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ON VISCOPLASTIC AND ELASTIC-PLASTIC OSCILLATORS*

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1. Introduction. The theory of plasticity oldest and best known to mathematicians is the traditional rate-independent theory developed in the early part of this century to model post-yield behavior of metals. Experimentalists, even in the early years of the subject, did not regard this as a very precise model even for metals (cf. the discussion by Bell [1]). In order to introduce experimentally observed rate-dependent behaviors, various models which include dissipative effects, among other corrections to the traditional model, have been proposed (cf. Cristescu and Suliciu [2]). Correspondingly, in order to make the analysis of rate-independent plasticity problems more tractable, analysts have introduced viscosities into the balance equations in various ways, in order to obtain solutions for the original equations by limiting arguments (cf., for example, Duvaut and Lions [3]). Since the models of viscoplasticity are experimentally more precise than rate-independent theories, and the resulting balance equations are much more tractable mathematically than those corresponding to the rate-independent theories it seems appropriate to utilize them in solving boundary-value problems of plasticity. In particular, any argument regarding the merits of viscoplastic versus rate-independent plastic models would be mollified by the observation that, in the limit of vanishing "viscosity", not only do the constitutive equations of viscoplasticity approach those of rate-independent theories (cf., e.g., Gurtin, Williams, and Suliciu [4]), but also the *solutions* of the corresponding equations approach solutions of the equations for the rate-independent model. In this paper we show, in the simplest possible geometry, that this is true for a reasonably general model of viscoplasticity. In particular, we choose a viscoplastic model in which a Maxwell-type viscosity forces return to the yield envelope, while within this envelope a linear elastic law governs. The model also includes work-hardening in a rather general form. We establish existence and uniqueness for the initial-value problem for both the viscoplastic model and the corresponding rate-independent model and demonstrate that the viscoplastic solutions approach

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(uniformly) the rate-independent solution. We believe that these results are all new for materials with work-hardening.[1]

2. The elastic-plastic oscillator. We consider the case of a uniform bar with one end fixed and a mass attached to the other end. This end is subject to a longitudinal force $f(t)$ at time $t \in [0, \infty)$. We presume the mass of the bar to be negligible so that it is reasonable that at each time t the stress $\sigma(t)$ and the strain $\varepsilon(t)$ are uniform throughout the bar. Thus we have a situation which may be described as "zero-dimensional", enabling us to study the response of the material by ordinary differential equations. Such problems have been described in the literature as "dangling spider problems" (eg., [6], [7]). The equation of motion for the mass in normalized form is

$$\dot{v} = f - \sigma, \tag{1}$$

where $v(t)$ is the velocity at time t. The equation

$$\dot{\varepsilon} = v \tag{2}$$

then is dictated by kinematics. To complete the system of equations we need only constitutive equations relating σ and ε. However, in order to include the effect of work-hardening, we introduce another quantity, the hardening parameter $\kappa(t)$. In our model, the relation between stress and strain is linear in the elastic zone $\Sigma(\kappa, \varepsilon) \subset \mathbf{R}$,

$$\dot{\sigma} = E\dot{\varepsilon} \quad \text{for } \sigma \in \Sigma(\kappa, \varepsilon). \tag{3}$$

Here E is a positive constant. We take the elastic zone Σ to be described as follows:

$$\Sigma(\kappa, \varepsilon) = \{\sigma \mid -Y(\kappa) + E_0\varepsilon < \sigma < Y(\kappa) + E_0\varepsilon\}. \tag{4}$$

Here Y is a positive isotone function and E_0 a positive constant. This model of the elastic zone incorporates both isotropic work-hardening (increasing the yield value via $Y(\kappa)$ as κ increases) and kinematic work-hardening (the term linear in ε introduces an idealized Bauschinger effect). The behavior of the work-hardening parameter is dictated by the requirement that κ represent a non-recoverable energy, as we will make precise below.

The behavior of σ on the rest of the domain will depend upon whether the material is taken to be rate-independent or not. For the classical rate-independent case, the only further requirements upon σ are that it stay in the closure of the elastic zone at all times and follow an elastic curve when possible: whenever $\sigma = \pm Y + E_0\varepsilon$,

$$\begin{cases} \dot{\sigma} = \pm Y'(\kappa)\dot{\kappa} + E_0\dot{\varepsilon} & \text{if } (\sigma - E_0\varepsilon)\dot{\varepsilon} > 0, \\ \dot{\sigma} = E\dot{\varepsilon} & \text{if } (\sigma - E_0\varepsilon)\dot{\varepsilon} \leq 0.^2 \end{cases} \tag{5}$$

[1] In fact, OWEN's result in [5] of uniqueness for a rate-independent model applies as well to a model with our slightly more general version of work-hardening, as we show in Sec. 5. OWEN points out that the proof of existence for a non-work-hardening rate-independent model by BUHITE and OWEN in [6] can be applied to his model; the same is true for our rate-independent model. Although one obtains a slightly stronger result by that method, the present method of approximation by viscoplastic solutions seems preferable to us, as it is more elementary.

[2] In fact, it is clear that in general if $(\hat{Y}\dot{\varepsilon}) \leq 0$ then either $\dot{\sigma}$ is not defined or is zero. The existence of transition points of this sort is one of the factors which makes analysis of rate-independent plasticity awkward.

If, on the other hand, the material is rate-dependent, specifically, viscoplastic, then the stress σ is allowed to travel outside the region $\mathrm{cl}(\Sigma)$ but is penalized by a viscous restoring effect when it does. We suppose that the viscous term is linear in the distance from σ to $\partial\Sigma$. Thus, if we define τ to be

$$\tau := \begin{cases} \sigma & \text{if } \sigma \in \Sigma, \\ \hat{Y} + E_0\varepsilon & \text{if } \sigma \notin \Sigma, \end{cases} \tag{6}$$

we require

$$\dot{\sigma} = E\dot{\varepsilon} - \mu[\sigma - \tau], \tag{7}$$

where μ is a positive constant, the (Maxwell) viscosity. In (6) we have made use of the function $\hat{Y}$ defined in the exterior of Σ by

$$\hat{Y} = \begin{cases} +Y(\kappa) & \text{if } \sigma \geq Y(\kappa) + E_0\varepsilon \\ -Y(\kappa) & \text{if } \sigma \leq -Y(\kappa) + E_0\varepsilon. \end{cases} \tag{8}$$

We assume that the dynamic elastic modulus E is larger than the static elastic modulus E_0:

$$E_0 < E.$$

We also suppose that Y is of class C^1 and so

$$Y' \geq 0.$$

Our definition of τ ensures that it must obey the equations

$$\begin{aligned} &\dot{\tau} = E\dot{\varepsilon} && \text{when } \sigma \in \Sigma \\ &\begin{cases} \dot{\tau} = \hat{Y}'(\kappa)\dot{\kappa} + E_0\dot{\varepsilon} & \text{if } \dot{\varepsilon}\hat{Y} > 0 \\ \dot{\tau} = E\dot{\varepsilon} & \text{if } \dot{\varepsilon}\hat{Y} \leq 0 \end{cases} && \text{when } \sigma \notin \Sigma. \end{aligned}$$

It is important to notice that the models are very closely related: formally, Equation (7) reduces to Equation $(5)_2$ whenever the viscosity μ becomes infinite, for then the equation ensures that $\sigma = \tau$. We could subsume both models in one by adding to the equations for the viscoplastic model the equation governing τ: then we would describe the rate-independent model by replacing the equation for σ by the equation $\sigma = \tau$.

Finally, we introduce the equation for $\dot{\kappa}$. First, note that we may define the plastic strain $\alpha(t)$ at time t to be the value of strain to which the current stress-strain point would relax along an elastic curve (see Fig. 1)[3]:

$$\sigma =: E(\varepsilon - \alpha). \tag{9}$$

Looking at Fig. 1, we observe, following ZYCZKOWSKI [8, Ch. I, §§8.7, 8.8], that the unshaded area represents unrecoverable work; the (ε, σ)–$(\varepsilon, 0)$–$(\alpha, 0)$ triangle represents an elastic contribution corresponding to the modulus E; likewise, the shaded triangle with side $(0, 0)$–$(\alpha, 0)$ represents an elastic energy corresponding to the modulus E_0. Note that the shaded area can be written as

$$\frac{\sigma^2}{2E} + \frac{E_0 E}{2(E - E_0)}\alpha^2$$

[3]In fact, even in the rate-independent case, α may not be accessible by a continuation of the process.

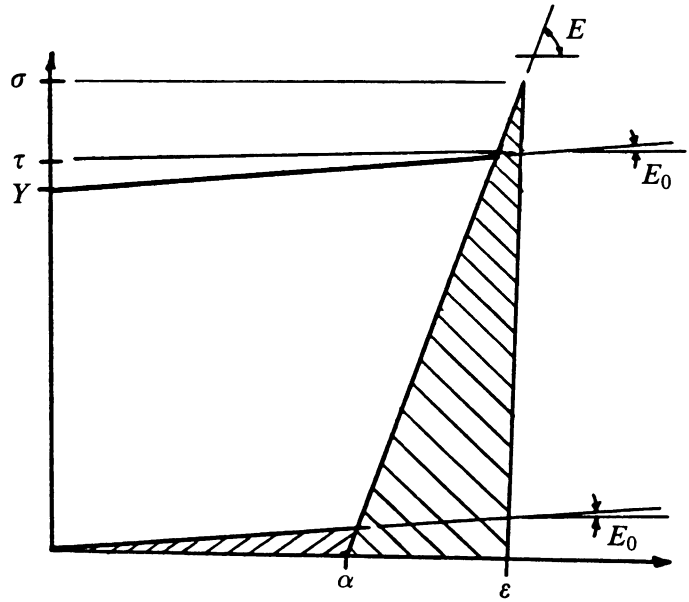

FIG. 1

since the stress value at the intersection of the two lower lines is $E_0 E\alpha/(E - E_0)$. Now, we may use the definition of α to write the local rate of working as

$$\sigma\dot{\varepsilon} = \sigma\left[\frac{1}{E}\dot{\sigma} + \dot{\alpha}\right] = \frac{d}{dt}\left[\frac{\sigma^2}{2E}\right] + \sigma\dot{\alpha}.$$

Returning to Fig. 1, we see that σ can be expressed as

$$\sigma = E(\varepsilon - \alpha) = (\sigma - \tau) + \hat{Y} + E_0\varepsilon,$$

so that

$$(E - E_0)\varepsilon = (\sigma - \tau) + \hat{Y} + E\alpha,$$

and finally we obtain

$$\sigma = \frac{E}{E - E_0}[(\sigma - \tau) + \hat{Y}] + \frac{EE_0}{E - E_0}\alpha.$$

Hence we find

$$\sigma\dot{\varepsilon} = \frac{d}{dt}\left[\frac{\sigma^2}{2E} + \frac{EE_0\alpha^2}{2(E - E_0)}\right] + \frac{\mu}{E - E_0}(\sigma - \tau)^2 + \frac{E}{E - E_0}\hat{Y}\dot{\alpha},$$

observing that whenever $\sigma \neq \tau$, $E\dot{\alpha} = \mu(\sigma - \tau)$. Thus, having isolated a viscous dissipation term, we generalize the corresponding computation of ZYCZKOWSKI to rate-dependent plasticity by defining the work-hardening rate to be:

$$\dot{\kappa} = \frac{E}{E - E_0}\hat{Y}\dot{\alpha} \tag{10}$$

for both the rate-independent and the viscoplastic models. Finally,

$$\sigma\dot{\varepsilon} = \frac{d}{dt}\left[\frac{\sigma^2}{2E} + \frac{EE_0\alpha^2}{2(E-E_0)}\right] + \frac{\mu}{E-E_0}(\sigma-\tau)^2 + \dot{\kappa}. \tag{11}$$

Thus we have introduced the final variable, κ. The equation governing its evolution is

$$\dot{\kappa} = \begin{cases} 0 & \text{if } \sigma \in \Sigma \\ [E\dot{\varepsilon} - \dot{\sigma}]\hat{Y}/(E-E_0) & \text{otherwise.} \end{cases} \tag{12}$$

Note that (11) suggests the identification of the function ψ defined by

$$\psi(\sigma, \varepsilon, \kappa) = \frac{\sigma^2}{2E} + \frac{EE_0\alpha^2}{2(E-E_0)} + \kappa$$

as a free-energy for the system, since it satisfies

$$\sigma\dot{\varepsilon} - \dot{\psi} = \frac{\mu}{E-E_0}(\sigma-\tau)^2 \geq 0.$$

Remarks. The constitutive equation which we introduce specializes to classical forms as follows.

a) When $E_0 = 0$ and $Y(\kappa) = Y_0$ for all κ, the viscoplastic model is the model of SOKOLOVSKY, and the rate-independent model is the classical elastic-perfectly plastic model.

b) When $E_0 = 0$ and $Y'(\kappa) > 0$, the model is that of isotropic work-hardening; cf. [5].

c) When $0 < E_0 < E$ and $Y(\kappa) = Y_0$ for all κ, the model represents a model with kinematic work-hardening or an "idealized Bauschinger effect".

For a more detailed discussion of these models, see [2] for the viscoplastic case and [8] for the rate-independent case.

3. The systems of differential equations. Let $f \in L_1[0, \infty)$ be given. We consider systems of equations for the functions

$$v, \sigma, \varepsilon, \kappa,$$

which shall represent the solutions for the viscoplastic case, and for

$$v_\infty, \sigma_\infty, \varepsilon_\infty, \kappa_\infty,$$

which represent the solutions for the rate-independent case. The equations introduced in the previous section can be put into standard form by removing all rate terms from the right-hand sides. For the rate-independent case we obtain

$$\begin{aligned} \dot{v}_\infty &= f - \sigma_\infty, \\ \dot{\varepsilon}_\infty &= v_\infty, \\ \left.\begin{aligned} \dot{\sigma}_\infty &= Ev_\infty, \\ \dot{\kappa}_\infty &= 0, \end{aligned}\right\} &\quad \text{whenever } \sigma_\infty \in \Sigma, \end{aligned}$$

while

$$\begin{cases} \dot{\sigma}_\infty = \dfrac{E_0(E-E_0) + EYY'}{E - E_0 + YY'} v_\infty, & \text{if } (\sigma_\infty - E_0\varepsilon_\infty)v_\infty > 0, \\ \dot{\sigma}_\infty = Ev_\infty, & \text{if } (\sigma_\infty - E_0\varepsilon_\infty)v_\infty \leq 0, \end{cases}$$

and

$$\begin{cases} \dot{\kappa}_\infty = \dfrac{(E-E_0)\hat{Y}}{E-E_0+YY'}v_\infty, & \text{if } (\sigma_\infty - E_0\varepsilon_\infty)v_\infty > 0, \\ \dot{\kappa}_\infty = 0, & \text{if } (\sigma_\infty - E_0\varepsilon_\infty)v_\infty \le 0, \end{cases}$$
$$\text{whenever } \sigma_\infty \in \partial\Sigma.$$

For the viscoplastic case, we have

$$\begin{aligned} &\dot{v} = f - \sigma, \\ &\dot{\varepsilon} = v, \\ &\left.\begin{aligned} &\dot{\sigma} = Ev, \\ &\dot{\kappa} = 0, \\ &\tau = \sigma, \end{aligned}\right\} \quad \text{whenever } \sigma \in \Sigma, \end{aligned}$$

and

$$\left.\begin{aligned} &\dot{\sigma} = E\dot{\varepsilon} - \mu[\sigma - \tau], \\ &\dot{\kappa} = \frac{\mu\hat{Y}}{E-E_0}(\sigma - \tau), \\ &\tau = \hat{Y} + E_0\varepsilon, \end{aligned}\right\} \quad \text{whenever } \sigma \notin \Sigma.$$

To simplify the notation in these equations we introduce the following function:

$$E^0(\sigma, \varepsilon, \kappa, v) = \begin{cases} E & \text{if } \sigma \in \Sigma \text{ or } (\sigma - E_0\varepsilon)v \le 0, \\ E_0 & \text{otherwise.} \end{cases}$$

Using this, we present the equations in final form. First, for the rate-independent case,

$$\left.\begin{aligned} &\dot{v}_\infty = f - \sigma_\infty, \\ &\dot{\varepsilon}_\infty = v_\infty, \\ &\dot{\sigma}_\infty = \frac{E^0(E-E_0) + EYY'}{E-E_0+YY'}v_\infty \\ &\dot{\kappa}_\infty = \frac{(E-E^0)\hat{Y}}{E-E_0+YY'}v_\infty \end{aligned}\right\} \quad \text{a.e.,} \tag{13}$$

with the initial conditions

$$\begin{aligned} v_\infty(0) &= v_0, \\ \varepsilon_\infty(0) &= \varepsilon_0, \\ \sigma_\infty(0) &= \sigma_0, \\ \kappa_\infty(0) &= \kappa_0, \end{aligned}$$

which obey

$$\sigma_0 \in \operatorname{cl}\Sigma(\varepsilon_0, \kappa_0), \tag{14}$$

constitute the initial-value problem RIP.

For the viscoplastic case, the equations compress in a similar fashion if we note that $\sigma = \tau$ in Σ:

$$\left.\begin{aligned} \dot{v} &= f - \sigma, \\ \dot{\varepsilon} &= v, \\ \frac{1}{\mu}[\dot{\sigma} - E\dot{\varepsilon}] &= -[\sigma - \tau], \\ \frac{1}{\mu}\dot{\kappa} &= \hat{Y}[\sigma - \tau]/(E - E_0) \end{aligned}\right\} \quad \text{a.e.} \tag{15}$$

These, with the same initial conditions, constitute the initial-value problem VPP.

Our objective is to demonstrate the existence and uniqueness of solutions to both problems and to show that solutions to VPP converge to solutions of RIP.

4. Energy identity and a priori estimates. We call the combination

$$e = \frac{v^2}{2} + \frac{\sigma^2}{2E} + \frac{EE_0\alpha^2}{2(E^0 - E_0)} + \kappa \tag{16}$$

the total energy of the system for either RIP or VPP; here, as throughout this section, we drop the sub-∞ notation for solutions of the former.

LEMMA 4.1. Every solution of RIP or VPP satisfies

$$\dot{e} = fv - \mu(\sigma - \tau)^2/(E - E_0) \quad \text{a.e. in } [0, \infty).$$

Proof. From the equations for $\dot{v}$ and $\dot{\varepsilon}$ we have

$$\begin{aligned} \dot{e} &= v\dot{v} + \frac{1}{E}\sigma\dot{\sigma} + \frac{EE_0\alpha\dot{\alpha}}{E - E_0} + \dot{\kappa} \\ &= fv - \dot{\varepsilon}\sigma + \frac{1}{E}\sigma\dot{\sigma} + \frac{EE_0\alpha\dot{\alpha}}{E - E_0} + \dot{\kappa}, \end{aligned}$$

and the computations leading to the definition of $\dot{\kappa}$ reduce this to

$$\dot{e} = fv - \mu(\sigma - \tau)^2/(E - E_0),$$

as stated. □

Next, the energy can be used to establish some estimates for the magnitude of the solutions to either system.

LEMMA 4.2. For any solution of RIP or VPP, for all $t \in [0, \infty)$,

$$\sqrt{e(t)} \leq \sqrt{e(0)} + \frac{1}{\sqrt{2}} \int_0^t |f|.$$

Proof. From the previous lemma we have

$$\dot{e} \leq fv \leq |f|\,|v| = |f|\sqrt{v^2} \leq |f|\sqrt{2e}.$$

Thus,

$$(\sqrt{e})^{\cdot} \leq \frac{1}{\sqrt{2}}|f|$$

and the result follows. □

This implies that solutions of either system are bounded. First, each of σ, v, and α is bounded by a constant times the root of the energy e, while κ is bounded by e. Of course, since the bound is expressible in terms of the initial values and the constants, μ excepted, of the problem, *the bound is the same for both problems and is uniform in t and in μ.* Since $\varepsilon = \alpha + \sigma/E$, ε is bounded; since Y is smooth, $Y(\kappa)$ is bounded, and thus so is τ.

The evolution equations allow us to make estimates now for the derivatives of the various quantities. Since $\dot{\varepsilon} = v$ and $|\dot{\sigma}| \leq E|v|$, both of these are bounded. For RIP we observe that it follows, since YY' is positive, that $\dot{\kappa}$ is bounded. For VPP, we need further estimates. First, we find an estimate for $\sigma - \tau$. Note that

$$\dot{\tau} = YY'\mu|\sigma - \tau|/(E - E_0) + E^0\dot{\varepsilon},$$

using the definition of $\hat{Y}$, and

$$\dot{\sigma} = E\dot{\varepsilon} - \mu(\sigma - \tau)$$

so that

$$(\sigma - \tau)^{\cdot} = (E - E^0)\dot{\varepsilon} - \frac{\mu YY'}{E - E_0}(\sigma - \tau) - \mu(\sigma - \tau).$$

Then we find

$$|\sigma - \tau|^{\cdot} \leq (E - E_0)|\dot{\varepsilon}| - \mu|\sigma - \tau|,$$

which leads, by the usual calculation, to

$$|\sigma - \tau|(t) \leq \frac{1}{\mu}(E - E_0)(\sup|\dot{\varepsilon}|)(1 - e^{-\mu t}). \tag{17}$$

Now applying this to (15) shows that

$$\dot{\kappa} \leq Y \sup(|v|),$$

and

$$|\sigma - \tau|^{\cdot} \leq (E - E_0)\sup(|v|).$$

Of course, unless f is bounded, we do not have a corresponding estimate for $\dot{v}$.

We summarize the principal bounds in

PROPOSITION 4.3. Given any solutions of RIP and VPP, all of

$$v, \sigma, \varepsilon, \kappa, \tau, \dot{\sigma}, \dot{\varepsilon}, \dot{\tau}, \dot{\kappa}, v_\infty, \sigma_\infty, \varepsilon_\infty, \kappa_\infty, \dot{\sigma}_\infty, \dot{\varepsilon}_\infty, \dot{\kappa}_\infty$$

are bounded on $[0, \infty)$, uniformly with respect to t and μ. If, in addition, f is bounded, then $\dot{v}$ and $\dot{v}_\infty$ are bounded uniformly in t and μ.

To verify the role of τ as a rate-independent version of σ we can easily see from (17) that as $\mu \to \infty$ the two coincide.

PROPOSITION 4.4. For solutions of VPP

$$\lim_{\mu \to \infty}(\sigma - \tau) = 0$$

uniformly in t.

5. Uniqueness of solutions to RIP. We note that there exist unique solutions to problem VPP; this is easy to show based on the smoothness of the right-hand side of the corresponding equations and the energy estimates. The problem for RIP is more difficult: in the next section we shall establish existence for this problem by examining the limiting case of the viscoplastic problem, but first we establish uniqueness for RIP. A simple calculation shows that this problem is not easily amenable to energy arguments; in fact, the energy does decrease, but one must have recourse to careful interval-by-interval examination of the energy change (cf. [5] for details). In [5] OWEN has established such a result for a version of RIP with $E_0 = 0$; we now show that his argument may be applied in our case. OWEN proceeds by showing that the problem can be re-expressed as a problem in which the elastic zone is expressed as

$$S_-(\alpha) < \sigma < S_+(\alpha)$$

with $S_\pm$ having certain properties. Let us identify the function S_+ for our situation: since

$$\dot{\kappa} = \frac{EY(\kappa)}{E - E_0}\dot{\alpha},$$

so long as $\dot{\alpha} \geq 0$ we may solve for α as a function of κ. The relation is invertible and we obtain

$$\kappa = K(\alpha).$$

Then since $\varepsilon = \alpha + \sigma/E$, when $\sigma \in \partial\Sigma$,

$$\sigma = Y + E_0(\alpha + \sigma/E)$$

and

$$S_+(\alpha) := \frac{E}{E - E_0}[Y(K(\alpha)) + E_0\alpha].$$

This means that

$$S_+' = \frac{E}{E - E_0}\left[\frac{E}{E - E_0}YY' + E_0\right]$$

and

$$S_+'' = \frac{E^3}{(E - E_0)^3}Y(Y'^2 + YY'').$$

A similar computation applies to the alternative $\sigma = S_-(\alpha)$ and we can now apply the result of OWEN. For convenience in describing the geometry of the situation, let us divide $\operatorname{cl}\Sigma$ into "positive" and "negative" parts:

$$\sigma \in \mathfrak{P} \Leftrightarrow \frac{EE_0}{E - E_0}\alpha \leq \sigma \leq S_+(\alpha),$$
$$\sigma \in \mathfrak{N} \Leftrightarrow S_-(\alpha) \leq \sigma \leq \frac{EE_0}{E - E_0}\alpha.$$

The linear bound represents the straight line through the origin and of slope E_0 in the (σ, ε)-plane, and replaces the bound of 0 in OWEN's model, in which $E_0 = 0$.

PROPOSITION 5.1. Suppose that $Y'^2 + YY'' \leq 0$. Any solution of problem RIP during each part of the motion in which $\sigma \in \mathfrak{P}$ also solves the equations:

$$\dot{v} = f - \sigma,$$
$$\dot{\sigma} = \begin{cases} \dfrac{ES_+'(\alpha)}{E + S_+'(\alpha)}v & \text{if } \sigma = S_+(\alpha) \quad \text{and } v \geq 0 \\ Ev & \text{otherwise,} \end{cases} \tag{18}$$
$$\dot{\alpha} = \begin{cases} \dfrac{E}{E + S_+'(\alpha)}v & \text{if } \sigma = S_+(\alpha) \quad \text{and } v \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The first derivative of S_+ is positive-valued, while its second derivative is negative-valued. During each part of the motion in which $\sigma \in \mathfrak{N}$ the solutions of RIP also solve

$$\dot{v} = f - \sigma,$$
$$\dot{\sigma} = \begin{cases} \dfrac{ES'_-(\alpha)}{E + S'_-(\alpha)} v & \text{if } \sigma = S_-(\alpha) \quad \text{and } v \le 0 \\ Ev & \text{otherwise}, \end{cases}$$
$$\dot{\alpha} = \begin{cases} \dfrac{E}{E + S'_-(\alpha)} v & \text{if } \sigma = S_-(\alpha) \quad \text{and } v \le 0 \\ 0 & \text{otherwise}. \end{cases}$$

The first and second derivatives of S_- are positive-valued.

It now is easy to see that the computations in [5] apply to this variation of the problem; the only difference between the two formulations lies in the description of the sets $\mathfrak{P}$ and $\mathfrak{N}$ and in the fact that our S_+ is not always positive nor S_- always negative. However, it is easy to see that the arguments of [5] are unchanged by these differences. To correspond with OWEN's nomenclature let us note that it suffices to consider only the solution in $\mathfrak{P}$ and describe the set of equations (18) together with the conditions

$$\sigma(0) = \sigma_0, \qquad v(0) = v_0, \qquad \alpha(0) = \alpha_0$$

$(\kappa_0 = K(\alpha_0))$ as the auxiliary problem.

THEOREM 5.2 (OWEN). For each $(v_0, \sigma_0, \kappa_0)$ which satisfy (14) and each solution of RIP on a given interval $[0, T]$ there exists $T_a \in (0, T]$ such that the restriction of the functions v and σ to $[0, T_a]$ together with the function $\alpha\colon [0, T_a] \to \mathbf{R}$ defined by

$$\alpha(t) := \alpha_0 + \int_0^t (v - \dot{\sigma}/E) \qquad \forall t \in [0, T_a]$$

form a solution of the auxiliary problem. Moreover, distinct solutions of RIP yield distinct solutions of the auxiliary problem. Solutions of the auxiliary problem are unique and hence solutions of RIP are unique.

6. Existence, approach of VPP solutions to RIP solutions. Because the right-hand sides of the equations VPP are all locally Lipschitzian functions of their arguments, and because of our energy estimate, this system can easily be shown to have a unique solution. We collect our assumptions and formalize the result in the following.

PROPOSITION 6.1. Let Y be C^1, $Y' \ge 0$, $E > E_0 \ge 0$, and suppose that $f \in L^1[0, \infty)$. Then for each $\mu \in [0, \infty)$ there exists a unique solution in AC$[0, \infty)$ to VPP.

For the following discussion, we will also require that f is bounded. For each value of μ the set of solutions to VPP are Lipschitzian uniformly in μ. It follows that for any choice of $T \in (0, \infty)$ we can select a sequence of solutions $((\sigma_n, \varepsilon_n, v_n, \kappa_n) | n \in \mathbf{N})$ with viscosity $\mu_n \to \infty$ which converge uniformly on $[0, T]$ to a set $(\sigma, \tau, \varepsilon, v, \kappa)$ of absolutely continuous functions. Moreover, since the derivatives are uniformly bounded, it follows that $(\dot{\sigma}_n, \dot{\tau}_n, \dot{\varepsilon}_n, \dot{v}_n, \dot{\kappa}_n) \rightharpoonup (\dot{\sigma}, \dot{\tau}, \dot{\varepsilon}, \dot{v}, \dot{\kappa})$, i.e., that the derivatives approach the derivatives of the limiting functions weakly in $L^1[0,T]$. Of course, since $\dot{\varepsilon}_n = v_n$, the convergence of $\dot{\varepsilon}_n$ to $\dot{\varepsilon}$ is uniform and hence

$$\dot{\varepsilon} = v.$$

Finally, note that we have established in Proposition 4.4 that

$$\lim |\sigma_n - \tau_n| = 0$$

(uniformly) and hence it is true that $\tau_n \to \sigma$ and $\dot{\tau}_n \rightharpoonup \dot{\sigma}$.

Now we wish to establish that the limiting functions are solutions to RIP. First, since σ is the limit of τ_n, it is clear that

$$|\sigma - E_0\varepsilon| \leq Y(\kappa).$$

Now, since

$$\dot{v}_n - \dot{v}_m = \sigma_n - \sigma_m,$$

it is clear that $\dot{v}_n$ approaches $\dot{v}$ in $L^1[0,T]$ and hence pointwise a.e. so that

$$\dot{v} = f - \sigma, \quad \text{a.e.}$$

Now we define the elastic zone Σ for $(\sigma, \tau, \varepsilon, v, \kappa)$ as usual and consider a time t_0 at which $\sigma \in \Sigma$. Since

$$|\sigma(t_0) - E_0\varepsilon(t_0)| < |Y(\kappa(t_0))|,$$

it follows that for all sufficiently large n the same equation holds for σ_n, ε_n, and κ_n. Thus we have for all t in some neighborhood of t_0,

$$\dot{\sigma}_n(t) = E\dot{\varepsilon}_n(t),$$
$$\dot{\kappa}_n(t) = 0.$$

Then it is clear that

$$\dot{\sigma}(t_0) = E\dot{\varepsilon}(t_0),$$
$$\dot{\kappa}(t_0) = 0.$$

Finally, let us define

$$\mathfrak{T} = \{t | \sigma(t) = Y(\kappa(t)) + E_0\varepsilon(t)\}.$$

This set is closed. Since the collection of isolated points of the set is of measure zero, we need not be concerned with the derivatives of σ and κ there. On the other hand, at an accumulation point t of $\mathfrak{T}$, there is by definition a sequence of points of $\mathfrak{T}$ which approach t. At each value t_m in the sequence we have $\sigma(t_m) = E_0\varepsilon(t_m) + Y(\kappa(t_m))$ and hence

$$\dot{\sigma}(t) = E_0\dot{\varepsilon}(t) + Y'(\kappa(t))\dot{\kappa}(t).$$

Finally we consider that since $\kappa_n \to \kappa$ uniformly and $\dot{\kappa}_n \rightharpoonup \dot{\kappa}$,

$$\frac{Y(\kappa_n)}{E - E_0}(E\dot{\varepsilon}_n - \dot{\sigma}_n) \rightharpoonup \dot{\kappa},$$

while the former also has weak limit

$$\frac{Y(\kappa)}{E - E_0}(E\dot{\varepsilon} - \dot{\sigma}),$$

so we can conclude that almost everywhere

$$\dot{\kappa} = \frac{Y(\kappa)}{E - E_0}(E\dot{\varepsilon} - \dot{\sigma}),$$

which means, of course, that κ satisfies the required equations.

Thus we have a set of solutions on $[0, T]$ to the problem RIP, obtained as a limit of a sequence of solutions of the viscoplastic problem. However, it follows from the uniqueness for RIP that in fact

$$\lim_{\mu \to \infty} (\sigma_\mu, \varepsilon_\mu, v_\mu, \kappa_\mu) = (\sigma, \varepsilon, v, \kappa).$$

We summarize in

THEOREM 6.2. Suppose that Y is of class C^2, that $Y' \geq 0$, $Y'^2 + YY'' \leq 0$, that $f \in L_1[0, \infty)$ is bounded, and that $E > E_0$. Then there exists a unique solution to the problem RIP, and it is the uniform limit on any bounded interval of the solution to VPP as μ goes to infinity.

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Note added in proof. K. Wang of Carnegie Mellon has pointed out to us that the assumption made in Sec. 6 and reiterated in Theorem 6.2, that f is bounded, is not necessary. A straightforward argument based on the integrability of f, the convergence properties of (σ_n) and the balance equations shows that v exists as a uniform limit of (v_n); since $v_n \to v$ (strongly), of course $v_n \rightharpoonup v$.

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