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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF INHOMOGENEOUS SECOND-ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. This paper is concerned with the asymptotic behavior of solutions of a class of inhomogeneous second-order quasilinear partial differential equations in two independent variables defined over rectangular plane domains whose lengths greatly exceed their widths. Solutions to a Dirichlet problem for such equations are shown to be well approximated, away from the ends of the rectangle, by solutions to the corresponding one-dimensional problem for an ordinary differential equation on the cross-section of the rectangle. Applications to problems in geometry and nonlinear continuum mechanics are discussed.

1. Introduction. In this paper, we are concerned with investigating the asymptotic behavior of solutions of a class of inhomogeneous second-order quasilinear partial differential equations in two independent variables defined over rectangular plane domains whose lengths greatly exceed their widths. For such long 'thin' domains, one would anticipate that solutions to partial differential equations of this class might be well approximated by solutions to the corresponding one-dimensional problem for an ordinary differential equation on the cross-section of the rectangle. Such an approximation might be expected to be accurate sufficiently far away from the ends of the rectangle.

Questions of this kind have long been of concern in elasticity theory, where they arise in connection with using one-dimensional theories as approximations to two-dimensional theories. Similar issues arise in connection with Saint-Venant's principle, governing the spatial decay of end effects (see, e.g., [1, 2] for recent reviews).

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Here we are concerned with second-order quasilinear equations in two independent variables of the form

$$[\rho(q^2)u,_\alpha],_\alpha + 2k = 0, \qquad (q^2 \equiv u,_\beta u,_\beta) \tag{1.1}$$

where $k > 0$ is a constant. In (1.1) the usual summation convention is employed with subscripts preceded by a comma denoting partial differentiation with respect to the corresponding Cartesian coordinate.

Equations of the form (1.1) occur in geometry as well as in problems of nonlinear continuum mechanics. Thus, for example, if

$$\rho = (1 + q^2)^{-1/2}, \tag{1.2}$$

(1.1) is the equation of a surface of constant mean curvature k. This equation also occurs in a theory of torsional creep [3, 4]. Quasilinear equations of the form (1.1) also occur in nonlinear elasticity theory concerning deformations governed by states of finite anti-plane shear (see, e.g., [5, 6]). In this context, ρ is determined by the constitutive model governing material behavior, and $2k$ is the constant body force. A commonly used constitutive model gives rise to functions ρ of power-law form

$$\rho = \mu(1 + bq^2/n)^{n-1}, \qquad \mu, b, n > 0. \tag{1.3}$$

The case $n = 1$ in (1.3) corresponds to the neo-Hookean material for which ρ is constant and (1.1) is Poisson's equation. When $n = \frac{1}{2}$ in (1.3), Eq. (1.1) is reducible, by a change of scale, to the equation of a surface of constant mean curvature.

We consider a Dirichlet problem for (1.1) on the rectangular region $R^* = \{(x_1, x_2) \mid 0 < x_1 < 2l,\ 0 < x_2 < h\}$, where $l \gg h$. On the long sides of the rectangle, we have homogeneous boundary conditions so that

$$u(x_1, 0) = u(x_1, h) = 0, \qquad 0 \le x_1 \le 2l, \tag{1.4}$$

while the data at the ends $x_1 = 0$, $x_1 = 2l$ are assumed to be symmetrically distributed so that

$$u(0, x_2) = f(x_2), \qquad u(2l, x_2) = f(x_2), \qquad 0 \le x \le h, \tag{1.5}$$

where the prescribed function f is sufficiently smooth and satisfies $f(0) = f(h) = 0$. In view of the form of the differential equation (1.1) and the symmetry of the data in (1.5), we conclude that if (1.1) is elliptic, then the solution $u(x_1, x_2)$ is such that $u(x_1, x_2) = u(2l - x_1, x_2)$ so that

$$u,_1 = 0 \quad \text{on } x_1 = l, \qquad 0 \le x_2 \le h. \tag{1.6}$$

Furthermore, sufficiently far away from the ends $x_1 = 0$, $x_1 = 2l$, one might expect $u(x_1, x_2)$ to be well approximated by the solution $v(x_2)$ of the one-dimensional problem

$$\frac{d}{dx_2}\left[\rho(p^2)\frac{dv}{dx_2}\right] + 2k = 0, \qquad 0 \le x_2 \le h, \tag{1.7}$$

$$v(0) = 0, \qquad v(h) = 0, \tag{1.8}$$

where $p^2 \equiv (dv/dx_2)^2$. To investigate this issue, it is sufficient to confine attention to the half-rectangle $R = \{(x_1, x_2) \mid 0 < x_1 < l,\ 0 < x_2 < h\}$ and so we consider solutions of (1.1) on R subject to the boundary conditions

$$u(x_1, 0) = u(x_1, h) = 0, \qquad 0 \le x_1 \le l, \tag{1.9}$$

$$u_{,1} = 0 \quad \text{on } x_1 = l, \qquad 0 \le x_2 \le h, \tag{1.10}$$

$$u(0, x_2) = f(x_2), \qquad 0 \le x_2 \le h, \tag{1.11}$$

where $l \gg h$. We shall examine the spatial evolution of $u(x_1, x_2)$ to the solution $v(x_2)$ of the one-dimensional problem (1.7), (1.8). We assume the existence of classical solutions $u \in C^2(R) \cap C^1(\overline{R})$ of the problem (1.1), (1.9)–(1.11). For an extensive discussion of the existence issue for general second-order quasilinear elliptic equations, see, e.g., Serrin [7].

2. Energy decay estimates. In this section, we establish an energy-decay estimate for solutions of (1.1), (1.9)–(1.11). We show that the energy measure

$$E(z) = \int_{R_z} \rho(q^2) w_{,\alpha} w_{,\alpha}\, dA, \qquad (q^2 = u_{,\alpha} u_{,\alpha}), \tag{2.1}$$

contained in the subdomain $R_z = \{(x_1, x_2) \mid 0 \le z < x_1 < l,\ 0 < x_2 < h\}$ has exponential decay in z, where the function $w(x_1, x_2)$ is defined by

$$w(x_1, x_2) = u(x_1, x_2) - v(x_2), \tag{2.2}$$

and v is the solution to the one-dimensional problem (1.7), (1.8). Thus $E(z)$ can be viewed as a "weighted energy" associated with the difference (2.2) between $u(x_1, x_2)$ and $v(x_2)$.

The hypotheses that we make concerning ρ may be conveniently separated into two cases. In this section we shall be concerned with generalizations of equations of surfaces of constant mean curvature where ρ is given by (1.2). Thus we consider ρ for which $\rho^{-1} \ge m_2 > 0$, where m_2 is a constant. In Sec. 4, where a modified energy norm is used, we shall be concerned with equations of the form (1.1) where $\rho \ge m_1 > 0$, m_1 constant. To conform with terminology introduced in our previous work [8, 9, 10], we will designate by Case 2 the class of functions ρ to be considered in this section, while the functions ρ considered in Sec. 4 will be described by Case 1.

It is assumed henceforth in this section that there exist positive constants m_2, c_2, d_2 such that for all values of the arguments s, t we have

Case 2.

$$\rho^{-1} \ge m_2 > 0, \tag{2.3}$$

$$|\rho^{-1}(s^2) - \rho^{-1}(t^2)|\, |t| \rho(t^2) \le c_2 |t - s|, \qquad 0 < c_2 < 1, \tag{2.4}$$

and

$$|\rho^{-1}(s^2) - \rho^{-1}(t^2)| [\rho(t^2) + \rho(s^2)] \le d_2 \rho(t^2) \rho(s^2) |s^2 - t^2|. \tag{2.5}$$

In our applications, the argument t will be taken to be p, where $p = |dv/dx_2|$, and c_2, d_2 will, in general, depend on the maximum value of p on $[0, h]$. It will be seen in Sec. 3 that the equation of constant mean curvature ((1.1) with ρ given by (1.2)) satisfies these hypotheses. It should be noted that (2.3)–(2.5) do *not* require that Eq. (1.1) be elliptic, that is, $\rho + 2\rho' q^2 \geq 0$ ($\rho' \equiv d\rho/dq^2$) for all solutions u and at all points of R, even though the issues of concern here are of interest primarily for elliptic equations.

We now proceed to establish that, for functions ρ satisfying (2.3)–(2.5), the exponential decay estimate

$$E(z) \leq K_2 e^{-2(\pi/h)\nu_2 z}, \qquad 0 \leq z \leq l, \tag{2.6}$$

holds. Here $\nu_2 = \nu_2(p)$ is a function of p (to be determined) which depends on the form of ρ. The estimated rate of energy decay is thus $2\pi\nu_2/h$. The constant $K_2 = K_2(E_0, m_2, c_2, d_2, p)$ depends on the quantity $E_0 \equiv E(0)$, which is the total energy contained in the rectangle $R \equiv R_0$. It is shown in Sec. 5 how an upper bound for E_0 in terms of geometry, boundary-data, ρ, and v may be obtained.

The result (2.6) is established in two stages. First we derive the differential inequality

$$E'(z) + 2\kappa_2(z)E(z) \leq 0, \qquad z \geq 0, \tag{2.7}$$

where the prime denotes differentiation with respect to z. Here

$$\kappa_2(z) = \frac{m_2\pi(1 - c_2)}{B_2(z)} \tag{2.8}$$

and

$$B_2(z) = \int_0^h \rho^{-1}(q^2)\, dx_2. \tag{2.9}$$

The inequality (2.7) may be immediately integrated to yield

$$E(z) \leq E_0 \exp\left[-2\int_0^z \kappa_2(s)\, ds\right], \qquad z \geq 0. \tag{2.10}$$

The second stage of the proof consists of using the hypotheses (2.3), (2.5) to obtain a bound for the exponential term in (2.10) leading to the result (2.6).

To establish (2.7) we proceed as follows: If L_z denotes the line segment $x_1 = z$, $0 \leq x_2 \leq h$, we find, on using the divergence theorem and (1.1), (1.9), (1.10), (1.7) that

$$E(z) = -\int_{L_z} \rho(q^2) w w_{,1}\, dx_2 + \int_{R_z} [\rho(p^2) - \rho(q^2)] v_{,\beta} w_{,\beta}\, dA. \tag{2.11}$$

Denoting the second integral on the right in (2.11) by I, we use the hypothesis (2.4) with $s = q$, $t = p$, to obtain

$$I \leq c_2 \int_{R_z} |p - q| \rho(q^2) \left|\frac{v_{,\beta} w_{,\beta}}{p}\right| dA. \tag{2.12}$$

Now, since $v = v(x_2)$, we have

$$p^2 = v_{,2}^2 = v_{,\beta} v_{,\beta}, \qquad pq = (v_{,\beta} v_{,\beta})^{1/2}(u_{,\alpha} u_{,\alpha})^{1/2} \geq v_{,\beta} u_{,\beta}, \tag{2.13}$$

and so

$$|p - q| = (p^2 - 2pq + q^2)^{1/2} \leq (v_{,\beta} v_{,\beta} - 2v_{,\beta} u_{,\beta} + u_{,\beta} u_{,\beta})^{1/2} = (w_{,\beta} w_{,\beta})^{1/2}. \tag{2.14}$$

Thus (2.12) yields

$$I \leq c_2 \int_{R_z} \rho(q^2) w_{,\beta} w_{,\beta} \, dA, \tag{2.15}$$

which, by virtue of (2.1), can be written as

$$I \leq c_2 E(z). \tag{2.16}$$

Thus, (2.11), (2.16) show that

$$(1 - c_2)E(z) \leq -\int_{L_z} \rho(q^2) w w_{,1} \, dx_2, \tag{2.17}$$

$$\leq \left(\int_{L_z} \rho(q^2) w_{,1}^2 \, dx_2 \right)^{1/2} \left(\int_{L_z} \rho(q^2) w^2 \, dx_2 \right)^{1/2}, \tag{2.18}$$

on using Schwarz's inequality.

We now obtain an upper bound for the integral

$$J \equiv \int_{L_z} \rho(q^2) w^2 \, dx_2 \tag{2.19}$$

appearing on the right hand side of (2.18). Following a scheme introduced in [8], we make the change of variable, for fixed x_1,[1]

$$t = \int_0^{x_2} \rho^{-1}(x_1, \eta) \, d\eta, \tag{2.20}$$

so that $dt = \rho^{-1} \, dx_2$. It follows from (2.8), (2.9), and (2.2) that $w(0) = 0$, $w(B_2) = 0$, (where B_2 is given by (2.9) as follows from (2.20)), and so we have the well-known inequality (see, e.g., [11, p. 185])

$$\int_0^{B_2} w^2 \, dt \leq \frac{B_2^2}{\pi^2} \int_0^{B_2} \left(\frac{dw}{dt} \right)^2 dt. \tag{2.21}$$

But $dw/dt = w_{,2} \, dx_2/dt = \rho w_{,2}$ and so (2.21) yields

$$\int_{L_z} \rho w_{,2}^2 \, dx_2 \geq \frac{\pi^2}{B_2^2(z)} \int_{L_z} \rho^{-1} w^2 \, dx_2. \tag{2.22}$$

Since $\rho^{-1} \geq m_2 > 0$, by virtue of (2.3), we deduce from (2.22), (2.19) that

$$J \leq \frac{B_2^2}{m_2^2 \pi^2} \int_{L_z} \rho(q^2) w_{,2}^2 \, dx_2. \tag{2.23}$$

On inserting (2.23) into (2.18) and using the arithmetic-geometric mean inequality, we obtain

$$(1 - c_2)E(z) \leq \frac{B_2}{2m_2\pi} \int_{L_z} \rho(q^2) w_{,\alpha} w_{,\alpha} \, dx_2, \tag{2.24}$$

and so, by (2.1),

$$(1 - c_2)E(z) \leq \frac{B_2[-E'(z)]}{2m_2\pi}. \tag{2.25}$$

This establishes (2.7), and so (2.10), as desired.

[1]Here we regard ρ^{-1} evaluated on a solution u as a function of x_1 and x_2.

We now show that (2.10) yields the following decay estimate

$$E(z) \leq E_0 \exp\left\{\frac{-2\pi m_2(1-c_2)z^2}{\int_0^z B_2(s)\,ds}\right\}. \tag{2.26}$$

The result (2.26) is an immediate consequence of (2.10) and use of the inequality

$$\int_0^z B_2^{-1}(s)\,ds \geq \frac{z^2}{\int_0^z B_2(s)\,ds}, \tag{2.27}$$

which in turn follows from Schwarz's inequality (see [8, pp. 315–316]).

The final step in establishing (2.6) is to use the hypothesis (2.3), (2.5) to obtain an upper bound for the quantity $\int_0^z B_2(s)\,ds \equiv G_2(z)$ appearing in (2.26). It is only at this stage that the third hypothesis (2.5) in Case 2 is used. We have

$$G_2(z) = \int_0^z B_2(s)\,ds = \int_0^z \int_0^h \rho^{-1}(q^2)\,dA, \tag{2.28}$$

which we write as

$$G_2(z) = \int_0^z \int_0^h [\rho^{-1}(q^2) - \rho^{-1}(p^2)]\,dA + \int_0^z \int_0^h \rho^{-1}(p^2)\,dA. \tag{2.29}$$

On using (2.5), with $s = q$, $t = p$, we obtain

$$G_2(z) = d_2 \int_0^z \int_0^h \frac{\rho(p^2)\rho(q^2)}{[\rho(p^2) + \rho(q^2)]}|q^2 - p^2|\,dA + \int_0^z \int_0^h \rho^{-1}(p^2)\,dA \tag{2.30}$$

$$= d_2 \int_0^z \int_0^h \frac{|w_{,\alpha} w_{,\alpha} + 2v_{,\alpha} w_{,\alpha}|}{[\rho^{-1}(p^2) + \rho^{-1}(q^2)]}\,dA + \int_0^z \int_0^h \rho^{-1}(p^2)\,dA. \tag{2.31}$$

Using the arithmetic-geometric mean inequality in the form

$$2ab \leq \gamma a^2 + \frac{1}{\gamma}b^2, \qquad (\gamma > 0), \tag{2.32}$$

we obtain from (2.31)

$$\begin{aligned} G_2(z) \leq{}& d_2(1+\gamma)\int_0^z \int_0^h \frac{w_{,\alpha} w_{,\alpha}\,dA}{[\rho^{-1}(p^2) + \rho^{-1}(q^2)]} \\ &+ \frac{d_2}{\gamma}\int_0^z \int_0^h \frac{p^2\,dA}{[\rho^{-1}(p^2) + \rho^{-1}(q^2)]} + \int_0^z \int_0^h \rho^{-1}(p^2)\,dA. \end{aligned} \tag{2.33}$$

Since $\rho^{-1} > 0$, we drop $\rho^{-1}(p^2)$ in the first integral, $\rho^{-1}(q^2)$ in the second integral in (2.33) and obtain

$$\begin{aligned} G_2(z) &\leq d_2(1+\gamma)\int_0^z \int_0^h \rho(q^2) w_{,\alpha} w_{,\alpha}\,dA + \left\{\int_0^h \left[\frac{d_2}{\gamma}p^2\rho(p^2) + \rho^{-1}(p^2)\right]dx_2\right\} z \\ &\leq d_2(1+\gamma)E_0 + F(d_2, p^2, \gamma)hz, \qquad \gamma > 0, \end{aligned} \tag{2.34}$$

where the definition (2.1) has been used to obtain the last inequality in (2.34) and we have introduced the notation

$$F(d_2, p^2, \gamma) \equiv \frac{1}{h}\int_0^h \left[\frac{d_2 p^2}{\gamma}\rho(p^2) + \rho^{-1}(p^2)\right]dx_2. \tag{2.35}$$

The inequality (2.34), valid for arbitrary $\gamma > 0$, is the desired upper bound for $G_2(z)$.

To use (2.34) in (2.26), we write

$$\left(\int_0^z B_2(s)\,ds\right)^{-1} \equiv G_2^{-1}(z) \geq \{d_2(1+\gamma)E_0 + Fhz\}^{-1} \tag{2.36}$$

so that

$$z^2 G_2^{-1}(z) \geq \frac{z}{Fh}\left\{1 + \frac{d_2(1+\gamma)E_0}{Fhz}\right\}^{-1} \geq \frac{z}{Fh} - \frac{d_2(1+\gamma)E_0}{F^2h^2}. \tag{2.37}$$

On substitution from (2.37) into (2.26) we thus obtain the result

$$E(z) \leq K_2 e^{-2(\pi/h)\nu_2 z}, \qquad z \geq 0, \tag{2.38}$$

where

$$\nu_2 = \nu_2(p) = \frac{m_2(1-c_2)}{F(d_2, p^2, \gamma)}, \tag{2.39}$$

and

$$\begin{aligned} K_2 &= K_2(E_0, m_2, c_2, d_2, p) \\ &= E_0 \exp\left[\frac{2m_2\pi E_0 d_2(1-c_2)(1+\gamma)}{F^2(d_2, p^2, \gamma)h^2}\right]. \end{aligned} \tag{2.40}$$

Here $F(d_2, p^2, \gamma)$ is given by (2.35), $\gamma > 0$ is an arbitrary parameter and m_2, c_2, d_2 are the constants appearing in the hypotheses (2.3), (2.4), (2.5), respectively. This completes the proof of the decay estimate (2.6) in Case 2.

3. Discussion of results in Case 2. We now specialize the results of Sec. 2 to the case when

$$\rho = (1+q^2)^{-1/2} \tag{3.1}$$

so that (1.1) is the equation of a surface of constant mean curvature k. It is shown in the Appendix that when ρ is given by (3.1), the hypotheses (2.3)–(2.5) hold for $s = q$, $t = p$ if the constants m_2, c_2, d_2 are chosen to be

$$m_2 = 1, \qquad c_2 = p_m(1+p_m^2)^{-1/2}, \qquad d_2 = 1. \tag{3.2}$$

Here $p_m \equiv \max_{[0,h]} p$, and we recall that $p^2 = (dv/dx_2)^2$ where $v(x_2)$ is the solution of the one-dimensional problem (1.7), (1.8). For finite $p_m > 0$, we see from (3.2) that $0 < c_2 < 1$ as required in (2.4).

When ρ is given by (3.1), the solution $v(x_2)$ of (1.7), (1.8) may be found explicitly. Thus, (1.7) may be integrated once to yield (we write $v' = dv/dx_2$)

$$\rho(p^2)v' = -2kx_2 + \text{const}, \tag{3.3}$$

and so, when ρ has the form (3.1), we find from (3.3) that

$$\frac{v'}{\sqrt{1+p^2}} = k(h - 2x_2). \tag{3.4}$$

The constant of integration in (3.3) has been written down in (3.4) to conform with the symmetry of $v(x_2)$ about $x_2 = h/2$. On squaring both sides of (3.4) and solving for v' we obtain

$$v' = \frac{k(h-2x_2)}{\sqrt{1-k^2(h-2x_2)^2}}, \tag{3.5}$$

which is bounded and real if and only if

$$m \equiv kh < 1. \tag{3.6}$$

Thus, we see that a solution to the one-dimensional problem exists only if the curvature k is restricted to satisfy (3.6). It is well known that a solution of the Dirichlet problem (1.1), (1.4), (1.5) (with ρ given by (3.1)) will not exist unless the boundary data are suitably restricted and a geometric condition similar to (3.6) holds. For instance, it follows from a result of Bernstein [12] that no solution can exist if $kh > 2$. For a general discussion of existence criteria, see, e.g., Serrin [7].

Assuming that (3.6) holds, we integrate (3.5) to find

$$v = \left(\frac{1}{2k}\right)\left\{\sqrt{1 - k^2(h - 2x_2)^2} - \sqrt{1 - m^2}\right\}. \tag{3.7}$$

Recalling that $p = |v'|$, we obtain from (3.5) that

$$p_m = \max_{[0,h]} p = m(1 - m^2)^{-1/2}. \tag{3.8}$$

Thus, from (3.2), we obtain

$$c_2 = m \qquad (= kh). \tag{3.9}$$

By virtue of (3.6), we see that $0 < c_2 < 1$, as required by (2.4).

Next we evaluate the function $F(d_2, p^2, \gamma)$ given by (2.35). With ρ given by (3.1), we find, using (3.5), that

$$\int_0^h p^2 \rho(p^2)\, dx_2 = \frac{1}{2k}\{\sin^{-1}(m) - m(1 - m^2)^{1/2}\}, \tag{3.10}$$

$$\int_0^h \rho^{-1}(p^2)\, dx_2 = \frac{1}{k}\sin^{-1}(m), \tag{3.11}$$

so that, since $d_2 = 1$, we have

$$F(d_2, p^2, \gamma) = \frac{1}{2m\gamma}\{\sin^{-1}(m) - m(1 - m^2)^{1/2}\} + \frac{1}{m}\sin^{-1}(m) \equiv \mathcal{F}(m, \gamma). \tag{3.12}$$

Thus, by virtue of (3.2), (3.9), we obtain from (2.39) that the estimated decay rate ν_2 appearing in (2.38) is given by

$$\nu_2 = \nu_2(m) = (1 - m)/\mathcal{F}(m, \gamma). \tag{3.13}$$

The multiplicative constant K_2 appearing in (2.38), given by (2.40), now can be written as

$$K_2 = E_0 \exp\left[\frac{2\pi E_0(1 - m)(1 + \gamma)}{\mathcal{F}^2(m, \gamma)h^2}\right]. \tag{3.14}$$

The role of the arbitrary constant $\gamma > 0$, still to be chosen, is now clear. If one seeks to maximize the estimated decay rate $\nu_2(m)$ in (3.13), one would choose $\gamma = \varepsilon^{-1}$, $\varepsilon \ll 1$. As $\varepsilon \to 0$,

$$\nu_2(m) \to \frac{(1 - m)m}{\sin^{-1}(m)}. \tag{3.15}$$

However, the multiplicative constant K_2 then has the asymptotic form $O(\varepsilon^{-1})$ as $\varepsilon \to 0$. On the other hand, to minimize K_2 one would choose $\gamma = \varepsilon$, $\varepsilon \ll 1$ so that $K_2 \to E_0$ as $\varepsilon \to 0$. For this choice of γ however, (3.13) shows that $\nu_2 = O(\varepsilon)$ as $\varepsilon \to 0$. Of course, γ could be taken to be $\gamma = 1$ (so that (2.32) is the simplest form of the arithmetic-geometric mean inequality), preserving a balance between the estimated decay rate ν_2 and the multiplicative factor K_2 in the estimate (2.38).

It is of interest to consider the limiting behavior of the estimate (2.38), (3.13), (3.14) as $k \to 0$. As $k \to 0$ in (1.1), with ρ given by (3.1), Eq. (1.1) becomes the minimal surface equation. As $k \to 0$, (3.5) shows that $p = |v'| \to 0$, and (3.7) shows that $v \to 0$. On letting $m = kh \to 0$ formally in (3.15), we find

$$\nu_2 \to \nu_2(0) = 1, \tag{3.16}$$

and from (3.14),

$$K_2 \to E_0 \exp\left(\frac{2\pi E_0(1+\gamma)}{h^2}\right). \tag{3.17}$$

Thus, in the limit as $k \to 0$, (2.38) provides the estimate

$$E(z) \le E_0 \exp\left(\frac{2\pi E_0(1+\gamma)}{h^2}\right) e^{-2\pi z/h}, \tag{3.18}$$

(where γ can be chosen to be arbitrarily small) for solutions of the minimal surface equation on a rectangular domain subject to the boundary conditions (1.9)–(1.11). Since $v \to 0$ as $k \to 0$, $E(z)$ has the form

$$E(z) = \int_{R_z} q^2(1+q^2)^{1/2}\, dA. \tag{3.19}$$

A result of the form (3.18) for the minimal surface equation on a semi-infinite strip was established in [8]. A pointwise decay estimate, with the same estimated decay rate, was obtained earlier in [13] using arguments based on maximum principles. It is shown in [14] that this decay rate is, in fact, optimal.

4. Decay estimate in Case 1. We turn now to solutions of (1.1) where the function ρ is such that $\rho \ge m_1 > 0$, m_1 constant. A simple example is $\rho = (1+q^2)^{1/2}$, where we can choose $m_1 = 1$. The class of equations considered will be such that for all values of the arguments s, t there exist positive constants m_1, d_1, and a constant c_1 such that

Case 1.

$$\rho \ge m_1 > 0, \tag{4.1}$$

$$-[\rho(s^2) - \rho(t^2)](s^2 - t^2) \le c_1[\rho(s^2) + \rho(t^2)](s-t)^2, \tag{4.2}$$

and

$$|\rho(s^2) - \rho(t^2)| \le d_1[\rho(s^2) + \rho(t^2)]|s^2 - t^2|. \tag{4.3}$$

In treating Case 1, we find it convenient to use an energy measure different from that employed in Sec. 2 for Case 2. Thus we consider the functional

$$\tilde{E}(z) = \int_{R_z} [\rho(q^2) + \rho(p^2)] w_{,\alpha} w_{,\alpha}\, dA, \tag{4.4}$$

where w is still given by (2.2). We will now show that, for functions ρ satisfying (4.1)–(4.3), the exponential decay estimate

$$\tilde{E}(z) \le \tilde{K} e^{-2(\pi/h)\tilde{\nu} z}, \qquad z \ge 0, \tag{4.5}$$

holds. Here $\tilde{\nu} = \tilde{\nu}(p)$ and $\tilde{K} = \tilde{K}(\tilde{E}_0, m_1, c_1, d_1, p)$, where $\tilde{E}_0 = \tilde{E}(0)$ is the total energy contained in the rectangle R.

The proof of (4.5) is similar to that of (2.6) in Sec. 2. Thus we first show that

$$\tilde{E}'(z) + 2\kappa(z)\tilde{E}(z) \leq 0, \tag{4.6}$$

where

$$\kappa(z) = m_1\pi(1 - c_1)/2B(z), \tag{4.7}$$

and

$$B(z) = \int_0^h \rho(q^2)\, dx_2. \tag{4.8}$$

The inequality (4.6) may then be integrated to yield

$$\tilde{E}(z) \leq \tilde{E}_0 \exp\left[-2\int_0^z \kappa(s)\, ds\right], \qquad z \geq 0. \tag{4.9}$$

On using the divergence theorem, and (1.1), (1.7)–(1.10), we have

$$\tilde{E}(z) = -\int_{L_z} \rho(q^2) w w_{,1}\, dx_2 + \int_{R_z} [\rho(p^2)u_{,\alpha} - \rho(q^2)v_{,\alpha}]w_{,\alpha}\, dA, \tag{4.10}$$

which may be written as

$$\begin{aligned}\tilde{E}(z) = &-\int_{L_z} \rho(q^2) w w_{,1}\, dx_2 + \tfrac{1}{2}\int_{R_z} [\rho(q^2) + \rho(p^2)]w_{,\alpha} w_{,\alpha}\, dA \\ &+ \tfrac{1}{2}\int_{R_z} [\rho(p^2) - \rho(q^2)](u_{,\alpha} + v_{,\alpha})(u_{,\alpha} - v_{,\alpha})\, dA,\end{aligned} \tag{4.11}$$

on recalling from (2.2) that $w = u - v$. We now employ the hypothesis (4.2), with $s = q, t = p$, to obtain an upper bound for the third integral on the right in (4.11) and so find

$$\tilde{E}(z) \leq -\int_{L_z} \rho(q^2) w w_{,1}\, dx_2 + \frac{\tilde{E}(z)}{2} + \frac{c_1}{2}\int_{R_z} [\rho(q^2) + \rho(p^2)](q - p)^2\, dA, \tag{4.12}$$

where the definition (4.4) has also been used. Making use of the inequality (2.14) in the third integral on the right in (4.12) and recalling (4.4), we obtain

$$\begin{aligned}\tilde{E}(z) &\leq -2(1 - c_1)^{-1}\int_{L_z} \rho(q^2) w w_{,1}\, dx_2, \\ &\leq 2(1 - c_1)^{-1}\left(\int_{L_z} \rho(q^2) w^2\, dx_2\right)^{1/2}\left(\int_{L_z} \rho(q^2) w_{,1}^2\, dx_2\right)^{1/2},\end{aligned} \tag{4.13}$$

on using Schwarz's inequality.

An upper bound for the integral

$$\tilde{J} \equiv \int_{L_z} \rho(q^2) w^2\, dx_2 \tag{4.14}$$

appearing on the right hand side of (4.13) is now obtained. We make the change of variable, for fixed x_1,

$$\sigma = \int_0^{x_2} \rho(x_1, \eta)\, d\eta \tag{4.15}$$

so that $d\sigma = \rho(q^2)\,dx_2$. Since $w(0) = 0$, $w(B) = 0$ (where B is given by (4.8) as follows from (4.15) when $x_2 = h$), we have

$$\int_0^B w^2\,d\sigma \le \frac{B^2}{\pi^2}\int_0^B \left(\frac{dw}{d\sigma}\right)^2 d\sigma, \tag{4.16}$$

which can be written as

$$\tilde{J} = \int_{L_z} \rho(q^2) w^2\,dx_2 \le \frac{B^2}{\pi^2}\int_{L_z} \rho^{-1}(q^2) w_{,2}^2\,dx_2. \tag{4.17}$$

Since $\rho \ge m_1 \ge 0$, we deduce from (4.17) that

$$\tilde{J} \le \frac{B^2}{m_1^2\pi^2}\int_{L_z} \rho(q^2) w_{,2}^2\,dx_2. \tag{4.18}$$

On inserting (4.18) in (4.13) and using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}\tilde{E}(z) &\le (1-c_1)^{-1}\frac{B}{m_1\pi}\int_{L_z} \rho(q^2) w_{,\alpha} w_{,\alpha}\,dx_2 \\ &\le (1-c_1)^{-1}\frac{B}{m_1\pi}\int_{L_z} [\rho(q^2)+\rho(p^2)] w_{,\alpha} w_{,\alpha}\,dx_2,\end{aligned} \tag{4.19}$$

the last inequality following from the fact that $\rho(p^2) > 0$. Thus by (4.4), we write (4.19) as

$$\tilde{E}(z) \le \frac{(1-c_1)^{-1} B[-\tilde{E}'(z)]}{m_1\pi}. \tag{4.20}$$

This establishes (4.6), and so (4.9), as desired.

From (4.9), (4.7) and an obvious analog of (2.27) we thereby obtain

$$\tilde{E}(z) \le \tilde{E}_0\left\{\frac{-m_1\pi(1-c_1)z^2}{\int_0^z B(s)\,ds}\right\}. \tag{4.21}$$

It remains to find an upper bound for

$$G(z) \equiv \int_0^z B(s)\,ds = \int_0^z \int_0^h \rho(q^2)\,dA. \tag{4.22}$$

We write

$$G(z) = \int_0^z \int_0^h [\rho(q^2) - \rho(p^2)]\,dA + \int_0^z \int_0^h \rho(p^2)\,dA, \tag{4.23}$$

and so, on using (4.3) with $s = q$, $t = p$, we obtain

$$G(z) \le d_1 \int_0^z \int_0^h [\rho(q^2)+\rho(p^2)]|q^2 - p^2|\,dA + \int_0^z \int_0^h \rho(p^2)\,dA \tag{4.24}$$

$$\begin{aligned}&= d_1 \int_0^z \int_0^h [\rho(q^2)+\rho(p^2)]|w_{,\alpha} w_{,\alpha} + 2v_{,\alpha} w_{,\alpha}|\,dA \\ &\quad + \int_0^z \int_0^h \rho(p^2)\,dA.\end{aligned} \tag{4.25}$$

Using the arithmetic-geometric mean inequality in the form (2.32), we obtain from (4.25)

$$G(z) \leq d_1(1+\gamma)\int_0^z\int_0^h [\rho(q^2)+\rho(p^2)]w_{,\alpha}w_{,\alpha}\,dA + \frac{d_1}{\gamma}\int_0^z\int_0^h [\rho(q^2)+\rho(p^2)]p^2\,dA + \int_0^z\int_0^h \rho(p^2)\,dA \tag{4.26}$$

$$\leq d_1(1+\gamma)\int_0^z\int_0^h [\rho(q^2)+\rho(p^2)]w_{,\alpha}w_{,\alpha}\,dA + \frac{d_1 p_m^2}{\gamma}G(z) + \int_0^z\int_0^h \rho(p^2)(\gamma^{-1}d_1p^2+1)\,dA, \tag{4.27}$$

where

$$p_m = \max_{[o,h]} p. \tag{4.28}$$

Thus we have

$$\left(1-\frac{d_1p_m^2}{\gamma}\right)G(z) \leq d_1(1+\gamma)\tilde{E}_0 + \tilde{F}(d_1,p^2,\gamma)hz, \tag{4.29}$$

where the definition (4.4) has been used to obtain (4.29) from (4.27) and we have introduced the notation

$$\tilde{F}(d_1,p^2,\gamma) \equiv \frac{1}{h}\int_0^h \left[\frac{d_1p^2}{\gamma}\rho(p^2)+\rho(p^2)\right]dx_2. \tag{4.30}$$

The inequality (4.29), valid for arbitrary $\gamma > 0$, yields an upper bound for $G(z)$ in the form

$$G(z) \leq \left(1-\frac{d_1p_m^2}{\gamma}\right)^{-1}\{d_1(1+\gamma)\tilde{E}_0 + \tilde{F}hz\}, \tag{4.31}$$

provided γ is chosen so that

$$1-\frac{d_1p_m^2}{\gamma} > 0. \tag{4.32}$$

When (4.31) is inserted into (4.21) we may proceed as in (2.37) and obtain the desired result (4.5) where

$$\tilde{\nu} = \tilde{\nu}(p) = \frac{m_1(1-c_1)(1-d_1p_m^2/\gamma)}{2\tilde{F}(d_1,p^2,\gamma)} \tag{4.33}$$

and

$$\tilde{K} = \tilde{K}(\tilde{E}_0,m_1,c_1,d_1,p) = \tilde{E}_0\exp\left[\frac{m_1\pi\tilde{E}_0d_1(1-c_1)(1+\gamma)(1-d_1p_m^2/\gamma)}{\tilde{F}^2(d_1,p^2,\gamma)h^2}\right]. \tag{4.34}$$

This completes the proof of the decay estimate (4.5) in Case 1.

The parameter $\gamma > 0$ appearing in (4.33), (4.34) is arbitrary, except that (4.32) must be satisfied. To obtain the *largest* estimated decay rate $\tilde{\nu}$, we choose γ to maximize the right hand side of (4.33) with respect to γ. Such a choice for γ is given by

$$\gamma = \frac{d_1p_m^2}{\varepsilon}, \tag{4.35}$$

where $\varepsilon > 0$ is an arbitrary parameter. The choice (4.35) conforms to (4.32) provided

$$\varepsilon < 1. \tag{4.36}$$

With γ chosen as in (4.35), we obtain from (4.30) that

$$\tilde{F} = \tilde{F}(p^2, \varepsilon) = \frac{1}{h}\int_0^h \left[\frac{\varepsilon p^2 \rho(p^2)}{p_m^2} + \rho(p^2)\right] dx_2, \tag{4.37}$$

and so the decay rate $\tilde{\nu}$ follows from (4.33) as

$$\tilde{\nu} = \tilde{\nu}(p, \varepsilon) = \frac{m_1(1-c_1)(1-\varepsilon)}{2\tilde{F}(p^2, \varepsilon)}. \tag{4.38}$$

The multiplicative constant $\tilde{K}$, given by (4.34), now has the form

$$\tilde{K} = \tilde{E}_0 \exp\left[\frac{m_1 \pi \tilde{E}_0 d_1 (1-c_1)(1 + d_1 p_m^2/\varepsilon)(1-\varepsilon)}{\tilde{F}^2(p^2, \varepsilon) h^2}\right]. \tag{4.39}$$

The largest decay rate $\tilde{\nu}(p, \varepsilon)$ given by (4.38) would occur if ε were allowed to tend to zero, in which case

$$\tilde{\nu} \to \frac{m_1(1-c_1)h}{2\int_0^h \rho(p^2)\, dx_2}. \tag{4.40}$$

However, the multiplicative constant $\tilde{K}$ then has the asymptotic form $O(\varepsilon^{-1})$ as $\varepsilon \to 0$.

When the function ρ is given by

$$\rho = (1 + q^2)^{1/2}, \tag{4.41}$$

the hypotheses (4.1)–(4.3) of Case 1 can be shown to hold with $s = q$, $t = p$ if the constants m_1, c_1, d_1 are chosen to be

$$m_1 = 1, \qquad c_1 = 0, \qquad d_1 = \frac{1}{4} \tag{4.42}$$

(see Appendix). The solution $v(x_2)$ of (1.7), (1.8) may then be found as in Sec. 3. From (3.3) with ρ given by (4.41), we find

$$v' = [\{1 + 4k^2(h - 2x_2)^2\}^{1/2} - 1]^{1/2}/\sqrt{2}. \tag{4.43}$$

Note that, in contrast to the problem treated in Sec. 3, there is no restriction necessary on $m = kh$ for existence of the solution $v(x_2)$ when ρ is given by (4.41). From (4.43), we have

$$v(x_2) = \frac{1}{\sqrt{2}}\int_0^{x_2} [\{1 + 4k^2(h - 2s)^2\}^{1/2} - 1]^{1/2}\, ds. \tag{4.44}$$

The integral in (4.44) may be evaluated by using an algebraic substitution. Since the resulting expression is algebraically cumbersome, we shall not record it here.

Recalling that $p = |v'|$, we obtain from (4.43) that

$$p_m = \max_{[0,h]} p = \{[(1 + 4m^2)^{1/2} - 1]/2\}^{1/2}, \tag{4.45}$$

and so, in view of (4.42), the parameter γ given by (4.35) can be written as

$$\gamma = [(1 + 4m^2)^{1/2} - 1]/8\varepsilon, \qquad \varepsilon > 0. \tag{4.46}$$

The final expressions for $\tilde{\nu}, \tilde{K}$ given by (4.38), (4.39) can be found once the quantity $\tilde{F}(p^2, \varepsilon)$ given in (4.37) is evaluated.The procedure follows that carried out in Sec. 3 for Case 2. Since the calculations are algebraically cumbersome, we omit the details here.

5. Total energy bounds. The total energy $E_0 = E(0)$ contained in the rectangle R appears in the multiplicative constant K_2 in the estimate (2.38) in Case 2. Here we show how an upper bound for E_0 in terms of the geometry, boundary data, ρ, and v can be obtained. Bounds of this type have also been discussed in our previous work [8] (see also [1]).

We begin with the inequality (2.17) evaluated at $z = 0$. Thus we obtain

$$(1 - c_2)E_0 \leq -\int_{L_0} \rho(q^2)(u - v)u_{,1}\, dx_2, \tag{5.1}$$

which, in view of the boundary condition (1.11) may be written as

$$(1 - c_2)E_0 \leq -\int_{L_0} \rho(q^2)(f - v)u_{,1}\, dx_2, \tag{5.2}$$

where $f(x_2)$ is prescribed. For the special case when $\rho = (1 + q^2)^{-1/2}$, (5.2) provides an upper bound of the desired form immediately. In this case

$$|\rho(q^2)u_{,1}| \leq 1 \tag{5.3}$$

and so (5.2) yields

$$E_0 \leq (1 - c_2)^{-1} \int_{L_0} |f - v|\, dx_2, \tag{5.4}$$

where v is given explicitly in (3.7), and we recall from (3.9) that $c_2 = kh$.

For general ρ in Case 2, we proceed from (5.2) as follows. Denoting the right hand side of (5.2) by I^* we write I^* in the form

$$I^* = \int_{\partial R} e^{-\alpha x_1} \rho(q^2)(f - v)u_{,\beta}\, n_\beta\, ds, \tag{5.5}$$

where ∂R denotes the boundary of R, n_β the components of the outward unit normal on ∂R, and s is arc length on ∂R. A weight function $e^{-\alpha x_1}$ ($\alpha > 0$, an arbitrary constant) has been introduced in (5.5), the utility of which will become clear presently. The equivalence of (5.5) with the right hand side of (5.2) makes use of the boundary condition (1.6) and the fact that $f = v = 0$ on $x_2 = 0, h$ $(0 \leq x_1 \leq l)$. On applying the divergence theorem, assuming the differentiability of $f(x_2)$, using the differential equation (1.1) and the fact that $f - v$ depends only on x_2 we may write

$$\begin{aligned} I^* = &\int_R e^{-\alpha x_1}(-2k)(f - v)\, dA + \int_R e^{-\alpha x_1} \rho(q^2)(f - v)_{,\beta}\, u_{,\beta}\, dA \\ &- \alpha \int_R e^{-\alpha x_1} \rho(q^2)(f - v)w_{,1}\, dA. \end{aligned} \tag{5.6}$$

The second integral in (5.6) may be written as

$$\int_R e^{-\alpha x_1}\rho(q^2)(u_{,2}-v_{,2}+v_{,2})(f-v)_{,2}\,dA = \int_R e^{-\alpha x_1}\rho(q^2)w_{,2}(f-v)_{,2}\,dA + \int_R e^{-\alpha x_1}\rho(q^2)v_{,2}(f-v)_{,2}\,dA, \tag{5.7}$$

on recalling the definition of w in (2.2). Now

$$v_{,2}(f-v)_{,2} = v_{,2}f_{,2} - v_{,2}^2 \leq \left(\frac{\beta}{2}-1\right)v_{,2}^2 + \frac{1}{(2\beta)}f_{,2}^2, \tag{5.8}$$

the last inequality following from the arithmetic-geometric mean inequality (2.32) with weight $\gamma=\beta>0$. Choosing $\beta=2$, we find

$$v_{,2}(f-v)_{,2} \leq \frac{1}{4}f_{,2}^2. \tag{5.9}$$

On using (5.9) and Schwarz's inequality, the second and third integrals in (5.6) may then be bounded above by

$$\begin{aligned}&\left(\int_R \rho(q^2)w_{,2}^2\,dA\right)^{1/2}\left(\int_R e^{-2\alpha x_1}\rho(q^2)(f_{,2}-v_{,2})^2\,dA\right)^{1/2}\\ &\quad+\left(\int_R \rho(q^2)w_{,1}^2\,dA\right)^{1/2}\left(\int_R \alpha^2 e^{-2\alpha x_1}\rho(q^2)(f-v)^2\,dA\right)^{1/2}\\ &\quad+\left(\frac{1}{4}\right)\int_R e^{-\alpha x_1}\rho(q^2)f_{,2}^2\,dA \equiv H.\end{aligned} \tag{5.10}$$

We now apply the arithmetic-geometric mean inequality (2.32) with weight $\gamma=\delta>0$ to the first two terms in (5.10) and find that

$$\begin{aligned}H \leq{}& \frac{\delta}{2}\int_R \rho(q^2)w_{,\alpha}w_{,\alpha}\,dA + \frac{1}{2\delta}\int_R e^{-2\alpha x_1}\rho(q^2)(f_{,2}-v_{,2})^2\,dA\\ &+\frac{\alpha^2}{2\delta}\int_R e^{-2\alpha x_1}\rho(q^2)(f-v)^2\,dA + \frac{1}{4}\int_R e^{-\alpha x_1}\rho(q^2)f_{,2}^2\,dA.\end{aligned} \tag{5.11}$$

The first term on the right in (5.11) is just $(\delta/2)E_0$, by virtue of (2.1), and so from (5.11), (5.10), (5.5), (5.2) we obtain

$$\begin{aligned}\left(1-c_2-\frac{\delta}{2}\right)E_0 \leq{}& -2k\int_R e^{-\alpha x_1}(f-v)\,dA + \frac{1}{2\delta}\int_R e^{-2\alpha x_1}\rho(q^2)(f_{,2}-v_{,2})^2\,dA\\ &+\frac{\alpha^2}{2\delta}\int_R e^{-2\alpha x_1}\rho(q^2)(f-v)^2\,dA + \frac{1}{4}\int_R e^{-\alpha x_1}\rho(q^2)f_{,2}^2\,dA,\end{aligned} \tag{5.12}$$

for arbitrary $\alpha,\delta>0$. In Case 2, $\rho^{-1}\geq m_2>0$ and so $\rho\leq m_2^{-1}$. Using this bound for ρ in the last three integrals in (5.12), the integration with respect to x_1 may be carried out (since $f - v$ depends only on x_2) and so we find

$$\begin{aligned}\left(1 - c_2 - \frac{\delta}{2}\right) E_0 \le & - \left[\frac{2k(1 - e^{-\alpha l})}{\alpha}\right] \int_0^h (f - v)\, dx_2 \\ & + \left[\frac{(1 - e^{-2\alpha l})}{4\alpha\delta m_2}\right] \int_0^h (f_{,2} - v_{,2})^2\, dx_2 \\ & + \left[\frac{\alpha(1 - e^{-2\alpha l})}{4\delta m_2}\right] \int_0^h (f - v)^2\, dx_2 \\ & + \left[\frac{(1 - e^{-\alpha l})}{4\alpha m_2}\right] \int_0^h f_{,2}^2\, dx_2,\end{aligned} \tag{5.13}$$

for arbitrary $\alpha, \delta > 0$. If δ is now chosen such that

$$1 - c_2 - \frac{\delta}{2} > 0, \tag{5.14}$$

(5.13) provides the desired upper bound for E_0. A simple choice for α, δ would be to take

$$\alpha = 1, \quad \delta = 1 - c_2. \tag{5.15}$$

For simplicity in the final estimate, we then drop the exponential terms in the last three terms of (5.13) and obtain

$$\begin{aligned}E_0 \le 2(1 - c_2)^{-1} \Bigg\{ & -2k(1 - e^{-l}) \int_0^h (f - v)\, dx_2 \\ & + [(1 - c_2)4m_2]^{-1} \int_0^h (f_{,2} - v_{,2})^2\, dx_2 \\ & + [(1 - c_2)4m_2]^{-1} \int_0^h (f - v)^2\, dx_2 + (4m_2)^{-1} \int_0^h f_{,2}^2\, dx_2 \Bigg\}.\end{aligned} \tag{5.16}$$

Using similar arguments, bounds on $\tilde{E}_0$ in Case 1 may also be obtained. In this case, an *a priori* restriction on the boundary data is necessary in order to obtain a result comparable to (5.16). Similar considerations arise in the analysis carried out in [8]. For special classes of functions ρ, however, (e.g. $\rho = (1 + q^2)^{1/2}$) this *a priori* restriction can be eliminated.

6. Concluding remarks. The decay estimates established in Sec. 2, Case 2, and in Sec. 4, Case 1, can be shown to hold when (1.1) is generalized to the equation

$$[\rho(q^2)u_{,\alpha}]_{,\alpha} + g(u, x_2) = 0, \tag{6.1}$$

where g is monotone nonincreasing in its first argument. Thus the decay estimates (2.38) and (4.5) continue to hold, thereby ensuring the decay of solutions of (6.1), (1.9)–(1.11) (assumed to exist) to solutions $v(x_2)$ of the one-dimensional problem

$$\frac{d}{dx_2}\left[\rho(p^2)\frac{dv}{dx_2}\right] + g(v, x_2) = 0, \qquad 0 \le x_2 \le h, \tag{6.2}$$

$$v(0) = 0, \qquad v(h) = 0. \tag{6.3}$$

A special case of (6.1) in Case 2 which has attracted much interest occurs when

$$\rho = (1+q^2)^{-1/2}, \qquad g(u, x_2) = -cu, \tag{6.4}$$

where $c > 0$ is a constant. Equation (6.1) is then the equation of the fluid surface in a capillary tube. Investigations of associated one-dimensional problems have played an important role in the study of capillary surface phenomena [15].

Appendix.

Verification of (3.2) *and* (4.42). When ρ is given by (3.1), we have

$$\rho^{-1} = (1+q^2)^{1/2} \tag{A.1}$$

and so $\rho^{-1} \geq 1$ so that the constant m_2 in (2.3) can be taken as $m_2 = 1$, as stated in $(3.2)_1$. To verify that (2.4) holds with $s = q$, $t = p$, and with c_2 chosen as in (3.2), we must show that

$$|(1+q^2)^{1/2} - (1+p^2)^{1/2}|\, p(1+p^2)^{-1/2} \leq c_2|p-q|, \tag{A.2}$$

or equivalently,

$$|q^2 - p^2|[(1+q^2)^{1/2} + (1+p^2)^{1/2}]^{-1} p(1+p^2)^{-1/2} \leq c_2|p-q|. \tag{A.3}$$

Thus c_2 must be chosen such that

$$c_2 \geq (q+p)[(1+q^2)^{1/2} + (1+p^2)^{1/2}]^{-1} p(1+p^2)^{-1/2}. \tag{A.4}$$

Since

$$q + p \leq (1+q^2)^{1/2} + (1+p^2)^{1/2}, \tag{A.5}$$

(A.4) will hold provided

$$c_2 \geq p(1+p^2)^{-1/2}. \tag{A.6}$$

The right side of (A.6) is monotone increasing in p and so (A.6) will be satisfied for all p if we choose

$$c_2 = p_m(1+p_m^2)^{-1/2}, \qquad p_m = \max_{[0,h]} p. \tag{A.7}$$

This completes the verification of $(3.2)_2$.

To see that (2.5) holds with $s = q$, $t = p$, and with d_2 chosen as in $(3.2)_3$, we must show that

$$|(1+q^2)^{1/2} - (1+p^2)^{1/2}|[(1+p^2)^{-1/2} + (1+q^2)^{-1/2}] \leq d_2(1+p^2)^{-1/2}(1+q^2)^{-1/2}|q^2 - p^2|. \tag{A.8}$$

On multiplying both sides of (A.8) by $(1+p^2)^{1/2}(1+q^2)^{1/2}$, it is seen that (A.8) is equivalent to

$$|q^2 - p^2| \leq d_2|q^2 - p^2|, \tag{A.9}$$

and so we take

$$d_2 = 1, \tag{A.10}$$

as stated in $(3.2)_3$.

When ρ is given by (4.41), that is

$$\rho = (1+q^2)^{1/2}, \tag{A.11}$$

we have $\rho \geq 1$ and so the constant m_1 in (4.1) can be taken as $m_1 = 1$, as stated in $(4.42)_1$.

To see that (4.2) holds with $s = q$, $t = p$, and with $c_1 = 0$, we observe that the left hand side of (4.2), with ρ given by (A.11), reads

$$-[(1+q^2)^{1/2} - (1+p^2)^{1/2}](q^2 - p^2), \tag{A.12}$$

which can be written as

$$-(q^2 - p^2)^2/[(1+q^2)^{1/2} + (1+p^2)^{1/2}] \tag{A.13}$$

which is nonpositive for all p, q. But the right hand side of (4.2) is clearly nonnegative and so the constant c_1 in (4.2) can be chosen $c_1 = 0$. To verify that (4.3) holds with $s = q$, $t = p$, and with d_1 chosen as in $(4.42)_3$, we must show that

$$|(1+q^2)^{1/2} - (1+p^2)^{1/2}| \leq d_1[(1+q^2)^{1/2} + (1+p^2)^{1/2}]|q^2 - p^2|, \tag{A.14}$$

or equivalently,

$$[(1+q^2)^{1/2} + (1+p^2)^{1/2}]^{-2} \leq d_1. \tag{A.15}$$

Thus if we choose d_1 such that

$$d_1 \geq [(1+p^2)^{1/2} + 1]^{-2}, \tag{A.16}$$

(A.15) will be satisfied for all q. The maximum of the right hand side of (A.16) occurs when $p = 0$ and so if we choose

$$d_1 = \frac{1}{4}, \tag{A.17}$$

then (A.14) is satisfied for all p, q. This completes the verification of (4.42).

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