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EXPLICIT EXPRESSIONS OF BARNETT-LOTHE TENSORS AND THEIR ASSOCIATED TENSORS FOR ORTHOTROPIC MATERIALS*

By

CHANGSONG DONGYE (*Dalien Railway Institute, Dalien, The Peoples' Republic of China*)

AND

T. C. T. TING (*University of Illinois, Chicago, Illinois*)

Abstract. The three Barnett-Lothe tensors **S**, **H**, **L**, appear very often in the solutions to two-dimensional anisotropic elasticity problems. So do their associated tensors $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$ for line forces and dislocations and $\mathbf{S}(v)$, $\mathbf{H}(v)$, $\mathbf{L}(v)$ in the problem of surface waves. Explicit expressions of the components of these tensors are derived and presented for orthotropic materials in which the planes of material symmetry coincide with the coordinate planes. With minor modifications, the results for **S**, **H**, **L** and $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$ can be applied to orthotropic materials in which only the $x_3 = 0$ plane coincides with one of the planes of material symmetry.

1. Introduction. In a fixed rectangular coordinate system x_i, $i = 1, 2, 3$, let u_i and σ_{ij} be the displacement and stress, respectively. The stress-strain laws and the equations of equilibrium are

$$\sigma_{ij} = C_{ijks} u_{k,s}, \tag{1.1}$$

$$C_{ijks} u_{k,sj} = 0, \tag{1.2}$$

in which C_{ijks} are the elastic constants, repeated indices imply summation, and a comma stands for differentiation. We assume that C_{ijks} are fully symmetric and positive definite such that the strain energy is positive. Let

$$\left.\begin{aligned} Q_{ik}(\theta_= C_{ijks} n_j(\theta) n_s(\theta), \\ R_{ik}(\theta) = C_{ijks} n_j(\theta) m_s(\theta), \\ T_{ik}(\theta) = C_{ijks} m_j(\theta) m_s(\theta), \end{aligned}\right\} \tag{1.3}$$

$$n_i(\theta) = (\cos\theta, \sin\theta, 0), \qquad m_i(\theta) = (-\sin\theta, \cos\theta, 0),$$

where θ is a real parameter and, in matrix notation,

$$\left.\begin{aligned} &\mathbf{N}_1(\theta) = -\mathbf{T}^{-1}(\theta)\mathbf{R}^T(\theta), \qquad \mathbf{N}_2(\theta) = \mathbf{T}^{-1}(\theta), \\ &\mathbf{N}_3(\theta) = \mathbf{R}(\theta)\mathbf{T}^{-1}(\theta)\mathbf{R}^T(\theta) - \mathbf{Q}(\theta). \end{aligned}\right\} \tag{1.4}$$

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The superscript T stands for the transpose. Define the incomplete integrals

$$\left.\begin{aligned}\hat{\mathbf{S}}(\theta) &= \frac{1}{\pi}\int_0^\theta \mathbf{N}_1(\theta')\,d\theta',\\ \hat{\mathbf{H}}(\theta) &= \frac{1}{\pi}\int_0^\theta \mathbf{N}_2(\theta')\,d\theta',\\ \hat{\mathbf{L}}(\theta) &= -\frac{1}{\pi}\int_0^\theta \mathbf{N}_3(\theta')\,d\theta',\end{aligned}\right\} \tag{1.5}$$

and the complete integrals

$$\mathbf{S} = \hat{\mathbf{S}}(\pi), \qquad \mathbf{H} = \hat{\mathbf{H}}(\pi), \qquad \mathbf{L} = \hat{\mathbf{L}}(\pi). \tag{1.6}$$

The three complete integrals **S**, **H**, **L** are the Barnett-Lothe tensors [1]. The tensor $-\mathbf{L}^{-1}$ was first introduced by Stroh who denoted it by **B** in [2]. The Barnett-Lothe tensors and their associated tensors $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$ appear often in the solutions to two-dimensional anisotropic elasticity problems [1, 3–15].

Other related tensors $\mathbf{S}(v)$, $\mathbf{H}(v)$, $\mathbf{L}(v)$, where v is the real wave speed, appear in the problem of steady state moving line dislocations and Rayleigh surface waves [4, 7–9]. We generalize $\mathbf{Q}(\theta)$, $\mathbf{R}(\theta)$, $\mathbf{T}(\theta)$ of (1.3) by

$$\begin{aligned}\mathbf{Q}(\theta, v) &= \mathbf{Q}(\theta) - \rho v^2 \cos^2\theta \mathbf{I},\\ \mathbf{R}(\theta, v) &= \mathbf{R}(\theta) + \rho v^2 \cos\theta \sin\theta \mathbf{I},\\ \mathbf{T}(\theta, v) &= \mathbf{T}(\theta) - \rho v^2 \sin^2\theta \mathbf{I},\end{aligned}$$

where ρ is the mass density and **I** is the 3×3 identity matrix. If we define $\mathbf{N}_i(\theta, v)$ in terms of $\mathbf{Q}(\theta, v)$, $\mathbf{R}(\theta, v)$, $\mathbf{T}(\theta, v)$ similar to that of (1.4) we have

$$\left.\begin{aligned}\mathbf{S}(v) &= \frac{1}{\pi}\int_0^\pi \mathbf{N}_1(\theta, v)\,d\theta,\\ \mathbf{H}(v) &= \frac{1}{\pi}\int_0^\pi \mathbf{N}_2(\theta, v)\,d\theta,\\ \mathbf{L}(v) &= -\frac{1}{\pi}\int_0^\pi \mathbf{N}(\theta, v)\,d\theta.\end{aligned}\right\} \tag{1.7}$$

We see that **S**, **H**, **L** are a special case of $\mathbf{S}(v)$, $\mathbf{H}(v)$, $\mathbf{L}(v)$ with $v = 0$.

Explicit expressions of $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$ for isotropic materials can be obtained easily by integrating (1.5). For $\mathbf{S}(v)$, explicit expressions are available for orthotropic materials [16], cubic materials [8, 16], and for transversely isotropic materials [17] is which the axis of symmetry is in the (x_1, x_2) plane or the (x_1, x_3) plane. Recently, Chadwick [18] obtained explicit expressions of $\mathbf{S}(v)$ for monoclinic materials for which the plane of symmetry is at $x_3 = 0$.

In this paper we give explicit expressions for **S**, **H**, **L**, $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$, $\mathbf{S}(v)$, $\mathbf{H}(v)$ and $\mathbf{L}(v)$ for orthotropic materials. We obtain the expressions not by integrations, but by identities which relate these tensors to certain combinations of eigenvalues and eigenvectors of 6×6 matrices **N** of elastic constants to be defined later. Basic equations and the eigenvalues and eigenvectors of the elastic constants are presented in Sec. 2 and 3. Since **S**, **H**, **L** are the simplest and are special cases of $\hat{S}(\theta)$, $\hat{\mathbf{H}}(\theta)$,

$\hat{\mathbf{L}}(\theta)$, when $\theta = \pi$, or $\mathbf{S}(v)$, $\mathbf{H}(v)$, $\mathbf{L}(v)$, when $v = 0$, we present the derivations of $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$ in Sec. 4. This serves as a basis for the derivations of the more complicated tensors $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$ in Sec. 5. In the derivations we assume that the planes of material symmetry of the orthotropic material coincide with the coordinate planes. In Sec. 6 we present modifications required when only the $x_3 = 0$ plane coincides with one of the planes of material symmetry. Finally in Sec. 7 we derive explicit expressions for the components of $\mathbf{S}(v)$, $\mathbf{H}(v)$, $\mathbf{L}(v)$ for orthotropic materials whose planes of material symmetry are the coordinate planes.

2. Basic equations. Instead of the integral representation for $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$, we consider an algebraic representation which avoids the integration. Let p_α and $\mathbf{a}_\alpha$, ($\alpha = 1, 2, \ldots, 6$) be the eigenvalues and eigenvectors of the following equation

$$\{\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}\}\mathbf{a} = \mathbf{0}, \tag{2.1}$$

$$Q_{ik} = C_{i1k1}, \qquad R_{ik} = C_{i1k2}, \qquad T_{ik} = C_{i2k2}. \tag{2.2}$$

We see that $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ are $\mathbf{Q}(\theta)$, $\mathbf{R}(\theta)$, $\mathbf{T}(\theta)$ of (1.3) with $\theta = 0$. Introducing the new vector

$$\mathbf{b} = (\mathbf{R}^T + p\mathbf{T})\mathbf{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a}, \tag{2.3}$$

in which the second equality comes from (2.1), the two equations in (2.3) can be recast in the standard eigenrelation

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}, \qquad \mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{bmatrix}, \ \boldsymbol{\xi} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix},$$

where $\mathbf{N}_i$, $i = 1, 2, 3$ are $\mathbf{N}_i(\theta)$ of (1.4) with $\theta = 0$. Since p_α cannot be real if the strain energy is positive [19], p_α come in three pairs of complex conjugates. We let

$$p_{\alpha+3} = \bar{p}_\alpha, \qquad \operatorname{Im} p_\alpha > 0, \qquad \alpha = 1, 2, 3, \tag{2.4}$$

where an overbar denotes the complex conjugate and Im stands for the imaginary part. If we introduce the 3×3 matrices

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \qquad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3],$$

in which $\mathbf{b}_\alpha$ is related to $\mathbf{a}_\alpha$ through (2.3), it can be shown that [7]

$$\left.\begin{aligned} &\mathbf{S} = i(2\mathbf{A}\mathbf{B}^T - \mathbf{I}), \\ &\mathbf{H} = 2i\mathbf{A}\mathbf{A}^T, \qquad \mathbf{L} = -2i\mathbf{B}\mathbf{B}^T, \end{aligned}\right\} \tag{2.5}$$

provided $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$ are normalized such that

$$2\mathbf{a}_\alpha^T\mathbf{b}_\alpha = 1, \qquad \alpha \text{ not summed.} \tag{2.6}$$

It should be pointed out that (2.5) hold when $\mathbf{N}$ is simple or semisimple. Modifications required when $\mathbf{N}$ is nonsemisimple can be found in [20]. Assuming that $\mathbf{N}$ is simple, we will use (2.5) to derive $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$, explicitly in terms of the elastic constants C_{ijks}. After we have obtained the results which do not contain p_α, $\mathbf{a}_\alpha$, $\mathbf{b}_\alpha$ explicitly, the problems associated with nonsemisimple $\mathbf{N}$ disappear and the results apply to nonsemisimple $\mathbf{N}$ as well. Equations (2.5) show that we need the eigenvectors $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$. This is presented in the next section.

In closing this section, we note that the three tensors **S**, **H**, **L** are not independent of each other. They are related by

$$\mathbf{HL} - \mathbf{SS} = \mathbf{I}, \qquad \mathbf{SH} + \mathbf{HS}^T = \mathbf{0}, \qquad \mathbf{LS} + \mathbf{S}^T\mathbf{L} = \mathbf{0}. \tag{2.7}$$

More studies of the relationships between these tensors and their structure and invariance properties can be found in [21, 22].

3. The eigenvalues and eigenvectors for orthotropic materials. Using the contracted notations, we replace C_{ijks} by a 6×6 symmetric matrix $C_{\alpha\beta}$. Assuming that the planes of material symmetry of the orthotropic material coincide with the coordinate planes, $C_{\alpha\beta}$ vanishes whenever α or β assumes the value of 4, 5, 6, except C_{44}, C_{55} and C_{66}. For the two-dimensional deformations considered here, the necessary and sufficient conditions for the strain energy to be positive are

$$\left.\begin{array}{c} C_{11} > 0, \qquad C_{44} > 0, \qquad C_{55} > 0, \qquad C_{66} > 0, \\ C_{11}C_{22} - C_{12}^2 > 0. \end{array}\right\} \tag{3.1}$$

Equations $(3.1)_1$ and $(3.1)_5$ imply that $C_{22} > 0$ and that

$$d_+ = (C_{11}C_{22})^{1/2} + C_{12} > 0, \qquad d_- = (C_{11}C_{22})^{1/2} - C_{12} > 0. \tag{3.2}$$

Equation (2.1) now has the expression

$$\begin{bmatrix} C_{11} + p^2C_{66} & p(C_{12} + C_{66}) & 0 \\ p(C_{12} + C_{66}) & C_{66} + pC_{22}^2 & 0 \\ 0 & 0 & C_{55} + p^2C_{44} \end{bmatrix} \mathbf{a} = \mathbf{0}. \tag{3.3}$$

The vanishing of the determinant of the 3×3 matrix on the left provides the eigenvalues p_1, p_2, p_3 which should have the positive imaginary part according to (2.4). We let

$$p_3 = i\left(\frac{C_{55}}{C_{44}}\right)^{1/2}. \tag{3.4}$$

Then p_1 and p_2 are the roots of

$$p^4 + 2\alpha p^2 + \beta^2 = 0, \tag{3.5}$$

in which, for convenience, α and β are given in (4.2) where we summarize the results for **S**, **H**, **L**. From (3.5) we have

$$p_1^2 + p_2^2 = -2\alpha, \qquad p_1p_2 = -\beta, \tag{3.6}$$

and hence

$$p_1 + p_2 = i[2(\alpha + \beta)]^{1/2}, \tag{3.7}$$

where $\alpha + \beta$ can be shown to be positive (see (5.5) and (4.2)).

The minus sign in $(3.6)_2$ can be justified as follows. We see from (3.5) that p_1^2 and p_2^2 are real or complex conjugates depending on $\alpha^2 \geq \beta^2$ or $\alpha^2 < \beta^2$. Noticing that the imaginary parts of p_1, p_2 are positive, we conclude that p_1, p_2 have the form

$$p_1 = i\rho_1, \qquad p_2 = i\rho_2, \qquad \rho_1, \rho_2 > 0,$$

when $\alpha^2 \geq \beta^2$ and

$$p_1 = w_- + iw_+, \qquad p_2 = -w_- + iw_+, \qquad w_+ > 0,$$

when $\alpha^2 < \beta^2$. The minus in $(3.6)_2$ is therefore justified.

Using the second row of the 3×3 matrix in (3.3) to construct the eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$, we have

$$\mathbf{A} = \begin{bmatrix} k_1(C_{66} + p_1^2 C_{22}) & k_2(C_{66} + p_2^2 C_{22}) & 0 \\ -k_1 p_1 (C_{12} + C_{66}) & -k_2 p_2 (C_{12} + C_{66}) & 0 \\ 0 & 0 & k_3 \end{bmatrix}, \tag{3.8}$$

where k_1, k_2, k_3 are complex constants to be determined and, by $(2.3)_1$,

$$\mathbf{B} = \begin{bmatrix} -k_1 p_1 C_{66}(C_{12} - p_1^2 C_{22}) & -k_2 p_2 C_{66}(C_{12} - p_2^2 C_{22}) & 0 \\ k_1 C_{66}(C_{12} - p_1^2 C_{22}) & k_2 C_{66}(C_{12} - p_2^2 C_{22}) & 0 \\ 0 & 0 & k_3 p_3 C_{44} \end{bmatrix}. \tag{3.9}$$

The normalization condition (2.6) for $\alpha = 1, 2, 3$ leads to

$$\left.\begin{aligned} -2k_1^2 p_1 C_{66}(C_{12} - p_1^2 C_{22})(C_{12} + 2C_{66} + p_1^2 C_{22}) = 1, \\ -2k_2^2 p_2 C_{66}(C_{12} - p_2^2 C_{22})(C_{12} + 2C_{66} + p_2^2 C_{22}) = 1, \\ 2k_3^2 p_3 C_{44} = 1. \end{aligned}\right\} \tag{3.10}$$

With k_1, k_2, k_3 given by (3.10), $\mathbf{A}$ and $\mathbf{B}$ contain the elastic constants $C_{\alpha\beta}$ and the eigenvalues p_α.

4. Explicit expressions for S, H, L. When we substitute (3.8) and (3.9) into (2.5), each component of the matrices **S**, **H**, **L** has the factors k_1^2, k_2^2, k_3^2 which can be eliminated by using (3.10). The results contain the terms $p_1 p_2$, $(p_1 + p_2)$, and $(p_1^2 + p_2^2)$ which are given in (3.6) and (3.7). Thus all components of **S**, **H**, **L** can be obtained in terms of $C_{\alpha\beta}$ only. We list below explicit expressions of **S**, **H**, **L**. Only the nonzero components are presented.

$$\left.\begin{aligned} S_{21} = \tau_+, \qquad S_{12} = -\beta^{-1} S_{21}, \\ L_{11} = d_+ \tau_+, \qquad L_{22} = \beta^{-1} L_{11}, \qquad L_{33} = (C_{44} C_{55})^{1/2}, \\ H_{11} = \beta^{-1} H_{22}, \qquad H_{22} = \frac{e_+ \tau_+}{C_{66}}, \qquad H_{33} = L_{33}^{-1}. \end{aligned}\right\} \tag{4.1}$$

All notations appeared above and those which will appear later on are defined below:

$$\left.\begin{aligned} 2\alpha = \{C_{11}C_{22} - C_{12}(C_{12} + 2C_{66})\}(C_{22}C_{66})^{-1}, \\ \beta = \left(\frac{C_{11}}{C_{22}}\right)^{1/2}, \qquad \tilde{C}_{12} = (C_{11}C_{22})^{1/2}, \\ d_\pm = \tilde{C}_{12} \pm C_{12}, \qquad \gamma_\pm = 2C_{66} + C_{12} \pm \tilde{C}_{12}, \\ \tau_+ = \left[\frac{C_{66}\, d_-}{C_{22}\gamma_+}\right]^{1/2}, \qquad \tau_- = \left[\frac{C_{66}\, d_+}{C_{22}|\gamma_-|}\right]^{1/2}, \\ 2w_+ = \left[\frac{d_-\gamma_+}{C_{22}C_{66}}\right]^{1/2}, \qquad 2w_- = \left[\frac{d_+|\gamma_-|}{C_{22}C_{66}}\right]^{1/2}, \\ e_+ = \frac{\tilde{C}_{12} + C_{66}}{\tilde{C}_{12} - C_{12}}, \qquad e_- = \frac{\tilde{C}_{12} - C_{66}}{\tilde{C}_{12} + C_{12}}. \end{aligned}\right\} \tag{4.2}$$

From (3.1) and (3.2), all quantities in (4.2) are positive except α, e_- and γ_- which can be negative. It can be shown that $\alpha > 0$ when $\gamma_- \le 0$.

5. Explicit expression for $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$. In this section we consider the functions $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$ defined in (1.5). They appeared, among others, in the problems of line forces and dislocations [3, 6, 10, 15] and the wedges [6, 12]. As before, we will employ an alternative representation for these tensors. We notice that $\mathbf{N}_i(\theta)$ are periodic in θ with periodicity π. Therefore

$$\hat{\mathbf{S}}(\theta+\pi) = \mathbf{S} + \hat{\mathbf{S}}(\theta), \qquad \hat{\mathbf{H}}(\theta+\pi) = \mathbf{H} + \hat{\mathbf{H}}(\theta), \qquad \hat{\mathbf{L}}(\theta+\pi) = \mathbf{L} + \hat{\mathbf{L}}(\theta), \tag{5.1}$$

and hence it suffices to find $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$, for $0 < \theta < \pi$. The alternate representations are (see Eq. (7.10) of [12])

$$\left.\begin{aligned} \hat{\mathbf{S}}(\theta) &= \frac{2}{\pi}\operatorname{Re}\{\mathbf{A}\boldsymbol{\Psi}(\theta)\mathbf{B}^T\}, \\ \hat{\mathbf{H}}(\theta) &= \frac{2}{\pi}\operatorname{Re}\{\mathbf{A}\boldsymbol{\Psi}(\theta)\mathbf{A}^T\}, \\ \hat{\mathbf{L}}(\theta) &= -\frac{2}{\pi}\operatorname{Re}\{\mathbf{B}\boldsymbol{\Psi}(\theta)\mathbf{B}^T\}, \end{aligned}\right\} \tag{5.2}$$

in which $\boldsymbol{\Psi}(\theta)$ is the diagonal matrix

$$\begin{aligned} \boldsymbol{\Psi}(\theta) &= \operatorname{diag}\{\ln\zeta_1(\theta), \ln\zeta_2(\theta), \ln\zeta_3(\theta)\}, \\ \zeta_\alpha(\theta) &= \cos\theta + p_\alpha \sin\theta. \end{aligned} \tag{5.3}$$

From the structure of $\mathbf{A}$, $\mathbf{B}$, $\boldsymbol{\Psi}(\theta)$, it is readily shown that the $\hat{S}_{13}(\theta)$, $\hat{S}_{23}(\theta)$, $\hat{S}_{31}(\theta)$, $\hat{S}_{32}(\theta)$ components of $\hat{\mathbf{S}}(\theta)$ vanish. The same components of $\hat{\mathbf{H}}(\theta)$ and $\hat{\mathbf{L}}(\theta)$ also vanish. For the nonzero components, $\hat{S}_{33}(\theta)$, $\hat{H}_{33}(\theta)$ and $\hat{L}_{33}(\theta)$ are very simple to calculate. The remaining nonzero components are $\hat{S}_{\alpha\beta}(\theta)$, $\hat{H}_{\alpha\beta}(\theta)$, $\hat{L}_{\alpha\beta}(\theta)$, $\alpha, \beta = 1, 2$, which contain the eigenvalues p_1, p_2 and $\ln\zeta_1, \ln\zeta_2$. They all have the form

$$f(p_1, p_2)\ln\zeta_1(\theta) + f(p_2, p_1)\ln\zeta_2(\theta),$$

where $f(p_1, p_2)$ is a function of p_1, p_2. This can be rewritten as

$$\frac{1}{2}\{f(p_1,p_2) + f(p_2,p_1)\}\ln[\zeta_1(\theta)\zeta_2(\theta)] + \frac{1}{2}\{f(p_1,p_2) - f(p_2,p_1)\}\ln\left[\frac{\zeta_1(\theta)}{\zeta_2(\theta)}\right]. \tag{5.4}$$

The calculations of $f(p_1,p_2)+f(p_2,p_1)$ are exactly the same as in the previous section. The calculations of $f(p_1,p_2)-f(p_2,p_1)$ involve p_1-p_2 and $p_1^2-p_2^2$ which are discussed next.

It can be shown that

$$\beta^2 - \alpha^2 = \frac{(C_{11}C_{22} - C_{12}^2)}{(C_{22}C_{66})^2}\gamma_+\gamma_-,$$

where $\gamma_\pm$ are defined in (4.2). From $(3.1)_5$ and the fact that $\gamma_+ > 0$, the inequality $\beta^2 \gtrless \alpha^2$ can be replaced by $\gamma_- \gtrless 0$. One can then show that, with w_+, w_- define in (4.2),

$$\begin{gathered} p_1 = w_- + iw_+, \qquad p_2 = -w_- + iw_+, \qquad \beta > |\alpha| \ge 0, \\ w_+^2 - w_-^2 = \alpha, \qquad w_+^2 + w_-^2 = \beta, \quad \text{if } \gamma_- > 0, \end{gathered}$$

and

$$p_1 = (w_+ + w_-)i, \qquad p_2 = (w_+ - w_-)i, \qquad \alpha \geq \beta > 0,$$
$$w_+^2 - w_-^2 = \beta, \qquad w_+^2 + w_-^2 = \alpha, \quad \text{if } \gamma_- \leq 0.$$

The combination $p_1 - p_2$ and $p_1^2 - p_2^2$ can now be obtained. Regardless of γ_- positive or negative, we have

$$p_1 + p_2 = 2iw_+, \qquad w_+^2 = \frac{(\alpha + \beta)}{2}. \tag{5.5}$$

We next investigate the logarithmic terms in (5.4). Using (5.3), (5.5) and $(3.6)_2$, we have

$$\begin{aligned}\zeta_1(\theta)\zeta_2(\theta) &= (\cos\theta + p_1 \sin\theta)(\cos\theta + p_2 \sin\theta) \\ &= \cos^2\theta - \beta \sin^2\theta + iw_+ \sin 2\theta.\end{aligned}$$

Hence,

$$\begin{aligned}\ln[\zeta_1(\theta)\zeta_2(\theta)] &= \frac{1}{2}\ln\{(\cos^2\theta - \beta\sin^2\theta)^2 + w_+^2 \sin^2 2\theta\} \\ &\quad + i\tan^{-1}\left[\frac{w_+ \sin 2\theta}{\cos^2\theta - \beta\sin^2\theta}\right].\end{aligned} \tag{5.6}$$

For $\ln[\zeta_1(\theta)/\zeta_2(\theta)]$, we need to consider $\gamma_- > 0$ and $\gamma_- < 0$ separately. When $\gamma_- > 0$,

$$\begin{aligned}\ln\zeta_1(\theta) &= \frac{1}{2}\ln(\cos^2\theta + \beta\sin^2\theta + w_- \sin 2\theta) \\ &\quad + i\tan^{-1}\left[\frac{w_+ \sin\theta}{\cos\theta + w_- \sin\theta}\right], \\ \ln\zeta_2(\theta) &= \frac{1}{2}\ln(\cos^2\theta + \beta\sin^2\theta - w_- \sin 2\theta) \\ &\quad + i\tan^{-1}\left[\frac{w_+ \sin\theta}{\cos\theta - w_- \sin\theta}\right].\end{aligned}$$

Employing the identities

$$\begin{aligned}\frac{1}{2}\ln\left[\frac{1+y}{1-y}\right] &= \tanh^{-1} y, \\ \tan^{-1} y_1 - \tan^{-1} y_2 &= \tan^{-1}\left[\frac{y_1 - y_2}{1 + y_1 y_2}\right],\end{aligned}$$

we obtain for $\gamma_- > 0$,

$$\ln\left[\frac{\zeta_1(\theta)}{\zeta_2(\theta)}\right] = \tanh^{-1}\Delta_1 - i\tan^{-1}\Delta_2, \tag{5.7}$$

where Δ_1, Δ_2 are given in (5.10). Similarly, it can be shown that for $\gamma_- < 0$,

$$\ln\left[\frac{\zeta_1(\theta)}{\zeta_2(\theta)}\right] = \tanh^{-1}\Delta_2 + i\tan^{-1}\Delta_1. \tag{5.8}$$

With the aid of (5.6) and (5.7) or (5.8) in the decomposition (5.4), the calculations of $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$ of (5.2) can be accomplished with less effort. We list below explicit expressions of the nonzero components of $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$, for $0 < \theta < \pi$:

$$\hat{S}_{11} = \Phi + \Gamma_2, \qquad \hat{S}_{22} = \Phi - \Gamma_2$$
$$\hat{S}_{21} = \Gamma_3 - \Gamma_1, \qquad \hat{S}_{12} = -\beta^{-1}(\Gamma_3 + \Gamma_1).$$
$$\hat{S}_{33} = \frac{1}{2\pi} \ln \left[\cos^2\theta + \frac{C_{55}}{C_{44}} \sin^2\theta\right],$$
$$\hat{L}_{11} = d_+\Gamma_3 + d_-\Gamma_1, \qquad \hat{L}_{22} = \beta^{-1}(d_+\Gamma_3 - d_-\Gamma_1),$$
$$\hat{L}_{12} = \hat{L}_{21} = 2C_{66}\Gamma_2, \qquad \hat{L}_{33} = (C_{44}C_{55})^{1/2}\delta_4,$$
$$\hat{H}_{11} = (\beta C_{66})^{-1}(e_+\Gamma_3 + e_-\Gamma_1), \qquad \hat{H}_{22} = C_{66}^{-1}(e_+\Gamma_3 - e_-\Gamma_1),$$
$$\hat{H}_{12} = \hat{H}_{21} = \frac{2(C_{12} + C_{66})}{C_{11}C_{22} - C_{12}^2}\Gamma_2, \qquad \hat{H}_{33} = (C_{44}C55)^{-1/2}\delta_4.$$

The new notations appearing above are

$$\Phi = \frac{1}{4\pi} \ln \left[(\cos^2\theta - \beta \sin^2\theta)^2 + w_+^2 \sin^2 2\theta\right]$$
$$\Gamma_1 = \delta_1\tau_-, \qquad \Gamma_2 = \left(\frac{C_{22}}{C_{66}}\right)\delta_2\tau_+\tau_-, \qquad \Gamma_3 = \delta_3\tau_+, \tag{5.9}$$

in which δ_i, $i = 1, 2, 3, 4$ are functions of θ defined as follows. Let

$$\Delta_1 = \frac{w_- \sin 2\theta}{\cos^2\theta + \beta \sin^2\theta}, \qquad \Delta_2 = \frac{2w_+w_- \sin^2\theta}{\cos^2\theta + \alpha \sin^2\theta}. \tag{5.10}$$

Then, for $0 \le \theta \le \pi$,

$$\delta_1 = \begin{cases} \frac{1}{2\pi} \tanh^{-1}\Delta_1, & \text{if } \gamma_- \ge 0, \\ \frac{1}{2\pi} \tan^{-1}\Delta_1, \quad -\frac{1}{4} < \delta_1 < \frac{1}{4}, & \text{if } \gamma_- < 0, \end{cases}$$
$$\delta_2 = \begin{cases} \frac{1}{2\pi} \tan^{-1}\Delta_2, \quad 0 \le \delta_2 < \frac{1}{2}, & \text{if } \gamma_- \ge 0, \\ \frac{1}{2\pi} \tanh^{-1}\Delta_2, & \text{if } \gamma_- \le 0, \end{cases}$$
$$\delta_3 = \frac{1}{2\pi} \tan^{-1}\left[\frac{w_+ \sin 2\theta}{\cos^2\theta - \beta \sin^2\theta}\right], \qquad \begin{array}{ll} 0 \le \delta_3 < \frac{1}{2}, & \text{for } 0 \le \theta < \frac{\pi}{2}, \\ \frac{1}{2} \le \delta_3 \le 1, & \text{for} \frac{\pi}{2} \le \theta \le \pi, \end{array}$$
$$\delta_4 = \frac{1}{\pi} \tan^{-1}\left[\left(\frac{C_{55}}{C_{44}}\right)^{1/2} \tan\theta,\right], \qquad \begin{array}{ll} 0 \le \delta_4 < \frac{1}{2}, & \text{for } 0 \le \theta < \frac{\pi}{2}, \\ \frac{1}{2} \le \delta_4 \le 1, & \text{for } \frac{\pi}{2} \le \theta \le \pi. \end{array}$$

Three special cases should be noted. In the first, when $\theta = \pi$, Φ, Γ_1, Γ_2 all vanish and $\Gamma_3 = \tau_+, \delta_4 = 1$. The results presented here reduce to (4.1).

The second special case is the degenerate case $\gamma_- = 0$, which corresponds to $p_1 = p_2$. When $\gamma_- = 0$, δ_1, δ_2 vanish while τ_- becomes infinite. A proper limit taken for Γ_1, Γ_2 in (5.9) leads to

$$\Gamma_1 = \frac{d_+ \sin 2\theta}{4\pi C_{22}(\cos^2\theta + \beta \sin^2\theta)}, \qquad \Gamma_2 = \frac{d_+d_- \sin^2\theta}{4\pi C_{22}(\cos^2\theta + \alpha \sin^2\theta)}.$$

The third special case is the case of isotropic materials for which $\gamma_- = 0$. If λ and μ are the Lamé constants,

$$\Gamma_1 = \frac{\lambda+\mu}{2\pi(\lambda+\mu)}\sin 2\theta, \qquad \Gamma_2 = \frac{\lambda+\mu}{\pi(\lambda+2\mu)}\sin^2\theta,$$

$$\Gamma_3 = \frac{\mu\theta}{\pi(\lambda+2\mu)}, \qquad e_+ = \frac{\lambda+3\mu}{2\mu}, \qquad e_- = \frac{1}{2}, \qquad \beta = 1,$$

$$d_+ = 2(\lambda+\mu), \qquad d_- = 2\mu, \qquad \Phi = 0, \qquad \delta_4 = \frac{\theta}{\pi}.$$

6. S, H, L, and $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$ referred to a rotated coordinate system. In applications, the planes of symmetry for orthotropic materials may not coincide with the coordinate planes. If the $x_3 = 0$ plane is the only plane which coincides with one of the planes of material symmetry, there is a simple relation by which $\mathbf{S}^*$, $\mathbf{H}^*$, $\mathbf{L}^*$, and $\hat{\mathbf{S}}^*(\theta^*)$, $\hat{\mathbf{H}}^*(\theta^*)$, $\hat{\mathbf{L}}^*(\theta^*)$ referred to this new coordinate system x_i^* can be obtained in terms of $\mathbf{S}$, $\mathbf{H}$, $\mathbf{L}$, $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$.

Let x_i^* be related to x_i by

$$x_i^* = \Omega_{ij}(\theta_0)x_j,$$

$$\Omega_{ij}(\theta_0) = \begin{bmatrix} \cos\theta_0 & \sin\theta_0 & 0 \\ -\sin\theta_0 & \cos\theta_0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus x_i^* is obtained by rotating the x_i coordinate system through an angle θ_0 about the x_3-axis. The fourth order tensor C_{ijks}^* referred to the x_i^* coordinate system is

$$C_{ijks}^* = \Omega_{ip}(\theta_0)\Omega_{jq}(\theta_0)\Omega_{kr}(\theta_0)\Omega_{st}(\theta_0)C_{pqrt}.$$

The 3×3 matrix $\mathbf{Q}^*(\theta^*)$ referred to the x_i^* coordinate system is

$$Q_{ik}^*(\theta^*) = C_{ijks}^* n_j(\theta^*) n_s(\theta^*),$$

Noticing that

$$\Omega_{jq}(\theta_0)n_j(\theta^*) = n_q(\theta_0+\theta^*),$$

we have

$$\mathbf{Q}^*(\theta^*) = \mathbf{\Omega}(\theta_0)\mathbf{Q}(\theta_0+\theta^*)\mathbf{\Omega}^T(\theta_0).$$

Similar equations are obtained for $\mathbf{R}^*(\theta^*)$ and $\mathbf{T}^*(\theta^*)$. It follows from the definition of $\mathbf{N}_i(\theta)$ in (1.4) that

$$\mathbf{N}_i^*(\theta^*) = \mathbf{\Omega}(\theta_0)\mathbf{N}_i(\theta_0+\theta^*)\mathbf{\Omega}^T(\theta_0).$$

Integration of both sides with respect to θ^* leads to

$$\left.\begin{aligned} \hat{\mathbf{S}}^*(\theta^*) &= \mathbf{\Omega}(\theta_0)[\hat{\mathbf{S}}(\theta_0+\theta^*) - \hat{\mathbf{S}}(\theta_0)]\mathbf{\Omega}^T(\theta_0), \\ \hat{\mathbf{H}}^*(\theta^*) &= \mathbf{\Omega}(\theta_0)[\hat{\mathbf{H}}(\theta_0+\theta^*) - \hat{\mathbf{H}}(\theta_0)]\mathbf{\Omega}^T(\theta_0), \\ \hat{\mathbf{L}}^*(\theta^*) &= \mathbf{\Omega}(\theta_0)[\hat{\mathbf{L}}(\theta_0+\theta^*) - \hat{\mathbf{L}}(\theta_0)]\mathbf{\Omega}^T(\theta_0). \end{aligned}\right\} \tag{6.1}$$

In particular, when $\theta^* = \pi$ we obtain from (5.1) and (6.1),

$$\left.\begin{aligned} \mathbf{S}^* &= \mathbf{\Omega}(\theta_0)\mathbf{S}\mathbf{\Omega}^T(\theta_0), \\ \mathbf{H}^* &= \mathbf{\Omega}(\theta_0)\mathbf{H}\mathbf{\Omega}^T(\theta_0), \\ \mathbf{L}^* &= \mathbf{\Omega}(\theta_0)\mathbf{L}\mathbf{\Omega}^T(\theta_0). \end{aligned}\right\} \tag{6.2}$$

Equations (6.1) and (6.2) allow us to find **S**, **H**, **L**, and $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$ for orthotropic materials for which only the $x_3 = 0$ plane coincides with a plane of material symmetry.

7. Explicit expressions for $\mathbf{S}(v)$, $\mathbf{H}(v)$, $\mathbf{L}(v)$. For a steady wave motion in the x_1-direction with a constant subsonic speed v, it can be shown [7] that an alternate representation of (1.7) is given by (2.5) provided that we replace $(2.2)_1$ by

$$Q_{ik} = C_{i1k1} - \rho v^2 \delta_{ik},$$

before evaluating the matrix **A** and **B**. Therefore, p_α, **A** and **B** depend on the real wave speed v. The calculations of $\mathbf{S}(v)$, $\mathbf{H}(v)$, $\mathbf{L}(v)$ parallel those of **S**, **H**, **L** in Sec. 3 and 4. The algebraic calculations are rather lengthy, but the final expressions can be substantially simplified [16, 18]. If we introduce the notations

$$\begin{aligned} C'_{ij} &= C_{ij} - \rho v^2 \delta_{ij}, \\ \tilde{C}_{ij} &= (C'_{ii} C_{jj})^{1/2}, \qquad i, j \text{ not summed,} \end{aligned}$$

in which δ_{ij} is the Kronecker delta and $\tilde{C}_{ij}$ has the property

$$\tilde{C}_{ik}\tilde{C}_{js} = \tilde{C}_{is}\tilde{C}_{jk},$$

the nonzero components of $\mathbf{S}(v)$, $\mathbf{H}(v)$, $\mathbf{L}(v)$ can be written as follows.

$$\left.\begin{aligned} L_{33}(v) = \tilde{C}_{54}, \qquad H_{33}(v) &= \tilde{C}_{54}^{-1}, \\ -\tilde{C}_{16} S_{12}(v) = \tilde{C}_{62} S_{21}(v) &= (\tilde{C}_{12} C_{66} - C_{12}\tilde{C}_{66}) W^{-1}, \\ \tilde{C}_{16} L_{22}(v) = \tilde{C}_{62} L_{11}(v) &= [(\tilde{C}_{12}^2 - C_{12}^2)\tilde{C}_{66} + (\tilde{C}_{66}^2 - C_{66}^2)\tilde{C}_{12}] W^{-1}, \\ \tilde{C}_{16} H_{11}(v) = \tilde{C}_{62} H_{22}(v) &= (\tilde{C}_{12} + \tilde{C}_{66}) W^{-1}, \\ W &= [(\tilde{C}_{12} + \tilde{C}_{66})^2 - (C_{12} + C_{66})^2]^{1/2}. \end{aligned}\right\} \tag{7.1}$$

When $v = 0$, we have $\tilde{C}_{66} = C_{66}$, $W = (\gamma_+ d_-)^{1/2}$, and the above expressions reduce to those of (4.1)

8. Concluding remarks. We have presented in this paper explicit expressions of **S**, **H**, **L**, $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$, and $\mathbf{S}(v)$, $\mathbf{H}(v)$, $\mathbf{L}(v)$ directly in terms of the elastic constants of orthotropic materials. The coordinate planes are assumed to coincide with the planes of material symmetry. If the material is referred to a coordinate system x_i^* which is obtained by rotating the x_i coordinate system an angle about the x_3-axis, a simple relation exists between $\mathbf{S}^*$, $\mathbf{H}^*$, $\mathbf{L}^*$, $\hat{\mathbf{S}}^*(\theta^*)$, $\hat{\mathbf{H}}^*(\theta^*)$, $\hat{\mathbf{L}}^*(\theta^*)$ and **S**, **H**, **L**, $\hat{\mathbf{S}}(\theta)$, $\hat{\mathbf{H}}(\theta)$, $\hat{\mathbf{L}}(\theta)$. No such relations exist between $\mathbf{S}^*(v)$, $\mathbf{H}^*(v)$, $\mathbf{L}^*(v)$ and $\mathbf{S}(v)$, $\mathbf{H}(v)$, $\mathbf{L}(v)$.

For **S**, **H**, **L** given in (4.1), we see the relations

$$\frac{-S_{21}}{S_{12}} = \frac{H_{22}}{H_{11}} = \frac{L_{11}}{L_{22}}, \qquad H_{33} = L_{33}^{-1}. \tag{8.1}$$

Equation $(2.7)_1$ provides the additional relation

$$H_{11} L_{11} - S_{12} S_{21} = 1, \tag{8.2}$$

which can be shown to be satisfied by **S**, **H**, **L** given in (4.1). The components of **S**(v), **H**(v), **L**(v) obtained in (7.1) also satisfy (8.1) and (8.2). Thus there are only four independent nonzero parameters for **S**, **H**, **L** or **S**(v), **H**(v), **L**(v).

One does not have to find the explicit expressions of **S**, **H**, **L**, or **S**(v), **H**(v), **L**(v) to realize that the number of independent parameters for orthotropic materials is four. It is shown in [23] that the number of independent parameters for monoclinic materials is five if the $x_3 = 0$ plane is the plane of symmetry and six if the $x_1 = 0$ or $x_2 = 0$ plane is the plane of symmetry. The number increases to nine for general anisotropic materials.

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References

[1] D. M. Barnett and J. Lothe, *Synthesis of the sextic and the integral formalism for dislocations, Green's function and surface waves in anisotropic elastic solids*, Phys. Norv. **7**, 13–19 (1973)

[2] A. N. Stroh, *Dislocations and cracks in anisotropic elasticity*, Phil Mag. **3**, 625–646 (1958)

[3] R. J. Asaro, J. P. Hirth, D. M. Barnett, and J. Lothe, *A further synthesis of sextic and integral theories for dislocations and line forces in anisotropic media*, Phys. Status Solidi B **60**, 261–271 (1973)

[4] D. M. Barnett and J. Lothe, *Consideration of the existence of surface wave (Rayleigh wave) solutions in anisotropic elastic crystals*, J. Phys. F. **4**, 671–686 (1974)

[5] D. M. Barnett and J. Lothe, *An image force theorem for dislocations in anisotropic bicrystals*, J. Phys. F. **4**, 1618–1635 (1974)

[6] D. M Barnett and J. Lothe, *Line force loadings on anisotropic half-spaces and wedges*, Phys. Norv. **8**, 13–22 (1975)

[7] P. Chadwick and G. D. Smith, *Foundations of the theory of surface waves in anisotropic elastic materials*, Adv. in Appl. Mech. **17**, 303–376 (1977)

[8] P. Chadwick and G. D. Smith, *Surface waves in cubic elastic materials*, Mechanics of Solids, The Rodney Hill 60th Anniversary Volume, edited by H. G. Hopkins and M. J. Sewell, Pergamon, Oxford, 47–100 (1982)

[9] P. Chadwick and D. A. Jarvis, *Surface waves in a pre-stressed elastic body*, Proc. Roy Soc. London A **366**, 517–536 (1979)

[10] T. C. T. Ting, *Line forces and dislocations in anisotropic elastic composite wedges and spaces*, Phys. Status Solidi, B **146**, 81–90 (1988)

[11] T. C. T. Ting, *Explicit solution and invariance of the singularities at an interface crack in anisotropic composites*, Int. J. Solids Structures **22**, 965–983 (1986)

[12] T. C. T. Ting *The critical angle of the anisotropic elastic wedge subject to uniform tractions*, J. Elasticity **20**, 113–130 (1988)

[13] Qianqian Li and T. C. T. Ting, *Line inclusions in anisotropic elastic solids*, J. Appl. Mech., in press (1989)

[14] Chyanbin Hwu and T. C. T. Ting, *Two-dimensional problems of the anisotropic elastic solids with an elliptic inclusion*, Quart. J. Mech. Appl. Math., in press (1989)

[15] H. O. K. Kirchner and J. Lothe, *Displacements and tractions along interfaces*, Phil. Mag. A **56**, 583–594 (1987)

[16] P. Chadwick and N. J. Wilson, *Surface waves in orthotropic and cubic elastic materials*, to appear, (1989)

[17] P. Chadwick, *Wave propagation in transversely isotropic elastic media.* I *Homogeneous plane waves.* II *Surface waves.* III *The special case $a_5 = 0$ and the inextensible limit*, Proc. Roy. Soc. London A **422**, 23–121 (1989)

[18] P. Chadwick, private communications

[19] J. D. Eshelby, W. T. Read, and W. Shockley, *Anisotropic elasticity with applications to dislocation theory*, Acta Metallurgica **1**, 251–259 (1953)

[20] T. C. T. Ting and Chyanbin Hwu, *Sextic formalism in anisotropic elasticity for almost non-semisimple matrix N*, Internat. J. Solids Structures **24**, 65–76 (1988)

[21] H. O. K. Kirchner and J. Lothe, *On the redundancy of the N matrix of anisotropic elasticity*, Phil. Mag. A **53**, L7–L10 (1986)

[22] P. Chadwick and T. C. T. Ting, *On the structure and invariance of the Barnett-Lothe tensors*, Quart. Appl. Math. **45**, 419–427 (1987)

[23] T. C. T. Ting, *The eigenvectors of S matrix and their relations with line dislocations and forces in anisotropic elastic solids*, Micromechanics and Inhomogeneity, The Toshio Mura Anniversary Volume, edited by G. J. Weng, M. Taya, and H. Abe, Springer-Verlag, New York, 1989, pp. 449–467