively. Letters $\mathbf{n}_j(\sigma_j)$, $j=1,\dots,m$ denote the inward drawn unit normals to Γ_j, $j=1,\dots,m$, respectively. We note that the coordinates in the neighbourhood of Γ_j and its diagrams are in the same form as in Sec. 3 of [4] with the interchanges $\sigma\leftrightarrow\sigma_j$, $n\leftrightarrow n_j$, $h\leftrightarrow h_j$, $I\leftrightarrow I_j$, $C(I)\leftrightarrow C(I_j)$ and $\delta\leftrightarrow\delta_j$. Thus, we have the same formulae (3.1)–(3.4) of Sec. 3 of [4] with the interchanges $c(\sigma)\leftrightarrow k_j(\sigma_j)$, $n\leftrightarrow n_j$, and $\mathbf{n}(\sigma)\leftrightarrow\mathbf{n}_j(\sigma_j)$.

6. Some local expansions. It now follows that the local expansions of the functions

$$K_0(sr_{\mathbf{x}\mathbf{y}}), \qquad \frac{\partial}{\partial n_{j\mathbf{y}}}K_0(sr_{\mathbf{x}\mathbf{y}}), \qquad j=1,\dots,m, \tag{6.1}$$

when the distance between $\mathbf{x}$ and $\mathbf{y}$ is small, are very similar to those obtained in Secs. 4, 5 of [4]. Consequently, for small/large impedances $\gamma_1,\dots,\gamma_m$ the local behaviour of the kernels

$$K_{\gamma_j}(\mathbf{y}',\mathbf{y}), \qquad {}^{*}K_{\gamma_j}(\mathbf{y}',\mathbf{y}), \tag{6.2}$$

$$K_{\gamma_j^{-1}}(\mathbf{y}',\mathbf{y}), \qquad {}^{*}K_{\gamma_j^{-1}}(\mathbf{y}',\mathbf{y}), \tag{6.3}$$

when the distance between $\mathbf{y}$ and $\mathbf{y}'$ is small, follows directly from the knowledge of the local expansions of the functions (6.1). This follows from the definition of e^{λ}-functions (see [4, 7]) in small domains $C(I_j)$, $j = 1, \ldots, m$. Thus, using methods similar to those obtained in Secs. 6–10 of [4], we can show that the functions (6.1) are e^{λ}-functions with degrees $\lambda = 0, -1$, respectively. Consequently, for small impedances γ_j the functions (6.2) are e^{λ}-functions with degrees $\lambda = 0, -1$ while for large impedances γ_j the functions (6.3) are e^{λ}-functions with degrees $\lambda = 0, 1$, respectively.

Definition. If $\mathbf{x}_1, \mathbf{x}_2$ are points in large domains $\Omega + \Gamma_j$, $j = 1, \ldots, m$, then we define

$$\hat{r}_{12} = \min_{\mathbf{y}}(r_{\mathbf{x}_1\mathbf{y}} + r_{\mathbf{x}_2\mathbf{y}}) \quad \text{if } \mathbf{y} \in \Gamma_j,\ j = 1, \ldots, k$$

and

$$\widehat{R}_{12} = \min_{\mathbf{y}}(r_{\mathbf{x}_1\mathbf{y}} + r_{\mathbf{x}_2\mathbf{y}}) \quad \text{if } \mathbf{y} \in \Gamma_j,\ j = k+1, \ldots, m.$$

An $E^{\lambda}(\mathbf{x}_1, \mathbf{x}_2; s)$-function is defined and infinitely differentiable with respect to $\mathbf{x}_1$ and $\mathbf{x}_2$ when these points belong to large domains $\Omega + \Gamma_j$ except when $\mathbf{x}_1 = \mathbf{x}_2 \in \Gamma_j$, $j = 1, \ldots, m$. Thus the E^{λ}-function has a similar local expansion of the e^{λ}-function (see [3, 4]).

By the help of Secs. 8, 9 in [4] it is easily seen that formula (4.4) is an $E^0(\mathbf{x}_1, \mathbf{x}_2; s)$-function and consequently

$$\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2) = \sum_{j=1}^{k} O\{[1 + |\log s\hat{r}_{12}|]e^{-A_j s\hat{r}_{12}}\} + \sum_{j=k+1}^{m} O\{[1 + |\log s\widehat{R}_{12}|]e^{-A_j s\widehat{R}_{12}}\} \quad (6.4)$$

which is valid for $s \to \infty$ and for small/large impendances $\gamma_1, \ldots, \gamma_m$ where $A_1, \ldots, A_m$ are positive constants. Formula (6.4) shows that $\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2)$ is exponentially small for $s \to \infty$. Similar statements are true in the other three cases.

With reference to Sec. 10 in [4], if the e^{λ}-expansions of the functions (6.1)–(6.3) are introduced into (4.4) and if we use formulae similar to (6.4), (6.9) of Sec. 6 in [4], we obtain the following local behaviour of $\overline{\chi}(\mathbf{x}_1, \mathbf{x}_2; s^2)$ when $\hat{r}_{12}$ or $\widehat{R}_{12}$ is small which is valid for $s \to \infty$ and for small/large impedances $\gamma_1, \ldots, \gamma_m$:

$$\overline{\chi}(\mathbf{x}_1, \mathbf{x}_2; s^2) = \sum_{j=1}^{m} \overline{\chi}_j(\mathbf{x}_1, \mathbf{x}_2; s^2), \quad (6.5)$$

where if $\mathbf{x}_1, \mathbf{x}_2$ belong to sufficiently small domains $C(I_j)$, $j = 1, \ldots, k$, then

$$\overline{\chi}_j(\mathbf{x}_1, \mathbf{x}_2; s^2) = \frac{1}{2\pi}\left\{1 - \gamma_j\left(\frac{\partial}{\partial \xi_1^2}\right)^{-1}\right\} K_0(s\hat{\rho}_{12}) + O(s^{-1}e^{-A_j s\hat{\rho}_{12}}), \quad (6.6)$$

while, if $\mathbf{x}_1, \mathbf{x}_2$ belong to sufficiently small domain $C(I_j)$, $j = k+1, \ldots, m$, then

$$\overline{\chi}_j(\mathbf{x}_1, \mathbf{x}_2; s^2) = -\frac{1}{2\pi}\left\{1 - \gamma_j^{-1}\left(\frac{\partial}{\partial \xi_1^2}\right)\right\} K_0(s\hat{\rho}_{12}) + O(s^{-1}e^{-A_j s\hat{\rho}_{12}}). \quad (6.7)$$

When $\hat{r}_{12} \geq \delta_j$, $j = 1, \ldots, k$ or $\widehat{R}_{12} \geq \delta_j$, $j = k+1, \ldots, m$ the function $\overline{\chi}(\mathbf{x}_1, \mathbf{x}_2; s^2)$ is of order $O(e^{-Ns})$ as $s \to \infty$, $N > 0$. Thus, since $\lim(\hat{r}_{12}/\hat{\rho}_{12}) = 1$ or $\lim(\widehat{R}_{12}/\hat{\rho}_{12}) = 1$

(see [4]) when $\hat{r}_{12}$ or $\widehat{R}_{12}$ tends to zero, then we have the asymptotic formulae (6.6) and (6.7) with $\hat{\rho}_{12}$ in the small domains cases being replaced by $\hat{r}_{12}$ or $\widehat{R}_{12}$ in the large domains $\Omega + \Gamma_j$, $j = 1, \ldots, m$. Similar formulae for the other three cases can be found.

7. Construction of our results. Since for $\xi^2 \geq h_j > 0$, $j = 1, \ldots, m$ the functions $\overline{\chi}_j(\mathbf{x}, \mathbf{x}; s^2)$ are of order $O(e^{-2sA_jh_j})$, the integral of the function $\overline{\chi}(\mathbf{x}, \mathbf{x}; s^2)$ over the domain Ω can be approximated in the following way (see (3.10)):

$$\overline{K}(s^2) = \sum_{j=1}^{m} \int_{\xi^2=0}^{h_j} \int_{\xi^1=0}^{L_j} \overline{\chi}_j(\mathbf{x}, \mathbf{x}; s^2)\{1 - k_j(\xi^1)\xi^2\}\, d\xi^1\, d\xi^2 + \sum_{j=1}^{m} O(e^{-2sA_jh_j}) \quad \text{as } s \to \infty. \tag{7.1}$$

If the e^{λ}-expansions of $\overline{\chi}_j(\mathbf{x}, \mathbf{x}; s^2)$, $j = 1, \ldots, m$ are introduced into (7.1), one obtains an asymptotic series of the form

$$\overline{K}(s^2) = \sum_{n=1}^{P} a_n s^{-n} + O(s^{-P-1}) \quad \text{as } s \to \infty, \tag{7.2}$$

where the coefficients a_n for all four cases are calculated from the e^{λ}-expansions by the help of formula (11.3) of Sec. 11 in [4].

Finally, on inverting Laplace transforms and using (3.6) we arrive at our results (2.1)–(2.3).

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