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NONLINEAR GRAVITY-WAVE GROUPS*

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Abstract. Groups of gravity waves of permanent form in deep water are investigated. The analysis provides a systematic procedure for determining the form of the group to any order of approximation, and a calculation is carried to the third order of the amplitude at least and, where it matters, to the fourth order. Closed formulas for the phase velocity c of the basic waves and the group velocity c_g are obtained. Inspection of the analytic procedure reveals that these formulas remain intact for all subsequent calculations to any order of approximation. These formulas are in terms of the group wavenumber ε which, to the attained order of approximation, is found to be proportional to the amplitude a and the square of the basic wavenumber k, but is, for any assigned k, a power series in a. It is found that c increases and c_g decreases with ε, in such a way that $2cc_g = g/k$, where g is the gravitational acceleration. The results are compared with the corresponding ones obtained by the cubic-Schrödinger-equation (CBE) approach, and wherever comparison is possible there is agreement. The CBE approach, however, does not give the variation of c_g with the amplitude.

The collision of wave groups with different group velocities is also investigated, and it is found that after the faster group has overtaken the slower one, both groups retain their original forms, without any phase shift for either group. The interaction terms eventually die down everywhere. When a group is reflected by a vertical boundary normal to its velocity, then the reflected group is, in time, just the continuation of its mirror image across the boundary, without any phase shift.

1. Introduction. The famous works of Cauchy (1815, cited in Lamb [9], pp. 17, 384) and Poisson (1816, cited in Lamb [9], p. 384) showed already early in the last century that gravity waves created by a concentrated impulse travel in groups with velocities dependent on their wavelengths, although in these works the wave groups are not separate, especially at the beginning. Later, isolated gravity-wave groups were observed by Russell (1844, cited in Lamb [9], p. 380). The first derivation of the group velocity of dispersive waves was given by Stokes (1876, cited in Lamb [9], p. 381), although Stokes' treatment necessarily involves infinitely many wave groups one after another.

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In an effort to construct a single wave group by a linear theory, Yih [11] obtained such a group of gravity waves in deep water, but as the free-surface pressure was eliminated by successive approximations, it was found that some residue of pressure always remained, and that residue finally becomes a concentrated force of oscillatory magnitude. It was then thought that perhaps nonlinearity could provide the elimination of free-surface pressure present in the linear theory, and the present work was begun.

As the work progressed, I became aware of the seminal papers of Benney and Newell [3], Zakharov [12], and Benney and Roskes [4] on the evolution of wave packets, and of the substantial papers treating water waves by Hasimoto and Ono [8], Djordjević and Redekopp [6, 7], and Ablowitz and Segur [1, 2]. When the superficial differences of the results of these latter authors are reconciled, and my results are compared with theirs, there is general agreement wherever comparison is possible. Since all these authors use the cubic Schrödinger equation, and that equation is based on a third-order approximation only, they have not produced a determination of the group velocity c_g as a function of the amplitude. Comparison of results for c_g is therefore impossible.

As will be shown in this paper, two closed formulas give the phase velocity c and group velocity c_g as functions of the group wavenumber ε, which in turn depends on the amplitude of the waves, though not in closed form. The two closed formulas remain intact at any higher order of approximation, as an inspection of the analytical procedure reveals. These formulas show that c increases and c_g decreases with ε (which in turn increases with the amplitude), and are the main contributions of this paper. Detailed comparison of the present work with previous results based on the cubic-Schrödinger-equation (CBE) approach will be made later.

One restriction of the CBE is that it applies only to a specific basic wavenumber. A single equation for studying the interaction of wave groups of different basic wavenumbers is not available. The approach used in this paper allows conclusions to be drawn on the persistence of nonlinear wave groups of different basic wavenumbers after interaction. This is another contribution of the present work that is of some significance.

2. Formation of the differential system. Consider two-dimensional gravity waves in deep water with an envelope that approaches the undisturbed free surface at infinity, which is at $y = 0$. The line $y = 0$ is the x-axis, and the waves and their envelope are supposed to propagate in the positive x-direction.

Irrotationality is assumed, so that a potential ϕ exists, which satisfies the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0, \tag{1}$$

where subscripts x and y indicate partial differentiation. At the free surface,

$$y = \eta(x, t), \tag{2}$$

with η approaching zero as x approaches infinity. The kinematic condition at the free surface is

$$\eta_t + \phi_x \eta_x = \phi_y, \tag{3}$$

and the dynamic condition there is

$$\phi_t + g\eta + \frac{\phi_x^2 + \phi_y^2}{2} = 0, \tag{4}$$

where g is the magnitude of the gravitational acceleration, acting in the direction of decreasing y. All the derivatives of ϕ in (3) and (4) are evaluated at $y = \eta$, and the zero on the right-hand side of (4) is a consequence of the fact that $\eta = 0 = \phi$ at $x = \infty$. The last boundary condition is

$$\phi \to 0 \quad \text{as} \quad y \to -\infty. \tag{5}$$

The differential system consists of (1), (3), (4), and (5), and is exact.

But since all the derivatives of ϕ in (3) and (4) are evaluated at $y = \eta$, further development of these conditions is necessary before a systematic calculation can begin. One needs a combined free-surface condition in terms of ϕ alone, in which all quantities are evaluated at $y = 0$ instead of $y = \eta$.

3. The combined free-surface condition. Let (4) be written as

$$-g\eta = L\phi = \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2). \tag{6}$$

This can be written as

$$-g\eta = L\phi + (L\phi)_y\eta + (L\phi)_{yy}\eta^2/2 + \cdots, \tag{7}$$

in which, and *henceforth*, the derivatives of ϕ are evaluated at $y = 0$. Iteration of (7) gives

$$\begin{aligned} -g\eta = L\phi &+ (L\phi)_y\left[-\frac{1}{g}\left\{L\phi + (L\phi)_y\left(-\frac{1}{g}L\phi\right)\right\}\right] \\ &+ \frac{1}{2}(L\phi)_{yy}\left[-\frac{1}{g}\left\{L\phi + (L\phi)_y\left(-\frac{1}{g}L\phi\right)\right\}\right]^2 + \cdots, \end{aligned} \tag{8}$$

where the quantity in the brackets is η carried to the second iteration only. To the third order in ϕ, (8) becomes

$$\begin{aligned} -g\eta &= L\phi - \frac{1}{g}L\phi(L\phi)_y + \frac{1}{2g^2}(\phi_t^2\phi_{ty})_y \\ &= L\phi - \frac{1}{2g}[(L\phi)^2]_y + \frac{1}{2g^2}(\phi_t^2\phi_{ty})_y. \end{aligned} \tag{9}$$

One now multiplies (3) by g and writes the result in terms of derivatives of ϕ evaluated at $y = 0$, by expansions about that point. When this is done to the third order in the magnitude of ϕ, one has, with all derivatives evaluated at $y = 0$,

$$
\begin{aligned}
g\left[\phi_y + \phi_{yy}\left(-\frac{1}{g}L\phi + \frac{1}{g^2}\phi_t\phi_{ty}\right) + \frac{1}{2g^2}\phi_t^2\phi_{yyy}\right] \\
&= -(L\phi)_t + \frac{1}{g}\left\{\phi_t(L\phi)_y + \frac{\phi_x^2+\phi_y^2}{2}\phi_{ty}\right\}_t - \frac{1}{2g^2}(\phi_t^2\phi_{ty})_{yt} \\
&\quad + (\phi_x + \phi_{xy}\eta)\left\{-(L\phi)_x + \frac{1}{2g}(\phi_t^2)_{yx}\right\} \\
&= -(L\phi)_t + \frac{1}{g}\left\{\phi_t(L\phi)_y + \frac{\phi_x^2+\phi_y^2}{2}\phi_{ty}\right\}_t - \frac{1}{2g^2}(\phi_t^2\phi_{ty})_{yt} \\
&\quad + \phi_x\left\{-(L\phi)_x + \frac{1}{2g}(\phi_t^2)_{yx}\right\} + \frac{\phi_t}{g}\phi_{xy}\phi_{xt},
\end{aligned} \tag{10}
$$

in which one appearance of η is allowed in an intermediate step to show the route of calculation. The first three terms on the right-hand side of (10), up to and including the term containing the factor $1/(2g^2)$, is η_t, and the terms that follow it represent $\phi_x\eta_x$ evaluated at $y = \eta$.

Equation (10) will be written in the form

$$I_1 + I_2 + I_3 = 0, \tag{11}$$

where

$$I_1 = g\phi_y + \phi_{tt}, \tag{12}$$

$$
\begin{aligned}
I_2 &= -\phi_t\phi_{yy} + \left(\frac{\phi_x^2+\phi_y^2}{2}\right)_t - \frac{1}{g}(\phi_t\phi_{ty})_t + \phi_x\phi_{tx} \\
&= (\phi_t\phi_x)_x + \left(\frac{\phi_x^2+\phi_y^2}{2}\right)_t - \frac{1}{g}(\phi_t\phi_{ty})_t,
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
I_3 &= -\frac{1}{2}\phi_{yy}(\phi_x^2+\phi_y^2) + \frac{1}{g}\phi_t\phi_{ty}\phi_{yy} + \frac{1}{2g}\phi_t^2\phi_{yyy} \\
&\quad - \frac{1}{2g}\left\{\phi_t(\phi_x^2+\phi_y^2)\right\}_{yt} + \frac{1}{2g^2}(\phi_t^2\phi_{ty})_{yt} + \frac{1}{2}\phi_x(\phi_x^2\phi_y^2)_x \\
&\quad - \frac{1}{2g}\phi_x(\phi_t^2)_{yx} - \frac{1}{g}\phi_t\phi_{xy}\phi_{xt} \\
&= \frac{1}{2}\left\{\phi_x(\phi_x^2+\phi_y^2)\right\}_x + \frac{1}{2g}(\phi_t^2\phi_{yy})_y - \frac{1}{2g}\left\{\phi_t(\phi_x^2+\phi_y^2)\right\}_{yt} \\
&\quad + \frac{1}{2g^2}(\phi_t^2\phi_{ty})_{yt} - \frac{1}{2g}\left\{\phi_x(\phi_t^2)_x\right\}_y \\
&= \frac{1}{2}\left\{\phi_x(\phi_x^2+\phi_y^2)\right\}_x - \frac{1}{2g}\left[(\phi_t^2\phi_x)_x + \left\{\phi_t(\phi_x^2+\phi_y^2)\right\}_t\right]_y \\
&\quad + \frac{1}{2g^2}(\phi_t^2\phi_{ty})_{yt}.
\end{aligned} \tag{14}
$$

In (13) and (14), (1) has been used whenever it is advantageous to replace ϕ_{yy} by $-\phi_{xx}$. With the I's so defined, (11) is the free-surface condition on ϕ. It is emphasized here once again that all the derivatives of ϕ appearing in (11) are evaluated at $y = 0$.

4. The solution of the differential system. Since ϕ must satisfy the Laplace equation, it will be assumed to be the real part of an analytic function of the complex variable

$$z = x + iy,$$

and it is convenient to use the $\hat{X}$ and $\hat{Y}$ defined as

$$\hat{X} = \varepsilon(z - c_g t), \qquad \hat{Y} = k(z - ct), \tag{15}$$

in which k and c are the wavenumber and phase velocity of the basic waves and ε and c_g are those of their envelope, or of the group. The value of c in the linear theory is

$$c_0 = (g/k)^{1/2}. \tag{16}$$

Given k, the task is to determine c, c_g, and ε for an isolated wave group as functions of the amplitude of the waves, which can be represented by the amplitude of the potential function ϕ, which will be denoted by $2a$ for convenience. Let

$$\varepsilon^2 = a^2(\alpha_1 + \alpha_2 a^2 + \alpha_3 a^4 + \cdots), \tag{17}$$

$$c^2 = c_0^2(1 + \beta_1 \varepsilon^2 + \beta_2 \varepsilon^4 + \cdots), \tag{18}$$

$$2cc_g = c_0^2(1 + \gamma_1 \varepsilon^2 + \gamma_2 \varepsilon^4 + \cdots). \tag{19}$$

The expansion (19) is for $2cc_g$ rather than for c_g^2 because it affords a certain convenience, as later developments will show.

Esthetics would demand, perhaps, that all quantities be made dimensionless. Yet, as the free-surface boundary condition (11) indicates, the calculation will be long and involved, and the dimensionless form of (11) may be quite burdensome. Keeping things in their dimensional forms also provides a certain check on the calculation, for when a mistake is made so that an equation becomes nonhomogeneous in dimension, that mistake will be detected. Thus one is prepared to pay the price that all the α's, β's, and γ's are dimensional, and have different dimensions when their numerical subscripts vary. Expansions (18) and (19) are in ε rather than a for a good reason that will appear later. Indeed, one will arrive at two closed formulas in terms of ε for c^2 and cc_g.

The expansion for ϕ is

$$\phi = a\phi_1 + a^2\phi_2 + a^3\phi_3 + \cdots, \tag{20}$$

in which ϕ_1, ϕ_2, etc. may themselves be sums involving coefficients containing powers of ε/a. These sums will not be exhibited now. As the calculations proceed, one will obtain the various terms in each of ϕ_1, ϕ_2, etc.

One starts by taking

$$\phi_1 = \hat{S}e^{-i\hat{Y}} + \hat{S}^* e^{-i\hat{Y}^*}, \tag{21}$$

where the asterisks indicate complex conjugates, and

$$\hat{S} = \operatorname{sech} \hat{X}. \tag{22}$$

It is immediately clear that $\hat{S}$ has singularities at $\hat{X} = \pm i(2n+1)\pi/2$. Those corresponding to the + sign are outside of the liquid. The remaining singularities can be removed, as explained in Yih [11]. This removal has negligible effect on the free-surface condition if k/ε is large compared with 1, as is the case for the phenomenon under study here. Higher singularities created at higher approximations can be similarly removed (Yih, [11]). At $y = 0$, (21) reduces to

$$\phi_1 = 2S \cos Y, \tag{23}$$

where

$$S = \operatorname{sech} X, \qquad X = \varepsilon(x - c_g t), \qquad Y = k(x - ct), \tag{24}$$

Using (21), one obtains, at $y = 0$,

$$I_1(a\phi_1) = a[2(gk - k^2c^2 + \varepsilon^2 c_g^2)S \cos Y - 2\varepsilon(g - 2kcc_g)ST \sin Y - 4\varepsilon^2 g^2 S^3 \cos Y]. \tag{25}$$

When evaluating I_2 and I_3, it is time-saving to remember that when no differentiation with respect to y is involved, one can use (23) directly and there is no need to invoke (21), and when differentiations with respect to y are involved, it is time-saving to replace ϕ_{yy} by $-\phi_{xx}$ whenever possible, for then one can again use (23) instead of (21). Since the calculation for I_2 and I_3 is lengthy, it will be presented in two subsections, to improve clarity.

4.1. Calculation for I_2. The following results for the various components of $I_2(\phi_1)$ can be readily verified, with the understanding that all the derivatives of ϕ_1 are evaluated at $y = 0$:

$$\phi_{1t} = 2kcS \sin Y + 2\varepsilon c_g ST \cos Y, \tag{26}$$

$$\phi_{1x} = 2kS \sin Y + 2\varepsilon ST \cos Y, \tag{27}$$

in which

$$T = \tanh X.$$

For the y-derivative of ϕ_1, one has to use (21), and the result is

$$\phi_{1y} = -2\varepsilon ST \sin Y + 2kS \cos Y, \tag{28}$$

which is again evaluated at $y = 0$. The calculation of $I_2(\phi_1)$ can now proceed in a straightforward manner. Detailed results for the various terms in I_2 will be given here for the convenience of anyone wishing to check the present calculations. First,

$$\phi_{1t}\phi_{1x} = -2[k^2cS^2(1 - \cos 2Y) + \varepsilon k(c + c_g)S^2T \sin 2Y + \varepsilon^2 c_g(S^2 - S^4)(1 + \cos 2Y)],$$

so that

$$(\phi_{1t}\phi_{1x})_x = -4k^3cS^2 \sin 2Y + 4\varepsilon k^2cS^2T - 4\varepsilon k^2(2c + c_g)S^2T \cos 2Y + T_h, \tag{29}$$

where T_h denotes higher-order terms given by

$$T_h = 4\varepsilon^2 k(c + 2c_g)S^2 \sin 2Y - 2\varepsilon^2 k(3c + 5c_g)S^4 \sin 2Y + 4\varepsilon^3 c_g(S^2T - 2S^4T)(1 + \cos 2Y).$$

For the next group of terms in I_2 one has

$$\phi_{1x}^2 + \phi_{1y}^2 = 4S^2[k^2 + \varepsilon^2(1 - S^2)], \tag{30}$$

so that

$$\left(\frac{\phi_{1x}^2 + \phi_{1y}^2}{2}\right)_t = 4\varepsilon c_g(k^2 + \varepsilon^2 - 2\varepsilon^2 S^2)S^2 T. \tag{31}$$

From (26) and (28) one has

$$\phi_{1t}\phi_{1ty} = 2k^3c^2S^2(1 - \cos Y) + 2\varepsilon k^2 c(c + 2c_g)S^2 T \sin 2Y + O(\varepsilon^2), \tag{32}$$

so that

$$(\phi_{1t}\phi_{1ty})_t = -4k^4c^3S^2 \sin 2Y + 4\varepsilon k^3c^2c_gS^2T - 4\varepsilon k^3c^2(c + 3c_g)S^2T\cos 2Y + O(\varepsilon^2). \tag{33}$$

From (29), (31), and (33) one obtains

$$I_2(\phi_1) = 4\varepsilon k^2 cS^2T + O(\varepsilon^2),$$

or

$$I_2(a\phi_1) = 4\varepsilon a^2k^2cS^2T + O(\varepsilon^2a^2). \tag{34}$$

4.2. Annihilation of $I_2(a\phi_1)$. One now seeks a function W so that (at $y = 0$)

$$\left(g\frac{\partial}{\partial y} + \frac{\partial^2}{\partial t^2}\right)W = -4\varepsilon a^2k^2cS^2T, \tag{35}$$

with the higher-order terms in (34) neglected. It can be readily verified that

$$W = -\frac{a^2k^2c}{g}\left[i(\hat{S}^2 - \hat{S}^{*2}) + \frac{2\varepsilon c_g^2}{g}(\hat{S}^2\hat{T} + \hat{S}^{*2}\hat{T}^*)\right], \tag{36}$$

from which one obtains, for $y = 0$,

$$W_t = O(\varepsilon^2a^2), \quad W_x = O(\varepsilon^2a^2), \quad W_{yt} = O(\varepsilon^2a^2), \quad W_y = -\frac{4\varepsilon a^2k^2c}{g}S^2T.$$

The contribution of W to I_2 is, when W is added to ϕ_1 given by (21),

$$\begin{aligned}\tilde{I}_2(a\phi_1, W) &= \left[(\phi_{1t}W_x + \phi_{1x}W_t)_x + (\phi_{1x}W_x + \phi_{1y}W_y)_t - \frac{1}{g}(\phi_{1t}W_{ty} + \phi_{1ty}W_t)_t\right] \\ &= (\phi_{1y}W_y)_t + O(\varepsilon^3a^3) = -\frac{8\varepsilon a^3k^4c^2}{g}S^3T\sin Y + O(\varepsilon^3a^3)\end{aligned} \tag{37}$$

if $I_2(W)$, being of higher order in a, is neglected at this stage of approximation. The amount given by (37) will be added to $I_3(a\phi_1)$ to be calculated below.

4.3. Calculation of $I_3(a\phi_1)$. Since the calculation is lengthy, it will be divided into several portions.

We have

$$\begin{aligned}I_3(a\phi_1) &= a^3 I_3(\phi_1), \\ I_3(\phi_1) &= \frac{1}{2}\left[J_1 - \frac{1}{g}(J_{21} + J_{22}) + \frac{1}{g^2}J_3\right],\end{aligned} \tag{38}$$

in which

$$J_1 = \left\{\phi_{1x}(\phi_{1x}^2 + \phi_{1y}^2)\right\}_x, \tag{39}$$

$$J_{21} = (\phi_{1t}^2\phi_{1x})_{xy}, \qquad J_{22} = \left\{\phi_{1t}(\phi_{1x}^2 + \phi_{1y}^2)\right\}_{ty}, \tag{40}$$

$$J_3 = (\phi_{1t}^2\phi_{1ty})_{yt}. \tag{41}$$

From (27) and (30), one obtains

$$\phi_{1x}(\phi_{1x}^2 + \phi_{1y}^2) = -8k^3S^3\sin Y - 8\varepsilon k^2S^3T\cos Y + O(\varepsilon^2),$$

so that

$$J_1 = -8k^4S^3\cos Y + 32\varepsilon k^3S^3T\sin Y + O(\varepsilon^2). \tag{42}$$

For J_{21}, one writes first of all

$$J_{21} = (H + K)_x, \tag{43}$$

where

$$H = 2\phi_{1t}\phi_{1ty}\phi_{1x}, \qquad K = \phi_{1t}^2\phi_{1yx}. \tag{44}$$

From (32) and (27) one has, with O now meaning "terms containing",

$$H = -12k^4c^2S^3\sin Y - 8\varepsilon k^3c(c + c_g)S^3T\cos Y + O(\varepsilon^2, \varepsilon\cos 3Y, \sin 3Y), \tag{45}$$

in which the terms containing $\cos 3Y$ or $\sin 3Y$ do not contribute at this stage of approximation. From (28) one has

$$\phi_{1xy} = -2k^2S\sin Y - 4\varepsilon kST\cos Y + O(\varepsilon^2), \tag{46}$$

and from this and (26) one obtains, with O used as in (45),

$$K = -6k^4c^2S^3\sin Y - 4\varepsilon k^3c(c + c_g)S^3T\cos Y + O(\varepsilon^2, \varepsilon\cos 3Y, \sin 3Y). \tag{47}$$

When H is added to K, and the result substituted into (43), one has

$$J_{21} = -18k^5c^2S^3\cos Y + 6\varepsilon k^4c(11c + 2c_g)S^3T\sin Y + O(\varepsilon^2, \varepsilon\cos 3Y, \sin 3Y). \tag{48}$$

In calculating J_{22}, one notes first of all that differentiation with respect to y is more troublesome, since it requires the form for ϕ_1 in terms of the complex variable z. Therefore one seeks to replace ϕ_{1yy} by $-\phi_{1yy}$ whenever possible, and writes

$$J_{22} = (M + N)_t, \tag{49}$$

$$M = \phi_{1ty}(\phi_{1x}^2 + \phi_{1y}^2), \qquad N = 2\phi_{1t}(\phi_{1x}^2\phi_{1xy}^2 - \phi_{1y}\phi_{1xx}). \tag{50}$$

From (30) and (32) one has

$$M = 8k^4cS^3\sin Y + 8\varepsilon k^3(c + c_g)S^3T\cos Y + O(\varepsilon^2).$$

From (27), (28), and (46),

$$\phi_{1x}\phi_{1xy} - \phi_{1y}\phi_{1xx} = 4k^3S^2 + O(\varepsilon^2),$$

and this gives, with (26),

$$N = 16k^4cS^3\sin Y + 16\varepsilon k^3c_gS^3T\cos Y + O(\varepsilon^2).$$

With M and N so determined, (49) gives

$$J_{22} = -13k^5c^2S^3\cos Y + 8\varepsilon k^4 c(12c_g + c)S^3T\sin Y + O(\varepsilon^2). \tag{51}$$

Adding (51) to (48), one has

$$J_{21} + J_{22} = -42k^5c^2S^3\cos Y + 2\varepsilon k^3 c(37c + 54c_g)S^3T\sin Y + O(\varepsilon^2). \tag{52}$$

For J_3, first write (again to substitute ϕ_{1xx} for ϕ_{1yy})

$$J_3 = (P - Q)_t, \tag{53}$$

where

$$P = 2\phi_{1t}\phi_{1ty}^2, \qquad Q = \phi_{1t}^2\phi_{1txx}. \tag{54}$$

From (28) one has

$$\phi_{1ty} = \phi_{1yt} = 2k^2cS\sin Y + 2\varepsilon k(c + c_g)ST\cos Y + O(\varepsilon^2),$$

which gives, together with (32),

$$P = 12k^5c^3S^3\sin Y + 4\varepsilon k^4c^2(2c + 3c_g)T\cos Y + O(\varepsilon^2, \varepsilon\cos 3Y, \sin 3Y). \tag{55}$$

From (27) one obtains

$$\phi_{1xxx} = -2k^3cS\sin Y - 2\varepsilon k^2(2c + c_g)ST\cos Y + O(\varepsilon^2),$$

and this, together with (26), gives

$$Q = -6k^5c^3S^3\sin Y - 2\varepsilon k^4c^2(2c + 3c_g)S^3T\cos Y + O(\varepsilon^2, \varepsilon\cos 3Y, \sin 3Y). \tag{56}$$

Substituting (55) and (56) into (53), one has, upon neglecting terms that do not contribute at this stage of approximation,

$$J_3 = -18k^6c^4S^3\cos Y + 12\varepsilon k^5c^3(c + 6c_g)S^3T\sin Y. \tag{57}$$

Finally, substituting (42), (52), and (57) into (38), one has,

$$\begin{aligned} I_3(\phi_1) = \frac{1}{2}\Bigg[&k^4\left(-8 + \frac{42kc^2}{g} - 18\frac{k^2c^4}{g^2}\right)S^3\cos Y \\ &+\varepsilon k^3\left\{32 - \frac{kc(74c + 108c_g)}{g} + \frac{12k^2c^3(c + 6c_g)}{g^2}\right\}S^3T\sin Y\Bigg], \end{aligned} \tag{58}$$

upon neglecting terms that do not contribute at this stage of approximation.

Since, from (18) and (19),

$$c^2 = c_0^2 + O(a^2) \quad \text{and} \quad c_g = c_0/2 + O(a^2),$$

(58) can be written

$$I_3(\phi_1) = 8k^4S^3\cos Y - 48\varepsilon k^3S^3T\sin Y, \tag{59}$$

upon neglecting terms of higher orders.

Equations (37) and (59) give the result

$$\tilde{I}_2(a\phi_1, W) + I_3(a\phi_1) = 8a^3k^4S^3\cos Y - 56\varepsilon a^3k^3S^3T\sin Y, \tag{60}$$

with higher-order terms neglected.

5. Treatment of the residue. The residue in the free-surface boundary condition, given by the sum of (25) and (60), is to be removed, and in the removal the variations of c, c_g, and ε with the amplitude (for the velocity potential) a are to be determined.

The equation (11) must be satisfied for various powers of S and T multiplied to $\cos Y$ or $\sin Y$. But now (11) reduces to

$$I_1(a\phi_1) + \tilde{I}_2(a\phi_1, W) + I_3(a\phi_1) = 0, \tag{61}$$

since $I_2(a\phi_1)$ has been annihilated by the introduction of W (which, as will be seen, constitutes part of ϕ_2). The last term in (60), which appears in (61), needs to be annihilated. For this purpose one notes that

$$\left(g\frac{\partial}{\partial y} + \frac{\partial^2}{\partial t^2}\right)(\widehat{S}\widehat{T}e^{-i\widehat{Y}}) = A\widehat{S}\widehat{T}e^{-i\widehat{Y}} + i\varepsilon B(-\widehat{S}\widehat{T}^2 + \widehat{S}^3)e^{-i\widehat{Y}} - 6\varepsilon^2 c_g^2 \widehat{S}^3\widehat{T}e^{-i\widehat{Y}}, \tag{62}$$

in which

$$A = gk - k^2c^2 + \varepsilon c_g^2, \qquad B = g - 2kcc_g. \tag{63}$$

To annihilate the last term in (60), which appears in (61), one adds the following term to ϕ:

$$Z = -\frac{14i}{3\varepsilon c_g^2} a^3k^3(\widehat{S}\widehat{T}e^{-i\widehat{Y}} - \widehat{S}^*\widehat{T}^*e^{i\widehat{Y}^*}). \tag{64}$$

From (64), (62), and the complex conjugate form of (62), one has, upon taking $y = 0$ on the right-hand side of the equation,

$$\left(g\frac{\partial}{\partial y} + \frac{\partial^2}{\partial t^2}\right)Z = 56\varepsilon a^3k^3S^3T\sin Y - \frac{28A}{3\varepsilon c_g^2}a^3k^3ST\sin Y + \frac{14B}{3c_g^2}a^3k^3(-2S + 4S^3)\cos Y. \tag{65}$$

Thus, as the last term in (60) is annihilated, other terms are created. Recalling (25), one has, from (61),

$$2A - \frac{28}{3c_g^2}a^2k^3B = 0, \tag{66}$$

$$-2\varepsilon B - \frac{28}{3\varepsilon c_g^2}a^2k^3A = 0, \tag{67}$$

$$-4\varepsilon^2c_g^2 + 8a^2k^4 + \frac{56}{3c_g^2}a^2k^3B = 0. \tag{68}$$

Equations (66) and (67) give, since they are obviously not linearly dependent, $A = 0 = B$, or

$$k^2c^2 = gk + \varepsilon^2 c_g^2, \qquad 2kcc_g = g, \tag{69}$$

and (68) gives then

$$\varepsilon^2 = \frac{2a^2k^4}{c_g^2}. \tag{70}$$

With higher-order terms neglected, this gives

$$\varepsilon^2 = \frac{8a^2k^4}{c_0^2} = \frac{8a^2k^5}{g}. \tag{71}$$

Equations (69) and (71) are the main results sought. It is important now to examine just to what order of a the calculation has been carried out. Note that the last term in (60) is of $O(a^4)$, and the concern arises that it is retained while other terms of $O(a^4)$ are neglected. But a close examination of the entire analytical procedure reveals that fourth-order terms can only arise from two sources:

(i) Equations (3) and (4), not accounted for in (11),

(ii) Other fourth-order terms in (11).

From source (i) one cannot obtain terms of $O(a^4)$ that contain $\sin Y$ or $\cos Y$ as a factor. From (ii) one can obtain terms of $O(a^4)$ by differentiation of $\widehat{S}$ or $\widehat{T}$ with respect to y, or S or T with respect to x or t, and, of course, also terms of fourth or higher orders by iteration. But the only term of fourth order obtained in this way that contains the factor $\sin Y$ or $\cos Y$ is the last term in (60). Since that term is important in establishing the second equation in (69), one can say that, in effect, the calculation has been carried to the fourth order in a, as far as the determinations of c and c_g are concerned. Indeed, the arguments in the following paragraph will establish that (69) stands intact at any order of approximation.

Denote the differential operator on the left-hand side of (65) by L_0. A simple calculation shows that

$$L_0(\widehat{S}^3\widehat{T}e^{-i\widehat{Y}}) = (A + 8\varepsilon^2 c_g^2)\widehat{S}^3\widehat{T}e^{-i\widehat{Y}} - 20\varepsilon^2 c_g^2\widehat{S}^5\widehat{T}e^{-i\widehat{Y}} + i\varepsilon B(-3\widehat{S}^3 + 4\widehat{S}^5)e^{-i\widehat{Y}}, \tag{72}$$

$$L_0(\widehat{S}^3 e^{-i\widehat{Y}}) = (A + 8\varepsilon^2 c_g^2)\widehat{S}^3 e^{-i\widehat{Y}} - 3i\varepsilon B c_g^2\widehat{S}^3\widehat{T}e^{-i\widehat{Y}} - 12\varepsilon^2 c_g^2\widehat{S}^5 e^{-i\widehat{Y}}. \tag{73}$$

Suppose one has a term (C real)

$$2CS^5T\sin Y$$

to annihilate in (17). Since this is equal to

$$iC(\widehat{S}^3\widehat{T}e^{-i\widehat{Y}} - \widehat{S}^{*3}\widehat{T}^*e^{-i\widehat{Y}^*})_{y=0},$$

one can use (72), and add a term of the form (C_1 is another real constant)

$$iC_1(\widehat{S}^5\widehat{T}e^{i\widehat{Y}} - \widehat{S}^{*5}\widehat{T}^*e^{i\widehat{Y}^*})$$

to ϕ to accomplish the purpose. But then one creates terms containing the factors

$$(i\widehat{S}^3\widehat{T}, \widehat{S}^3, \widehat{S}^5)e^{i\widehat{Y}}$$

and their complex conjugates. The term containing $\widehat{S}^3$, and its complex conjugate, go toward a higher-order approximation for ε^2, as a continuation of (68). The term containing $i\widehat{S}^3\widehat{T}$, and its complex conjugate, are removed by the process already illustrated by (62) to (69), and would only possibly contribute to the determination of ε^2. The terms containing $\widehat{S}^5$ and $(\widehat{S}^*)^5$ can be annihilated by using (73). In that process terms containing $\widehat{S}^3$ and $\widehat{S}^3\widehat{T}$ and their complex conjugates will be created, as has been shown. But these will only contribute to the further determination of ε^2. Throughout the process of residue annihilation, the arguments leading from (66) and (67) to (69) are unaffected. Now that one has shown (69) is valid for all orders of approximation, it is worthwhile to obtain c^2 and c_g^2 explicitly in terms of ε from (69). The results are

$$c^2 = \frac{c_0^2}{2}\left[1 + \left\{1 + (\varepsilon/k)^2\right\}^{1/2}\right], \tag{74}$$

$$c_g^2 = \frac{c_0^2}{2}\frac{1}{1 + \{1 + (\varepsilon/k)^2\}^{1/2}}. \tag{75}$$

From (74) the β's in (18) can be readily determined if needed. As to (19), the second equation in (69) shows that all γ's are zero. An expansion of c_g^2 in a power series in ε can be readily obtained from (75) if needed. That c_g decreases as ε increases (or as the amplitude increases) may seem strange, but this is not the first instance of such a situation. See Whitham ([10], p. 526, formula for the group velocity V).

6. Results for ϕ_2. With W and Z determined by (36) and (64), respectively, the ϕ_2 in (20) is determined, and is

$$\phi_2 = (W + Z)/a^2. \tag{76}$$

7. The shape of the group. The shape of the wave group is given by η, which can be evaluated from (6) or (8). To calculate η to $O(a^3)$, one has to include terms containing $\sin 3Y$ and $\cos 3Y$. This can be done but is lengthy. Here η will be given to $O(a^2)$ only. For this purpose one notes that on $y = 0$ (36) gives

$$W = O(a^2\varepsilon) = O(a^3),$$

and (64) gives

$$Z = -\frac{28}{3\varepsilon c_g^2}a^3k^3ST\sin Y. \tag{77}$$

To $O(a^2)$ then, (9) and (32) give, upon use of (20), (21), (30), (36), (76), and (77),

$$-g\eta = 2akc_0S\sin Y + 2a\varepsilon c_gST\cos Y + 2a^2k^2S^2\cos 2Y + \frac{28a^3k^4c}{3\varepsilon c_g^2}ST\cos Y.$$

When c and c_g are replaced by c_0 and $c_0/2$ respectively, and (16) and (71) are used, one has

$$-g\eta = 2akc_0S\sin Y + \frac{17}{3}a\varepsilon c_0ST\cos Y + 2a^2k^2S^2\cos 2Y, \tag{78}$$

in which ε is given by (71).

The leading item of η is (Fig. 1)

$$\eta = -\alpha S\sin Y,$$

where

$$\alpha = \frac{2akc_0}{g} = \frac{2a}{c_0} \tag{79}$$

is the amplitude of the group to $O(a)$, and is related to ε by

$$\varepsilon = \sqrt{2}k^2\alpha. \tag{80}$$

Thus the group is shorter for a larger amplitude and for a shorter basic wavelength.

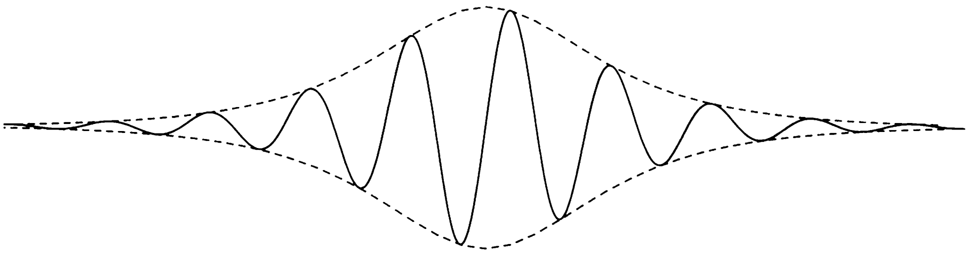

FIG. 1. Sketch of the gravity-wave group

8. Comparison with previous results. For the study of dispersive-wave groups and their stability, many previous investigators have derived the cubic Schrödinger equation (CBE). These are cited in the introduction of this paper. Among these, Hasimoto and Ono [8] derived it for gravity waves in a liquid of constant depth, and Djordjević and Redekopp [6] and Ablowitz and Segur ([2], henceforth referred to as AS) derived it for capillary-gravity waves in water of constant depth. In a later paper, Djordjević and Redekopp ([7], henceforth referred to as DR) derived it for gravity-capillary waves in water of slowly varying depth. From the results of these authors, among others, a CBE for gravity waves in water of infinite depth can be readily obtained. The coefficients of the CBE vary with the authors, and it is useful to reconcile the differences before comparing the results presented in this paper with their results.

The CBE obtained by Hasimoto and Ono [8] and that in AS, when capillary effects are neglected and the water depth is infinite and after a sign error in the definition of τ in (AS, 2.22) is corrected, are the same, except that the former is in dimensional and the latter is in dimensionless terms. The latter is (AS, p. 702)

$$iA_\tau + \lambda A_{\xi\xi} = \nu|A|^2 A, \tag{81}$$

in which A is no longer the A defined by (63), and

$$\lambda = -\tfrac{1}{8}, \qquad \nu = 2. \tag{82}$$

The form of CBE obtained by Djordjević and Redekopp ([6], [7]) is dimensional, and when reduced to the case of gravity waves in water of infinite depth, is

$$iA_\xi + \lambda A_{\tau\tau} = \nu|A|^2 A, \tag{83}$$

where

$$\lambda = -\frac{1}{g}, \qquad \nu = \frac{4k^4}{g}. \tag{84}$$

Ablowitz and Segur's τ and ξ correspond to ξ and τ in (83). The differences in the values of λ and ν are the result of differences in the scaling schemes. If the ξ in DR is doubled and the τ in DR is halved, as a close examination of the scaling schemes used in AS and DR would suggest, the new λ and ν for the DR form would be

$$\lambda = -\frac{1}{8g}, \qquad \nu = \frac{2k^4}{g}, \tag{85}$$

which then agree with (82), apart from the appearance of k and g in (85) as the consequence of the fact that (83) is in dimensional terms.

We then need to use only (81) and (82) for any purpose of comparison, keeping in mind that these are dimensionless. The solution of (81) given in AS is (AS, 3.4) which, after correction of an error, is

$$A = a|2\lambda/\nu|^{1/2}\,\mathrm{sech}\,\{a(\xi - 2\lambda b\tau)\}\exp\left\{ib\xi + i\lambda(a^2 - b^2)\tau\right\}, \tag{86}$$

where

$$\tau = \varepsilon^2(gk)^{1/2}t, \qquad \xi = \varepsilon k(x - \frac{c_0}{2}t). \tag{87}$$

In (AS, 3.4), the $2b\tau$ should be $2\lambda b\tau$, as in (86). The solution (86) is valid only for $\lambda/\nu < 0$, as is the case here, In regard to notation, the c_g in AS has been changed to $c_0/2$ in (87). The symbols x, t, g, and k mean the same things in AS as here, but a and ε mean different things, and λ, ν, b, τ, and ξ are not used in this paper. The A in AS is the dimensionless form of the $\tilde{A}$ in AS. My a therefore corresponds to their $A\varepsilon$, and my ε to their $a\varepsilon$. My α in (79) corresponds to their $2(2\lambda/\nu)^{1/2}a\varepsilon$, the factor 2 being there because $2A_{\max}\varepsilon$ is the amplitude of the dominant term in their ϕ. From (80),

$$\frac{\alpha}{\varepsilon} = \frac{1}{\sqrt{2}k^2}. \tag{88}$$

The k^2 in this equation arises because dimensional quantities are used here. The corresponding ratio in AS, from (AS, 3.4), is

$$\frac{2A_{\max}\varepsilon}{a\varepsilon} = 2|2\lambda/\nu|^{1/2}. \tag{89}$$

Since this is dimensionless, one equates this to $1/\sqrt{2}$ in (88), and thereby obtains

$$\nu = 2,$$

since $\lambda = -1/8$. This agrees with (82).

In AS, the authors did not say what their a is. To arrive at the dependence of c with amplitude, it is necessary to determine their a. This can be done from the definition of ε in (AS, 2.13a), which is

$$\varepsilon = k\hat{a}, \tag{90}$$

where $\hat{a}$ is the dimensional amplitude of the surface displacement. It was denoted by a in (AS, 2.13a). But since a is used in (AS, 3.4) to denote something else, I have supplied the symbol $\hat{a}$ to avoid confusion. This $\hat{a}$ is then given by

$$\hat{a} = \frac{2\varepsilon}{k}A_{\max}, \tag{91}$$

where the right-hand side is in AS notation, except the subscript. Comparing (90) with (91), one has

$$A_{\max} = \frac{1}{2}.$$

But from (86)

$$A_{\max} = \frac{a}{2\sqrt{2}}.$$

Therefore

$$a = \sqrt{2}. \tag{92}$$

As Ablowitz and Segur said in AS, one may take b in (86) equal to zero without loss in generality. Doing so, and combining (86) with the exponential factor assumed in AS for the basic waves, one has

$$2A \exp ik(x - c_0 t) = \operatorname{sech} a\xi \exp ik\left\{x - c_0\left(1 + \frac{\varepsilon^2}{4}\right)t\right\}, \tag{93}$$

which immediately shows that

$$c = c_0\left(1 + \frac{\varepsilon^2}{4}\right), \tag{94}$$

where, one emphasizes, ε is in AS notation. On the other hand, (69) and (71) give, to the order $O(\varepsilon^2)$ and in my notation,

$$c = c_0\left(1 + \frac{\varepsilon^2}{8k^2}\right) = c_0\left(1 + \frac{a^2k^3}{g}\right). \tag{95}$$

Since my a^2k^3/g is the $A^2_{\max}\varepsilon^2$ or $\varepsilon^2/4$ in AS, this formula is in agreement with (94). In this connection note that for Stokesian waves

$$c = c_0\left(1 + \frac{k^2\alpha^2}{2}\right) = c_0\left(1 + \frac{\varepsilon^2}{4k^2}\right). \tag{96}$$

There is no reason to expect agreement between (95) and (96).

If b is assumed different from zero, a straightforward calculation shows that the right-hand side of (93) becomes

$$\operatorname{sech}\left[ak'\varepsilon\left\{x - (c_g)'_0\left(1 - \frac{3b^2\varepsilon^2}{8}\right)t\right\}\right] \exp ik'\left\{x - c'_0\left(1 + \frac{\varepsilon^2}{4}\right)t\right\} \tag{97}$$

if terms of $O(\varepsilon^2)$ are dropped from the exponent and terms of $O(\varepsilon^4)$ are dropped from the argument of the sech function. In (97),

$$k' = (1 + b\varepsilon)k, \qquad c'_0 = (g/k')^{1/2} = 2(c_g)'_0.$$

To arrive at (97), one needs only to use the formulas obtained from the Taylor expansion (on going from k' to k),

$$kc_0 = k'c'_0\left(1 - \frac{b\varepsilon}{2} + \frac{3b^2\varepsilon^2}{8}\right) + O(\varepsilon^3), \tag{98}$$

$$c_0 = 3c'_0\left(1 - \frac{b\varepsilon}{2} + \frac{b^2\varepsilon^2}{8}\right) + O(\varepsilon^3). \tag{99}$$

It must *not* be argued from (97) that

$$c'_g = (c_g)'_0\left(1 - \frac{3b^2\varepsilon^2}{8}\right),$$

for the term of $O(\varepsilon^3)$ in the argument of the sech function in (97) would contribute terms of $O(\varepsilon^4)$ in the calculation, since the A given by (86) is to be multiplied by ε in the AS analysis, and that analysis has been carried only to terms of $O(\varepsilon^3)$. Throwing away terms of $O(\varepsilon^3)$ in the argument of the sech function in (97), one recovers the right-hand side of (93), with k replaced by k' and the c_0 and $(c_g)_0$ replaced by c'_0

and $(c_g)'_0$. Thus the CBE approach used in AS and indeed by all previous authors does not and cannot determine the effect of amplitude on c_g. For that determination one has to carry out the calculation to $O(\varepsilon^4)$, and that is what has been done in this paper.

In summary, the CBE approach does not provide a result for the variation of the group velocity with amplitude, for comparison with the result on c_g in this paper. Wherever comparisons are possible, e.g., in regard to the form of the wave envelope or the dependence of c (phase velocity of the basic waves) on the amplitude, agreement is found between the results of this paper and previous results obtained by the CBE approach.

The wave group found here may be considered the result of instability of Stokesian waves, as suggested by Hasimoto and Ono [8], who cited the computations of Chu and Mei [5] and of others to support that suggestion.

9. Interaction of wave groups. Let there be two wave groups, denoted by the subscripts 1 and 2, respectively, and let (with $i = 1$ or 2)

$$\begin{aligned} X_i &= \varepsilon_i(x - c_{gi}t - \xi_i), & Y_i &= k_i(x - c_it - x_i), \\ S_i &= \operatorname{sech} X_i, & T_i &= \tanh X_i, \\ \hat{X}_i &= \varepsilon_i(x + iy - c_{gi}t - \xi_i), & \hat{S}_i &= \operatorname{sech} \hat{X}_i, \quad \text{etc.,} \end{aligned}$$

and let

$$c_{g1} \neq c_{g2}.$$

The ξ_i and x_i are mere phase indicators. For the two groups,

$$\phi = a_1\phi_1 + a_2\phi_2 + a_1^2\phi_{11} + a_1a_2\phi_{12} + a_2^2\phi_{22} + a_1^3\phi_{111} + a_1^2a_2\phi_{112} + a_1a_2^2\phi_{122} + a_2^3\phi_{222} + \cdots.$$

The terms with only 1 for indices are for the first group and those with only indices 2 are for the second group. All other terms are interaction terms. From the analysis presented in the preceding sections, it is evident that ϕ_1 contains $\hat{S}_1$ and $\hat{S}_1^*$, and ϕ_2 contains $\hat{S}_2$ and $\hat{S}_2^*$. Nonlinear interaction will give rise to second-order terms containing $\hat{S}_1\hat{S}_2$, $\hat{S}_1\hat{S}_2^*$, and their complex conjugates, in what has been called the residue. These will be multiplied to exponential functions containing, among others, the factor $\exp(k_1 + k_2)y$, with positive k_1 and k_2. Because of the presence of the operator

$$L_0 = g\frac{\partial}{\partial y} + \frac{\partial^2}{\partial t^2}$$

in I_1 defined by (12), annihilation of these terms in the residue will call forth second-order terms containing the same $\hat{S}$-factors to constitute ϕ_{12}, and the process continues as indicated in the analysis presented in the foregoing sections. When finally (11) is satisfied at any order of approximation, one is left with three sets of terms: those purely for the first group, those purely for the second group, and hybrid terms, for ϕ or for the displacement η. The hybrid terms contain both $\hat{S}_1$ (or its complex conjugate), and $\hat{S}_2$ (or its complex conjugate), and possibly $\hat{T}_1$, $\hat{T}_2$ and their complex conjugates as well. Thus in time they must vanish everywhere, because $c_{g1} \neq c_{g2}$, and where $\hat{S}_1$ is of order 1 the factor $\hat{S}_2$ will vanish as time increases, and vice versa.

The conclusion is then that the two groups will interact and emerge intact, with not even a shift of phase for either group after interaction, and that the interaction terms eventually vanish everywhere. Recall that in the present analysis c_g varies with both k and ε. Whatever k and ε are, it is always the group with the greater c_g that overtakes the slower group. That is for overtaking. For head-on collision c_{g1} can be equal to c_{g2}.

I am aware that researchers using the CBE have asserted that collision of wave groups leaves the group shapes intact but with "possible" phase shifts after collision. But since the CBE applies only to wave groups with the same basic wavenumber k, and the solution of the CBE for any wave group for a given k gives the same linear group velocity for that k, there can be no overtaking in the first place. As to head-on collision, the foregoing analysis seems to rule out any phase shifts of wave groups after collision. Note that the arguments for no phase shifts given in this paper do not apply to ordinary solitary waves, which definitely suffer phase shifts after collision.

Since normal reflection can be considered as equivalent in all physical aspects to the head-on collision of two identical wave groups propagating in opposite directions, one concludes that a gravity-wave group is reflected by a vertical boundary normal to its path with no change of shape, no shift in phase, and no change of group velocity when it is considered as a continuation of its mirror image after collision.

10. Conclusions. From the foregoing analysis one reaches the following conclusions:

1. A procedure has been devised by which the permanent shape of a group of gravity waves in deep water and the attending irrotational flow can be systematically determined to any order of approximation.

2. The phase velocity c of the individual waves in any group and the group velocity c_g are determined by two closed formulas, (74) and (75), in terms of the group wavenumber ε. These formulas remain intact for all orders of approximation, although ε^2 varies as a power series in a^2 (starting with that power), where a is proportional to the amplitude of the waves.

3. The phase velocity c increases with ε and the group velocity c_g decreases with ε in such a way that their product is constant. The increase of c with ε is in agreement with the corresponding result from the cubic-Schrödinger-equation approach. But since that approach, being an analysis to the third order of the amplitude only, is incapable of determining the dependence of the group velocity c_g on the amplitude, no comparison can be made with the CBE approach in regard to c_g.

4. The group wavenumber ε, given by (80), is proportional to the amplitude of the surface displacement and to k^2, k being the wavenumber of the basic waves.

5. Collision of groups of gravity waves leaves the groups intact, with not even any phase shifts. In particular, a group approaching a vertical boundary normal to its velocity of advance will be, after the transients have died out, just the continuation of its mirror image across the boundary.

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