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ON THE COMPLETENESS OF THE PAPKOVICH-NEUBER SOLUTION*

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1. Introduction. The displacement equation of equilibrium without body force in the linear theory of homogeneous and isotropic elasticity in a region D is written as

$$\nabla^2 \mathbf{u} + \frac{1}{1-2\nu} \nabla(\nabla \cdot \mathbf{u}) = 0 \tag{1}$$

where $\mathbf{u}$ is the displacement field and ν denotes Poisson's ratio.

Papkovich [1] and Neuber [2] independently gave the general solution to (1) as

$$\mathbf{u} = \alpha \mathbf{B} - \nabla(\mathbf{x} \cdot \mathbf{B} + \Phi), \tag{2}$$

where α denotes $4(1-\nu)$ and $\mathbf{B}$ and Φ are harmonic, i.e.,

$$\nabla^2 \mathbf{B} = 0, \qquad \nabla^2 \Phi = 0. \tag{3}$$

The solution has been proved to be complete by Mindlin [3] and then by Gurtin [4] for an infinite region with suitable decay behavior of $\mathbf{u}$ at infinity. A summary of the development can be found in the book by Gurtin [5].

It was observed and unconvincingly proved by both Papkovich and Neuber that any one of the above four harmonic functions can be omitted in the above equation (2) without affecting its generality. The removal of some of the four harmonic functions of (2) as such is known as the completeness problem of the Papkovich-Neuber solution. This removal is of practical importance in the computation of the solutions to particular problems as well as of theoretical interest. Eubanks and Sternberg [6] gave the first correct proof that the function B_3 can be omitted when the region D under consideration is x^3-convex, i.e., every line segment parallel to x^3 joining any two points of D lies totally in D. The proof for the omission of Φ was built up gradually from the conjectural work of Slobodyansky [7] through Sokolnikoff's [8] and finally to the correct proof of Eubanks and Sternberg [6]. The completed version of the proof by Eubanks and Sternberg showed that the function Φ can be omitted when α is a noninteger and D is star-shaped with respect to the origin O, i.e., every line segment joining any point of D to the origin O lies totally in D and the representation without Φ is incomplete in this region D when α is an integer. Stippes [9] re-examined the completeness problems using the boundary value theory of harmonic functions, among those problems is the omission of Φ when the region D is interior to, exterior

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to, or between smooth, star-shaped surfaces (a surface is defined to be star-shaped when its normal vector is never at a right angle to the radial vector). Stippes's work on the omission of Φ is based on the theory of Fredholm equations as given by the book of Mikhlin [10]. Tran-Cong and Steven [11] gave a simplified proof to the same completeness problem without Φ in regions star-shaped with respect to the origin or infinity, using line integrals and the solution to a partial differential equation on a spherical surface. For an interesting review of older literature, the readers are refered to the introduction of the paper by Eubanks and Sternberg mentioned above.

It will be shown here that the Papkovich-Neuber solution is complete in terms of three harmonic functions B_1, B_2, and Φ in a region D which is x^3-convex with respect to a surface S where S can intersect more than once a line drawn parallel to the x^3-axis. (This is more general than Eubanks's and Sternberg's result which requires that the region D can only be x^3-convex.) For example, the region D can be the space occupied by a helical spring. The results are obtained simply by the use of curvilinear coordinates. The solution without Φ is shown to be incomplete when the region D is not x^3-convex. It will also be shown, using the same technique, that when α is a noninteger, the Papkovich-Neuber solution is complete in terms of three harmonic functions B_1, B_2, and B_3 in a region D which is radially convex with respect to a surface S. This is an extension of the technique used in [11]. The surface ∂D of D here needs not be regular as in Stippes's method and some finite parts of it can even be tangential to the radial lines while the surface S can intersect more than once a radial line drawn from the origin. The new method is presented along an argument parallel to that given for the omission of B_3. For the case where S is a closed surface, corresponding to the boundary of a star-shaped region, it gives a simpler route to obtain Stippes's result for the case of a region bounded by two star-shaped surfaces and the result for a region external to a star-shaped surface. For the case where S is not a closed surface, it gives a new result not considered previously. In this case, the method inevitablly leads to an application of the Fredholm theorems. The result is applicable to a number of geometries which are not star-shaped. For example, D can be the region occupied by an elastic foil of finite size, oriented parallel to the x^3-axis having its cross section in the x^1x^2-plane being a logarithmic spiral based on the origin O. The exceptional case with an integer value of α is also considered for a region radially convex with respect to a surface S. The solution without Φ is shown to be incomplete when the region D is not radially convex. Finally, a concise version of Stippes's boundary value method using Fredholm's equations (such as given in [10]), is represented for both cases of completeness without B_3 and without Φ, respectively, and the method is re-examined. An inconsistency is found between the results of this method and those of its preceding sections.

2. Completeness in terms of B_1, B_2, and Φ. The common notation of tensor calculus is adopted here as it allows us to change from one coordinate system to another without having to redefine various quantities. The cartesian coordinates of the Euclidean space are denoted by (x^1, x^2, x^3). A bar on top of a quantity denotes that same quantity as a function of the coordinate system $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$, i.e.,

$$\overline{\psi}(\overline{x}^1, \overline{x}^2, \overline{x}^3) \equiv \psi(x^1(\overline{x}^1, \overline{x}^2, \overline{x}^3), x^2(\overline{x}^1, \overline{x}^2, \overline{x}^3), x^3(\overline{x}^1, \overline{x}^2, \overline{x}^3)).$$

The summation convention is used so that repeated indices in the same term denote a sum over all allowed values of indices. Superscripts and subscripts are used for contravariant and covariant variables, respectively. For example, x^i is the component of the contravariant vector $\mathbf{x}$.

A region D is called x^3-convex with respect to a surface S in it if every point of D can be continuously projected along a line segment L in the x^3-direction to one, and no more than one, point on S such that L lies totally in D.

We rewrite the solution (2) in indicial form as

$$u_i = \alpha B_i - \frac{\partial}{\partial x^i}(B_i x^i + \Phi) \tag{4}$$

where B_i is the ith component of the covariant vector $\mathbf{B}$.

The necessary and sufficient condition for the omission of the harmonic function B_3 (Eubanks and Sternberg [6]) is that we have a solution ψ to the equations

$$\frac{\partial \psi}{\partial x^3} = B_3 \quad \text{and} \quad \nabla^2 \psi = 0. \tag{5a,b}$$

The function B_3 is to be replaced by the scalar function $\partial\psi/\partial x^3$ to give the same displacement.

Here we generalise the result such that the surface S can intersect more than once a line drawn parallel to the x^3 axis. This is done by using a general curvilinear coordinate system $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$. We can form a coordinate system $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$ such that any point of D can be joined by a line of constant $(\overline{x}^1, \overline{x}^2)$ to the surface S. The variable $\overline{x}^3$ is chosen to be $\overline{x}^3 = x^3$, and the variables $\overline{x}^1$ and $\overline{x}^2$ are chosen such that the metric tensor is given by

$$g \equiv \begin{pmatrix} \overline{g}_{11} & \overline{g}_{12} & \overline{g}_{13} \\ \overline{g}_{21} & \overline{g}_{22} & \overline{g}_{23} \\ \overline{g}_{31} & \overline{g}_{32} & \overline{g}_{33} \end{pmatrix} = \begin{pmatrix} \overline{m}(\overline{x}^1, \overline{x}^2) & 0 & 0 \\ 0 & \overline{m}(\overline{x}^1, \overline{x}^2) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

The surface S comprises points of the form $(\overline{x}^1, \overline{x}^2, \overline{x}^3 = \overline{f}(\overline{x}^1, \overline{x}^2))$. The single valued function $\overline{f}(\overline{x}^1, \overline{x}^2)$ is of class $C^{2+\beta}$, $\beta > 0$ in S, that is, its second derivatives satisfy the Hölder condition of order β in the projection A of S onto the surface $\overline{x}^1\overline{x}^2$.

We choose the function $\overline{\psi}$ to be

$$\overline{\psi}(\overline{x}^1, \overline{x}^2, \overline{x}^3) = \int_{\overline{f}(\overline{x}^1, \overline{x}^2)}^{\overline{x}^3} \overline{B}_3 \, d\overline{x}^3 + \overline{\chi}(\overline{x}^1, \overline{x}^2) \tag{7}$$

where the twice differentiable function $\overline{f}(\overline{x}^1, \overline{x}^2)$ is the function describing the surface S in the coordinates $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$ as mentioned above. Its Laplacian is given in the curvilinear coordinate system $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$ as

$$\begin{aligned} \nabla^2 \overline{\psi} &\equiv \frac{1}{(\det g)^{1/2}} \frac{\partial}{\partial \overline{x}^i} \left((\det g)^{1/2} g^{ij} \frac{\partial}{\partial \overline{x}^j} \overline{\psi} \right) \\ &= \frac{\partial^2 \overline{\psi}}{(\partial \overline{x}^3)^2} + \frac{1}{\overline{m}(\overline{x}^1, \overline{x}^2)} \left(\frac{\partial^2}{(\partial \overline{x}^1)^2} + \frac{\partial^2}{(\partial \overline{x}^2)^2} \right) \overline{\psi}, \end{aligned}$$

which gives

$$\frac{\partial}{\partial \overline{x}^3}\nabla^2(\overline{\psi}-\overline{\chi}) = \nabla^2\frac{\partial}{\partial \overline{x}^3}(\overline{\psi}-\overline{\chi}) = \nabla^2\overline{B}_3 = 0.$$

Hence

$$\left[\frac{\partial^2}{(\partial \overline{x}^3)^2} + \frac{1}{\overline{m}(\overline{x}^1,\overline{x}^2)}\left(\frac{\partial^2}{(\partial \overline{x}^1)^2} + \frac{\partial^2}{(\partial \overline{x}^2)^2}\right)\right](\overline{\psi}-\overline{\chi}) = \overline{h}(\overline{x}^1,\overline{x}^2) \tag{8}$$

where $\overline{h}$ can be directly calculated to be

$$\begin{aligned}
&\nabla^2(\overline{\psi}-\overline{\chi})\\
&= \int_{\overline{f}(\overline{x}^1,\overline{x}^2)}^{\overline{x}^3}\left[\frac{\partial^2}{(\partial \overline{x}^3)^2} + \frac{1}{\overline{m}(\overline{x}^1,\overline{x}^2)}\left(\frac{\partial^2}{(\partial \overline{x}^1)^2} + \frac{\partial^2}{(\partial \overline{x}^2)^2}\right)\right]\overline{B}_3\, d\overline{x}^3 + \left(\frac{\partial \overline{B}_3}{\partial \overline{x}^3}\right)_{\overline{x}^3=\overline{f}(\overline{x}^1,\overline{x}^2)}\\
&\quad - \frac{1}{\overline{m}(\overline{x}^1,\overline{x}^2)}\left(2\frac{\partial \overline{f}}{\partial \overline{x}^1}\frac{\partial \overline{B}_3}{\partial \overline{x}^1} + 2\frac{\partial \overline{f}}{\partial \overline{x}^2}\frac{\partial \overline{B}_3}{\partial \overline{x}^2} + \overline{B}_3\left[\frac{\partial^2 \overline{f}}{(\partial \overline{x}^1)^2} + \frac{\partial^2 \overline{f}}{(\partial \overline{x}^2)^2}\right]\right)_{\overline{x}^3=\overline{f}(\overline{x}^1,\overline{x}^2)}\\
&= \left(\frac{\partial \overline{B}_3}{\partial \overline{x}^3}\right)_{\overline{x}^3=\overline{f}} - \frac{1}{\overline{m}}\left(2\frac{\partial \overline{f}}{\partial \overline{x}^1}\frac{\partial \overline{B}_3}{\partial \overline{x}^1} + 2\frac{\partial \overline{f}}{\partial \overline{x}^2}\frac{\partial \overline{B}_3}{\partial \overline{x}^2} + \overline{B}_3\left[\frac{\partial^2 \overline{f}}{(\partial \overline{x}^1)^2} + \frac{\partial^2 \overline{f}}{(\partial \overline{x}^2)^2}\right]\right)_{\overline{x}^3=\overline{f}}\\
&\equiv -\overline{h}(\overline{x}^1,\overline{x}^2).
\end{aligned} \tag{9}$$

For the function $\overline{\psi}$ to be harmonic, we need only to make $\overline{\chi}$ a solution to the equation

$$\frac{1}{\overline{m}(\overline{x}^1,\overline{x}^2)}\left(\frac{\partial^2}{(\partial \overline{x}^1)^2} + \frac{\partial^2}{(\partial \overline{x}^2)^2}\right)\overline{\chi}(\overline{x}^1,\overline{x}^2) = \overline{h}(\overline{x}^1,\overline{x}^2). \tag{10}$$

The function $\overline{h}(\overline{x}^1,\overline{x}^2)$ satisfies the Hölder condition of order β due to our assumption on the behavior of $\overline{f}(\overline{x}^1,\overline{x}^2)$. The solution χ is given by classical potential theory for finite S as

$$\overline{\chi}(\overline{y}^1,\overline{y}^2) = \frac{1}{2}\int\int_S \ln((\overline{x}^1-\overline{y}^1)^2 + (\overline{x}^2-\overline{y}^2)^2)\overline{m}(\overline{x}^1,\overline{x}^2)\overline{h}(\overline{x}^1,\overline{x}^2)\, d\overline{x}^1\, d\overline{x}^2, \tag{11}$$

and the completeness is proved.

By choosing $\overline{x}^1 = (\frac{1}{2})\ln((x^1)^2 + (x^2)^2)$, $\overline{x}^2 = \arctan(x^1/x^2)$, and $\overline{f} = \overline{f}(\overline{x}^2)$, we can form a helical surface S. The result is that the Papkovich-Neuber solution in terms of B_1, B_2, Φ is complete for the region D occupied by an elastic helical coil spring with its axis oriented along the x_3 direction. This tricky geometry is one of many others not covered by the results by Eubanks and Sternberg. (Their work also did not pay attention to the requirement that the lower limit of the integral to define their function H, which is equivalent to $\overline{\psi}$ here, needs to be of class $C^{2+\beta}$.)

3. Incompleteness in terms of B_1, B_2, and Φ in regions not x^3-convex. Consider a region D inside the sphere centered on the origin O and of radius $R > 3$. This sphere has a small spherical cut of radius ε, centered on the point $(x^1 = 0,\, x^2 = 0,\, x^3 = 1)$. Let the Papkovich-Neuber solution take the form

$$u_i = \alpha\delta_i^3\frac{1}{((x^1)^2 + (x^2)^2 + (x^3-1)^2)^{1/2}} - \frac{\partial}{\partial x^i}\left(x^3\frac{1}{((x^1)^2 + (x^2)^2 + (x^3-1)^2)^{1/2}}\right), \tag{12}$$

where δ_i^3 is the Kronecker delta function, then this solution cannot be represented in terms of three harmonic functions B_1, B_2, and Φ.

We have a solution to (5) given as

$$\psi(x^1,x^2,x^3) = \int_0^{x^3} \frac{1}{((x^1)^2+(x^2)^2+(x^3-1)^2)^{1/2}}\,dx^3 + \chi(x^1,x^2) \tag{13}$$

which is harmonic for every point of distance less than unity from the plane x^1x^2. This function ψ tends to infinity close to the semi-infinite straight line ($x^1 = 0$, $x^2 = 0$, $x^3 > 1$).

Suppose that there is another harmonic function η which is defined in D and satisfies Eq. (5). This harmonic function η is bounded on the disc $((x^1)^2+(x^2)^2 < 1$, $x^3 = 2)$. Equation (5) then gives

$$\frac{\partial}{\partial x^3}(\eta - \psi) = 0. \tag{14}$$

This leads to the following equality inside the unit sphere centered on O

$$\eta(x^1,x^2,x^3) = \psi(x^1,x^2,x^3) + \zeta(x^1,x^2) \tag{15}$$

where ζ is defined and harmonic with respect to x^1 and x^2, in the circular disc $(x^1)^2+(x^2)^2 \le R^2$. Since the harmonic function on the right-hand side of Eq. (15) is equal to the harmonic function η inside the unit sphere centered on O, it must be equal to this harmonic function η in all of D, by continuation theorem (Kellog [12, p. 259]). Therefore, the function η is unbounded close to the semi-infinite straight line ($x^1 = 0$, $x^2 = 0$, $x^3 > 1$). This is contrary to our initial assumption on η. The incompleteness of the Papkovich-Neuber solution in a nonconvex region has been proved.

4. Completeness in terms of B_1, B_2, and B_3. A region D is called radially convex with respect to a surface S in it if every point of D can be continuously projected along a line segment L in the radial direction to one, and no more than one, point on S such that L lies totally in D.

In this case we can form a coordinate system $(\overline{x}^1,\overline{x}^2,\overline{x}^3)$ from (x^1,x^2,x^3) such that any point of D can be joined by a line of constant $(\overline{x}^1,\overline{x}^2)$ to the surface S and $\overline{x}^3$ is chosen to be $\overline{x}^3 = ((x^1)^2+(x^2)^2+(x^3)^2)^{1/2}$. Similar to the case of Sec. 2, the surface S comprises points of the form $(\overline{x}^1,\overline{x}^2,\overline{x}^3 = \overline{f}(\overline{x}^1,\overline{x}^2))$ where the single-valued function $\overline{f}(\overline{x}^1,\overline{x}^2)$ is of class $C^{2+\beta}$, $\beta > 0$ in S.

Equation (4) can also be written in the coordinates $(\overline{x}^1,\overline{x}^2,\overline{x}^3)$ as

$$\overline{u}_i = \alpha\overline{B}_i - \frac{\partial}{\partial \overline{x}_i}(\overline{B}_3\overline{x}^3 + \overline{\Phi}) \tag{16}$$

which simplifies our subsequent calculations.

The necessary and sufficient condition for the omission of the harmonic function $\overline{\Phi}$ when $0 \ne \alpha \ne 2$ is that [11] we have a solution ψ to the equations

$$\alpha\psi - x^i\frac{\partial\psi}{\partial x^i} = \Phi \quad \text{and} \quad \nabla^2\psi = 0. \tag{17a,b}$$

The vector function **B** to replace the scalar function $\nabla\Phi$ to give the same displacement is $\nabla\psi$.

In the coordinate system $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$, Eq. (17a) is written as

$$\alpha\overline{\psi} - \overline{x}^3 \frac{\partial\overline{\psi}}{\partial\overline{x}^3} = \overline{\Phi} \quad \text{and} \quad \nabla^2\overline{\psi} = 0. \tag{18a,b}$$

By putting $\overline{\overline{x}}^1 = \overline{x}^1$, $\overline{\overline{x}}^2 = \overline{x}^2$, and $\overline{\overline{x}}^3 = \ln(\overline{x}^3)$ we have a simplified equation to determine ψ as below

$$\frac{\partial}{\partial\overline{\overline{x}}^3}(\overline{\overline{\psi}} e^{-\alpha\overline{\overline{x}}^3}) = -\alpha e^{-\alpha\overline{\overline{x}}^3}\overline{\overline{\Phi}}$$

which gives the solution ψ as

$$\overline{\psi}(\overline{x}^3)^{-\alpha} = -\alpha \int_{\overline{f}(\overline{x}^1,\overline{x}^2)}^{\overline{x}^3} (\overline{x}^3)^{-1-\alpha}\overline{\Phi}\, d\overline{x}^3 + \overline{\chi}(\overline{x}^1, \overline{x}^2) \tag{19}$$

where $\overline{\chi}(\overline{x}^1, \overline{x}^2)$ is an arbitrary function of only $\overline{x}^1$ and $\overline{x}^2$.

We have thus found a function ψ satisfying Eq. (18a). It remains only to prove that we can select the function $\chi(\overline{x}^1, \overline{x}^2)$ such that ψ is harmonic in the region D. For the special case where $\overline{f}(\overline{x}^1, \overline{x}^2)$ is equal to a constant c, i.e., S is on a spherical surface, direct calculations [11] give

$$\begin{aligned}
\nabla^2\overline{\psi} &= -\alpha\nabla^2\left[(\overline{x}^3)^\alpha \int_c (\overline{x}^3)^{-1-\alpha}\overline{\Phi}\, d\overline{x}^3\right] + \nabla^2[(\overline{x}^3)^\alpha\overline{\chi}(\overline{x}^1, \overline{x}^2)] \\
&= (\overline{x}^3)^{(\alpha-2)}\left[-\alpha\left((\alpha+1)c^{-\alpha}\overline{\Phi}(\overline{x}^1, \overline{x}^2, c) + c^{-\alpha+1}\left(\frac{\partial\overline{\Phi}}{\partial\overline{x}^3}\right)_{\overline{x}^3=c}\right)\right. \\
&\qquad \left. + (\alpha+1)\alpha\overline{\chi} + \nabla_*^2\overline{\chi}\right],
\end{aligned} \tag{20}$$

which enables c to be set at convenient values to make ψ harmonic. For the general case where $\overline{f}(\overline{x}^1, \overline{x}^2)$ is not a constant, we note that

$$0 = \nabla^2\Phi = \nabla^2\left(\alpha\psi - x^i\frac{\partial\psi}{\partial x^i}\right) = \left[(\alpha-2) - x^i\frac{\partial}{\partial x^i}\right]\nabla^2\psi$$

which gives, similarly to the previous calculations,

$$\frac{\partial}{\partial\overline{\overline{x}}^3}(\nabla^2\overline{\overline{\psi}} e^{-\alpha\overline{\overline{x}}^3}) = 0$$

or

$$\nabla^2\overline{\psi} = -(\overline{x}^3)^{\alpha-2}\overline{h}(\overline{x}^1, \overline{x}^2) \tag{21}$$

where $\overline{h}(\overline{x}^1, \overline{x}^2)$ is a function of $\overline{x}^1$ and $\overline{x}^2$. The function $\overline{h}(\overline{x}^1, \overline{x}^2)$ can also be directly calculated as in Sec. 2 but it suffices to say that $\overline{h}(\overline{x}^1, \overline{x}^2)$ satisfies the Hölder condition of order β due to our assumption on the property of $\overline{f}(\overline{x}^1, \overline{x}^2)$. It is clear that ψ is harmonic if we can make $\overline{\chi}(\overline{x}^1, \overline{x}^2)$ a solution to the equation

$$\nabla_*^2\overline{\chi} + \alpha(\alpha+1)\overline{\chi}(\overline{x}^1, \overline{x}^2) = \overline{h}(\overline{x}^1, \overline{x}^2) \tag{22}$$

where ∇_*^2 denotes

$$\nabla_*^2 \equiv r^2\nabla^2 - \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right).$$

There are at least two methods of solving for the solution $\overline{\chi}$ of Eq. (22) which are applicable to two different cases, as in the following.

4.1 *Solution when S is the boundary of a star-shaped region.* In this case, we choose the contravariant variables $\overline{x}^1$ and $\overline{x}^2$ to be $\overline{x}^1 = \theta \equiv (x^2/|x^2|) \times \operatorname{Arccos}(x^1/((x^1)^2 + (x^2)^2)^{1/2})$ and $\overline{x}^2 = \gamma \equiv \operatorname{Arcsin}(x^3/((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2})$. As S is a star-shaped boundary of a star-shaped region E, we have the function $\overline{f}$ being single-valued for every given value of $(\overline{x}^1, \overline{x}^2)$. We limit this study to only the case of a noninteger value of α. The following proof is valid only for $\alpha > -1$, which corresponds to $\nu < \frac{5}{4}$.

We define the following functions $P_n(\mathbf{x}, \mathbf{y})$

$$P_{-1}(\mathbf{x}, \mathbf{y}) = 0 \qquad P_0(\mathbf{x}, \mathbf{y}) = \frac{1}{|\mathbf{x}|}$$

and

$$P_n(\mathbf{x}, \mathbf{y}) = \sum_{i_1=1}^{3}\sum_{i_2=1}^{3}\cdots\sum_{i_n=1}^{3}(-1)^n y^{i_1}y^{i_2}\cdots y^{i_n}\frac{\partial^n}{\partial x^{i_1}\partial x^{i_2}\cdots\partial x^{i_n}}\frac{1}{|\mathbf{x}|} \tag{23}$$

and construct the function $\xi(\mathbf{x})$ by

$$\xi(\mathbf{x}) = \int_{-\pi}^{\pi}\int_0^{2\pi}\int_1^{\infty}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|} - \sum_{i=-1}^{[\alpha-]}P_i(\mathbf{x}, \mathbf{y})\right)|\mathbf{y}|^{-\alpha-1}\overline{h}(\theta, \gamma)\cos\gamma d|\mathbf{y}|\,d\theta\,d\gamma \tag{24}$$

where $[\alpha-]$ denotes the largest signed integer number still less than α. We note that the quantity inside the large parentheses of Eq. (24) is of order $O(|\mathbf{y}|^{[\alpha-]})$ as $|\mathbf{y}|$ tends to infinity and is of order $O(|\mathbf{x}|^{-[\alpha-]-2})$ as $|\mathbf{x}|$ tends to infinity. Therefore, we have $\xi(\mathbf{x})$ defined for all $|\mathbf{x}| \neq 0$. The function $\xi(\mathbf{x})$ is continuous at every point outside the sphere except at infinity and satisfies

$$\nabla^2\overline{\xi} = \nabla^2\int_{-\pi}^{\pi}\int_0^{2\pi}\int_1^{\infty}\frac{1}{|\mathbf{x}-\mathbf{y}|}|\mathbf{y}|^{-\alpha-1}\overline{h}(\theta, \gamma)\cos\gamma d|\mathbf{y}|\,d\theta\,d\gamma = (\overline{x}^3)^{-\alpha-3}\overline{h}(\theta, \gamma), \tag{25}$$

and we also have

$$\xi(\mathbf{x}) = O(|\mathbf{x}|^{-[\alpha-]-2}) \quad \text{as } |\mathbf{x}| \to \infty. \tag{26}$$

By virtue of Eq. (26), we can define a function $\overline{\chi}(\overline{x}^1, \overline{x}^2)$ by

$$\overline{\chi}(\overline{x}^1, \overline{x}^2) = \lim_{R\to\infty}\left[\frac{1}{\ln R}\int_1^R \overline{\xi}(\overline{x}^1, \overline{x}^2, \overline{x}^3)(\overline{x}^3)^{\alpha}\,d\overline{x}^3\right], \tag{27}$$

which satisfies

$$\begin{aligned}&\nabla_*^2\overline{\chi}(\overline{x}^1, \overline{x}^2) + \alpha(\alpha+1)\overline{\chi}(\overline{x}^1, \overline{x}^2)\\ &\quad = \overline{h}(\overline{x}^1, \overline{x}^2) - \lim_{R\to\infty}\left[\frac{1}{\ln R}\left((\overline{x}^3)^{\alpha+2}\frac{\partial\overline{\xi}}{\partial\overline{x}^3} + \alpha(\alpha+1)(\overline{x}^3)^{\alpha+1}\overline{\xi}\right)_{\overline{x}^3=1}^{R}\right].\end{aligned} \tag{28}$$

From Eq. (26), it is easy to see that the last term of Eq. (28) vanishes as R tends to infinity, and $\overline{\chi}$ becomes a solution to Eq. (22). As a matter of interest, we can also put $\alpha^* = -\alpha$ to prove that the proof also holds for $\alpha < -1$.

This result corresponds with Stippes's results regarding the region between two star-shaped surfaces.

4.2 *Solution using Fredholm's equation.* If the function $\overline{f}(\overline{x}^1, \overline{x}^2)$ does not describe the boundary of a star-shaped region E then the technique of the previous subsection does not work. This is caused by the use of the intermediate function $\overline{\xi}(\theta, \gamma)$. Therefore, we need to use a technique employing only $\overline{x}^1$ and $\overline{x}^2$ for the general case.

We introduce the variable $\eta = \ln |[1+\tan(\gamma/2)]/[1-\tan(\gamma/2)]|$ so that the Laplacian operator becomes

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{(e^{2\eta}+1)^2}{4r^2e^{2\eta}}\left(\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial^2\eta}\right) \tag{29}$$

and Eq. (22) becomes

$$\frac{4e^{2\eta}}{(e^{2\eta}+1)^2}\overline{h}(\theta,\eta) = \alpha(\alpha+1)\frac{4e^{2\eta}}{(e^{2\eta}+1)^2}\overline{\chi}(\theta,\eta) + \left(\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\eta^2}\right)\overline{\chi}(\theta,\eta). \tag{30}$$

As the surface S is finite in the plane (θ, η), we can define

$$\overline{\chi}(\theta,\eta) = \int\int_S \frac{1}{2}\ln((\theta-\overline{y}^1)^2 + (\eta-\overline{y}^2)^2)\overline{\xi}(\overline{y}^1,\overline{y}^2)\,d\overline{y}^1\,d\overline{y}^2 \tag{31}$$

to have Eq. (30) transformed into

$$\overline{\xi}(\overline{x}) + \lambda\int\int_S \overline{K}(\overline{x},\overline{y})\overline{\xi}(\overline{y})\,d\overline{y}^1\,d\overline{y}^2 = \overline{F}(\overline{x}) \tag{32}$$

where

$$\lambda = \alpha(\alpha+1) \qquad \overline{K}(\overline{x},\overline{y}) = \frac{2e^{2\overline{x}^2}}{(e^{2\overline{x}^2}+1)^2}\ln((\overline{x}^1-\overline{y}^1)^2 + (\overline{x}^2-\overline{y}^2)^2)$$

and

$$\overline{F}(\overline{x}) = \frac{4e^{2\overline{x}^2}}{(e^{2\overline{x}^2}+1)^2}\overline{h}(\overline{x}), \tag{33}$$

which is an integral equation with a weak singularity and has the Fredholm theorems applicable. For noncharacteristic values of λ, it is solvable and the solution is unique. But it is still a problem to determine the countable set of characteristic values (with its only possible accumulation point at infinity). We will determine this set of characteristic values by an indirect method.

We note that if S is only part of a closed star-shaped surface we can extend it to have a closed star-shaped surface of the kind considered in the previous section and then prove that (22) has a solution to whatever $\overline{h}(\overline{x}^1, \overline{x}^2) \in C^\beta$, $\beta > 0$ on S when α is a noninteger. (Alternatively, we can also arrive at the same result by applying exactly the method of the previous subsection to only the solid angle $\Omega \leq 4\pi$ sustained by S.) Therefore, a noninteger cannot be a characteristic value of the Fredholm equation (32). On the other hand when α is an integer, the surface spherical harmonic functions readily give the nontrivial solutions to the homogeneous equation (22).

Therefore, when S is only part of a closed star-shaped surface, the set of characteristic values of Eq. (32) are all values of the form $\lambda = (\alpha + 1)\alpha$ where α is an integer.

When S is more than part of a closed star-shaped surface, it is obvious that the set of characteristic values to Eq. (32) still contains the set of values $\lambda = (\alpha+1)\alpha$ where α is an integer. We will prove that it contains nothing else: Let α be a noninteger giving a characteristic value of Eq. (32). The homogeneous equation (22) must then have a nontrivial solution $\overline{\xi}$. There must be a surface S_1 in S which can be considered part of a closed star-shaped surface on which $\overline{\xi}$ is nontrivial. Therefore S_1 has a characteristic value $\lambda = (\alpha + 1)\alpha$ where α is a noninteger. This is contrary to the preceding result. Therefore we conclude that the equation (32) on a star-shaped surface has all its characteristic values given by $\lambda = (\alpha + 1)\alpha$ where α is an integer.

When the surface S intersects the x^3-axis, Eq. (32) cannot be used to prove that Eq. (22) is equivalent to a Fredholm equation since Eq. (30) ceases to apply to this case. However, we still can prove that Eq. (22) with zero on the right-hand side has a nontrivial solution when α is an integer, and Eq. (22) has a unique solution for an arbitrarily given function $\overline{h} \in C^\beta$ on the right-hand side: We note that the spherical surface harmonic of degree α is a solution to the homogeneous equation (22) when α is an integer. When α has a noninteger value, we divide S into smaller parts S_n's each of which is in a quadrant limited by the three axes x^1, x^2, x^3, does not intersect an appropriately chosen straight line through the origin, and therefore corresponds to a Fredholm equation which has a unique solution to any function $\overline{h} \in C^\beta$. It remains only to prove that the solutions on each part of S_n are a continuation of the solution on adjacent surfaces. Let two adjacent surfaces S_k and S_m have the solutions $\overline{\xi}_k$ and $\overline{\xi}_m$ to a given function $\overline{h} \in C^\beta$. We choose a part S_μ of S overlapping both S_k and S_m. This surface S_μ also does not intersect an appropriately chosen straight line through the origin, therefore has the equation (22) on it corresponding to a Fredholm equation. Since $\overline{\xi}_k$, $\overline{\xi}_m$, $\overline{\xi}_\mu$ are unique solutions on the three overlapping surfaces S_k, S_m, S_μ, respectively, for a noninteger α, they are the continuation of each other.

Consequently, the Papkovich-Neuber solution in terms of three harmonic functions B_1, B_2, B_3 is complete when α is a noninteger and the region D is radially convex with respect to a surface S described by $\overline{f}(\overline{x}^1, \overline{x}^2)$.

The result of this section is applicable to tricky geometries such as the case of an elastic foil coiled into a logarithmic spiral shape given by $(x^1)^2 + (x^2)^2 = \exp(2\theta)$ where θ is the angle from the x^1 axis to the radial line, i.e., $\theta = \arctan(x^2/x^1)$. This is covered by neither the results by Eubanks and Sternberg nor by those by Stippes.

4.3 *Solution to* (18) *when α is an integer.* When α is an integer and the region D is star-shaped, the solution has been shown by Eubanks and Sternberg [6] to be incomplete. (It is easy to see that the representation remains complete if Φ is only limited to be a harmonic function of degree α.) Their conclusion was reconfirmed by later studies ([9] and [11]).

When D is star-shaped with respect to infinity, i.e., every semi-infinite radial line drawn from a point of D lies totally in D, and Φ is of order $O(|x|^{-1})$ as $\mathbf{x}$ tends to infinity then Φ can be omitted. This was proved in [11] by letting c of Eq. (20) tend to infinity and $\overline{\chi}$ be identically zero. To prove that the term

$$\left(\frac{\partial \overline{\Phi}}{\partial \overline{x}^3}\right)_{\overline{x}^3=c}$$

is of order $O(|\mathbf{x}|^{-2})$ as $\mathbf{x}$ tends to infinity, we note that Φ can be turned into the Newtonian potential of a distribution in the star-shaped region. Therefore its radial derivative has the required order of vanishing at infinity. Hence the Papkovich-Neuber solution without Φ is complete in a region D complement to a finite star-shaped region, for integer as well as noninteger values of α. This is more general than Stippes's result for a region external to a smooth star-shaped surface.

Finally, we consider the completeness for the finite, regular region D being the region between two disjointed, closed star-shaped surfaces S_0 and S_1. The surface S_0 encloses S_1 which, in turn, encloses the origin O. As any potential Φ inside D can be turned into the Newtonian potential of the distribution of charges on S_0 and S_1, it can be written as the sum of a potential Φ_0, due to the distribution on S_0, and Φ_1, due to that on S_1. The harmonic functions Φ_0 and Φ_1 are defined on the star-shaped region inside S_0 and on the star-shaped region outside S_1, respectively. Therefore we can apply the preceding results for the omission of Φ to these two terms individually. The requirement that S_0 and S_1 are disjointed can also be relaxed to give a more general result: The Papkovich-Neuber solution in a finite region D bounded by two star-shaped surfaces is still complete when Φ is limited to be a harmonic function of degree α. Again, the result is more general than Stippes's result for a region between two smooth star-shaped surfaces.

5. Incompleteness in terms of B_1, B_2, and B_3 in regions not radially-convex. We use the same region D of Sec. 3, which is inside the sphere centered on the origin O and of radius $R > 3$. This sphere has a small spherical cut of radius ε, centered on the point $(x^1 = 0,\ x^2 = 0,\ x^3 = 1)$. The Papkovich-Neuber solution of the form

$$\mathbf{u} = -\nabla\left(\frac{1}{((x^1)^2 + (x^2)^2 + (x^3 - 1)^2)^{1/2}}\right), \tag{34}$$

cannot be represented in terms of three harmonic functions B_1, B_2, and B_3.

To prove the incompleteness, we form a solution to (16) as

$$\overline{\psi}(\overline{x}^3)^{-\alpha} = -\alpha\int_0^{\overline{x}^3} (\overline{x}^3)^{-1-\alpha}\overline{\Phi}\, d\overline{x}^3 + \overline{\chi}(\overline{x}^1, \overline{x}^2) \tag{35}$$

which is harmonic for every point of distance less than unity from the origin. This function ψ tends to infinity close to the semi-infinite straight line $(x^1 = 0,\ x^2 = 0,\ x^3 > 1)$. An analogous argument to that used in Sec. 3 then proves that the representation is incomplete in terms of only three harmonic functions B_1, B_2, and B_3. (This incompleteness has been considered previously by this author in his Ph.D. thesis, 1979.)

6. Completeness by boundary value method. With the preceding sections giving an adequate insight into the nature of the completeness problem, we now examine a sophisticated but fairly general boundary method as pioneered by Stippes. The results from this method are compared against those of the preceding sections.

We note that when ψ is a harmonic function satisfying

$$\nabla^2 \psi = 0 \quad \text{in } D \tag{36}$$

the following two functions

$$\frac{\partial \psi}{\partial x^3} \quad \text{and} \quad \alpha\psi - \mathbf{x} \cdot \nabla \psi$$

are also harmonic in D. Since the given function B_3 and Φ of Secs. 2 and 4 are harmonic, we only need to have

$$\frac{\partial \psi}{\partial x^3} = B_3 \quad \text{on } \partial D \quad \text{or} \quad \alpha\psi - \mathbf{x} \cdot \nabla \psi = \Phi \quad \text{on } \partial D \tag{37a,b}$$

respectively. The result is that

The Papkovich-Neuber solution is complete without B_3 or Φ according to whether or not we can find a harmonic function ψ in D satisfying (36) and either (37a) or (37b), respectively.

Following Stippes, we note that (37a,b) gives the potential ψ in D of a single layer of charge density σ on the surface ∂D, i.e.,

$$\psi(\mathbf{x}) = \int\int_{\partial D} \frac{\sigma(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \, dS(\mathbf{y}). \tag{38}$$

The equations for the surface charge σ are derived from Eqs. (37a) and (37b) as

$$2\pi\sigma(\mathbf{x})\mathbf{e}_3 \cdot \mathbf{n}(\mathbf{x}) + \int\int_{\partial D} -\frac{\mathbf{e}_3 \cdot [\mathbf{x} - \mathbf{y}]}{|\mathbf{x} - \mathbf{y}|^3} \sigma(\mathbf{y}) \, dS(\mathbf{y}) = B_3(\mathbf{x}) \tag{39a}$$

and

$$2\pi\sigma(\mathbf{x})\mathbf{x} \cdot \mathbf{n}(\mathbf{x}) + \int\int_{\partial D} \left(-\frac{\mathbf{x} \cdot [\mathbf{x} - \mathbf{y}]}{|\mathbf{x} - \mathbf{y}|^3} - \frac{\alpha}{|\mathbf{x} - \mathbf{y}|} \right) \sigma(\mathbf{y}) \, dS(\mathbf{y}) = \Phi(\mathbf{x}) \tag{39b}$$

respectively. They are singular Fredholm equations on two-dimensional surfaces, according to Stippes (based on [10]), when $\mathbf{e}_3 \cdot \mathbf{n} \neq 0$ on all ∂D for (39a) and $\mathbf{x} \cdot \mathbf{n}(\mathbf{x}) \neq 0$ on all ∂D for (39b).

An elegant way to see whether Eqs. (39a,b) can have a solution for any arbitrarily given harmonic function in the right-hand side is to examine their suitably chosen conjugate equations. We first divide Eqs. (39a) and (39b) by $(\mathbf{e}_3 \cdot \mathbf{n}(\mathbf{x}))^{1/2}$ and $(\mathbf{x} \cdot \mathbf{n}(\mathbf{x}))^{1/2}$, respectively, and take their conjugates. The resulting equations are then multiplied back by one of the above respective factors. The final equations are

$$2\pi\tau(\mathbf{x})\mathbf{e}_3 \cdot \mathbf{n}(\mathbf{x}) + \int\int_{\partial D} \frac{\mathbf{e}_3 \cdot [\mathbf{x} - \mathbf{y}]}{|\mathbf{x} - \mathbf{y}|^3} \tau(\mathbf{y}) \, dS(\mathbf{y}) = 0 \tag{40a}$$

and

$$2\pi\tau(\mathbf{x})\mathbf{x} \cdot \mathbf{n}(\mathbf{x}) + \int\int_{\partial D} \left(\frac{\mathbf{x} \cdot [\mathbf{x} - \mathbf{y}]}{|\mathbf{x} - \mathbf{y}|^3} - \frac{\alpha + 1}{|\mathbf{x} - \mathbf{y}|} \right) \tau(\mathbf{y}) \, dS(\mathbf{y}) = 0. \tag{40b}$$

A moment of reflection shows that Eqs. (40a,b) are the equations at ∂D on the side of $C(D)$, where $C(D)$ denotes the complement of D in the three-dimensional space, for the surface distribution $\tau(\mathbf{x})$. For convenience, we shall call a surface which is never tangential to the x^3 direction of an x^3-facing surface.

The potential ξ of this charge distribution $\tau(\mathbf{x})$ on ∂D is continuous in the whole space and satisfies

$$\xi(\mathbf{x}) = \int\int_{\partial D} \frac{\tau(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, dS(\mathbf{y}) \quad \text{and} \quad \nabla^2\xi = 0 \quad \text{in } C(\partial D), \tag{41}$$

where $C(\partial D)$ denotes the complement of ∂D in the three-dimensional space, and also one of the following equations

$$\frac{\partial \xi}{\partial x^3} = 0 \quad \text{in } C(D) \quad \text{or} \quad (\alpha+1)\xi + \mathbf{x}\cdot\nabla\xi = 0 \quad \text{in } C(D) \tag{42a,b}$$

respectively. These equations allow us to determine whether Eqs. (37a,b) are solvable.

To extend the theory to regular surfaces (defined by [12, p. 112]), which are either x^3-facing or star-shaped, we note firstly that the results of potential theories for surface distributions of single charge layers are applicable when the distribution satisfies the Hölder condition of some order γ on S, $\gamma > 0$, and S is a regular surface element. Therefore Eqs. (37) and (42) are applicable to regular surfaces except at the edges. Secondly, we note that the theory of singular equations on a surface allows the surface to be a regular surface, and this surface may be made up of unconnected surfaces. Therefore Eqs. (39) and (40) are also applicable on regular surfaces. (The discontinuity of the kernels only makes the distributions $\sigma(\mathbf{x})$ and $\tau(\mathbf{x})$ discontinuous across the edges.) Neither of the above problems with the edges affects our application of the method to the general case of a region D bounded by regular surfaces which are either all x^3-facing or all star-shaped.

We can now apply the method to the two situations examined previously.

6.1 *Completeness without B_3.* Consider a finite, nonperiphractic region D, bounded by a regular x^3-facing surface S_0. Its boundary $\partial D = S_0$ is finite and the potential ξ must vanish at infinity. Putting $\zeta = \partial\xi/\partial x^3$, we have ζ harmonic in $C(D)$ and ζ vanishes on ∂D and at infinity. Therefore, we must have ζ identically zero in $C(D)$, giving

$$\xi(\mathbf{x}) = p(x^1, x^2) \quad \text{in } C(D),$$

where p is a function of only x^1 and x^2. Since ξ vanishes as x^3 tends to infinity, this function p must be identically zero, i.e.,

$$\xi(\mathbf{x}) = 0 \quad \text{in } C(D).$$

As $\xi(\mathbf{x})$ is continuous in the whole space, we have $\xi(\mathbf{x}) = 0$ on both sides of ∂D. The harmonicity of ξ in D then gives

$$\xi(\mathbf{x}) = 0 \quad \text{in } D.$$

For this function ξ, we must have $\tau(\mathbf{x})$ identically zero on ∂D. Hence, the Papkovich-Neuber solution without B_3 is complete in this case.

Consider now the other case where D is the finite but periphractic region bounded by two closed, x^3-facing surfaces S_0 and S_1. The surfaces S_0 and S_1 are the external and internal boundaries of D, respectively. In this case, we have ξ identically zero on the part of $C(D)$ outside S_0 but ξ can be nontrivial inside S_0. Indeed, S_1 can be the surface of an inner conductor with a distribution of charge $\tau(\mathbf{x})$ while S_0 can be the inner surface of an infinite outer conductor with opposite total charge to give a nontrivial solution $\tau(\mathbf{x})$ to (40a). The Papkovich-Neuber solution without B_3 is incomplete in this case. This appears to be similar to the situation considered in Sec. 3.

6.2 *Completeness without* Φ. We consider here only the case of a finite regular region D being the region between two disjointed closed, star-shaped, regular surfaces S_0 and S_1. The surface S_0 encloses the surface S_1 which, in turn, encloses the origin. We further limit the study to the case of only noninteger values of α.

Since ∂D is finite, ξ vanishes in the part of $C(D)$ at infinity. By the argument of Sec. 4 with α replaced by $-(\alpha+1)$, ξ must have the form $r^{-(\alpha+1)}\chi(\theta,\gamma)$. To be harmonic, ξ must be identically zero outside a large sphere for noninteger values of α. The analytic continuation theorem then extends the value of ξ to the boundary ∂D of $C(D)$.

Also by the argument of Sec. 4, ξ must have the form $r^{-(\alpha+1)}\chi(\theta,\gamma)$. Therefore ξ must be identically zero on the part of $C(D)$ inside S_1. Since ξ is zero on both boundaries of D, ξ must be identically zero in D. Therefore the solution $\tau(\mathbf{x})$ to Eq. (40b) must be trivial. This is Stippes's result.

6.3 *Inconsistency with previous results.* We note that the result here for the case of incompleteness without B_3 is different from that of Sec. 3. By the removal of only a cylinder of radius ε having as its centerline the line segment joining the two points $(x^1=0,\ x^2=0,\ x^3=1)$ and $(x^1=3,\ x^2=0,\ x^3=1)$, from the region D of Sec. 3, we have a nonperipractic region D_1 which is not x^3-convex. According to the results here, the representation in D_1 in terms of three harmonic functions is complete while it should not be complete according to the result of Sec. 3: An inconsistency has been arrived at!

This seems surprising as Fredholm's equations have been successfully used in solving the first (Dirichlet's) and second (Neumann's) boundary value problems of harmonic functions. However, a comparison between Eqs. (39) and those of the first and second boundary value problems shows that the kernels here have weak singularities of order 2 while those of the first and second boundary value problems are of order 1 (due to the finite curvature of the boundary ∂D and also the orientation in the normal direction of the derivative of the harmonic functions there). A study from S. G. Mikhlin's book reveals that the proof for Fredholm's theorems in the book is only applicable to singular equations whose kernels have singularities only of order less than 2, when the integrals are over two-dimensional surfaces. Therefore, Mikhlin's theory of Fredholm's equations is not, strictly speaking, applicable to the reasoning past Eqs. (39).

7. Conclusions. The Papkovich-Neuber solution has been shown to be complete in terms of three harmonic functions using the line integral methods as in Sec. 2 and 4. It is shown to be incomplete in Sec. 3 and 5. The convexity requirement in this problem is with respect to a surface rather than with respect to a point. This requirement on the convexity in the x^3-direction and the radial direction is also an intrinsic property of the problem and not an artificial limitation caused by the use of a particular method. The application of the boundary value method, as pioneered by Stippes, to these completeness problems requires more than Mikhlin's version of Fredholm's theorems. The main results of this work are summarized as

(1) The Papkovich-Neuber solution in terms of three harmonic functions B_1, B_2, and Φ is complete when the region D under consideration is x^3-convex with respect to a surface S described by $(x^1, x^2, x^3 = \overline{x}^3 = \overline{f}(\overline{x}^1, \overline{x}^2))$ where $\overline{f}(\overline{x}^1, \overline{x}^2) \in C^{2+\beta}$, $\beta > 0$ in S, i.e., the second derivatives of $\overline{f}(\overline{x}^1, \overline{x}^2)$ satisfy the Hölder condition of order β at every point of S. S is assumed to be finite with respect to $(\overline{x}^1, \overline{x}^2)$.

(2) The solution in terms of three harmonic functions B_1, B_2, and Φ is incomplete when the region D under consideration is not x^3-convex with respect to any surface S described above.

(3) The equation

$$\frac{1}{\cos^2\gamma}\left(\frac{\partial^2}{\partial\theta^2} + \cos\gamma\frac{\partial}{\partial\gamma}\left(\cos\gamma\frac{\partial}{\partial\gamma}\right)\right)\overline{\chi} + \alpha(\alpha+1)\overline{\chi}(\theta,\gamma) = \overline{h}(\theta,\gamma)$$

on a finite surface S with $\overline{h}(\theta,\gamma) \in C^{\beta}$, $\beta > 0$ on S, has a unique solution when α is a noninteger. When α is an integer, its homogeneous equation has a nontrivial solution. When the surface S described by $(\overline{x}^1 \equiv \theta,\ \overline{x}^2 \equiv \gamma,\ \overline{x}^3 = \overline{f}(\overline{x}^1, \overline{x}^2))$ does not intersect an appropriately chosen straight line through the origin of the space (x^1, x^2, x^3) having $(\overline{x}^1 \equiv \theta,\ \overline{x}^2 \equiv \gamma,\ \overline{x}^3 \equiv ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2})$ as its spherical coordinates, this equation corresponds to a Fredholm equation.

(4) The solution in terms of three harmonic functions B_1, B_2, and B_3 is complete when α is a noninteger and the region D under consideration is radially convex with respect to a surface S described by $(\overline{x}^1, \overline{x}^2, \overline{x}^3 = \overline{f}(\overline{x}^1, \overline{x}^2))$ where $\overline{f}(\overline{x}^1, \overline{x}^2) \in C^{2+\beta}$ on S and $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$ denotes the spherical coordinates.

(5) The solution in terms of three harmonic functions B_1, B_2, and B_3 is incomplete when the region D under consideration is not radially convex with respect to any surface S described above.

(6) When the region D under consideration is the exterior of a finite star-shaped region, the solution in terms of B_1, B_2, and B_3 is complete whether α is an integer or a noninteger.

(7) When the region D under consideration is finite and is either a region star-shaped with respect to the origin or a region bounded by two star-shaped surfaces, the solution remains complete when Φ is limited to being only a harmonic function of degree α.

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