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PATTERNS OF SHIP WAVES II. GRAVITY-CAPILLARY WAVES*

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In a previous paper [4], Yih's formulas [3] were used to obtain patterns of gravity waves, or of capillary waves in a thin fluid sheet, created by a moving disturbance. In this paper the effects of surface tension are taken into account in finding the patterns of capillary-gravity waves in deep water with a free surface created by a moving disturbance, and much more extensive results than those of Rayleigh [2] have been obtained. The most important feature of the waves is that there are capillary waves *behind* the disturbance, which have very short wavelengths at high values of the speed U of the disturbance and which are confined to a wedge of an angle that decreases as U increases. Of interest too is the existence of two cusps in the phase lines on either side of the centerline at high values of U (relative to a minimum wave velocity defined in the paper) for those waves which are entirely behind the disturbance.

1. Introduction. Explicit formulas for phase lines of any kind of dispersive waves created by a point disturbance moving in a fluid with a free surface were given by Yih [3]. These formulas are in terms of the parameter k, which is the local wavenumber. The point disturbance is an idealized representation of a ship, for instance, so that the formulas are useful for determining the pattern of waves far enough away from the ship. Yih's formulas were used by Yih and Zhu [4] to obtain patterns of ship waves in deep water (Kelvin waves), in water of finite depth, in a stratified ocean, and in the wake of a ship, as well as patterns of waves in a thin sheet caused by a moving point disturbance. But capillary-gravity waves were not treated in [4]. It will be treated in this paper, and many patterns of capillary-gravity waves caused by a moving disturbance will be presented.

The main reason for giving capillary-gravity waves a closer examination is that the treatment by Rayleigh [2], as quoted in Lamb ([1], pp. 469–471), is very sketchy and calls for a new calculation after more than a century, especially in view of the relevance of the problem to remote sensing. As will be seen, one important feature of capillary-gravity waves caused by a moving disturbance is that there are (predominantly) capillary waves *behind* the disturbance. This point has not been stressed in

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Lamb's book, but explains the presence of short waves within a narrow wedge which are often found in photographs obtained by remote sensing in the wake of a ship.

2. Analysis. Let the point disturbance move with speed U in the horizontal direction of decreasing x. The y-axis is also horizontal, and is normal to the x-axis. As in [4], ρ denotes the density of the fluid, g denotes the gravitational acceleration, T denotes surface tension, and we shall continue to use U^2/g as the length scale, so that the local wavenumber k will continue to be measured in units of g/U^2. The x and y components of the wavenumber vector k will again be denoted by ξ and η, so that

$$\xi^2 + \eta^2 = k^2. \tag{1}$$

With this in mind, the requirement that the local wave velocity must be equal to the component of the velocity of the disturbance normal to the wave front (or the phase line) is

$$\xi = F(k) = k(1/k + \sigma k)^{1/2}, \tag{2}$$

which is Eq. (6) in [4] for the present problem. The σ in (2) is

$$\sigma = \frac{Tg}{\rho U^4} = \frac{1}{4}\left(\frac{c_{\text{min}}}{U}\right)^4, \tag{3}$$

where $c_{\text{min}} = 2(Tg/\rho)^{1/2}$ is the minimum value of the wave velocity c calculated from the dimensional dispersion equation, given by

$$c^2 = \frac{g}{\hat{k}} + \frac{T}{p}\hat{k}, \tag{4}$$

in which $\hat{k}$ denotes the dimensional local wavenumber. The $\hat{k}$ for c_{min} is

$$\hat{k}_{\text{cr}} = \{\rho g/T)^{1/2}. \tag{5}$$

If $\hat{k} > \hat{k}_{\text{cr}}$, or

$$k > (1/\sigma)^{1/2},$$

the waves are predominantly capillary waves. If

$$k < (1/\sigma)^{1/2},$$

the waves are predominantly gravity waves. The word "predominantly" may from time to time be omitted in the rest of this paper for brevity.

Yih's formulas for phase lines are (see Eqs. (19) and (20) in [1])

$$y = -\frac{aF'}{k(F - kF')}(k^2 - F^2)^{1/2}, \tag{6}$$

$$x = \frac{a(k - FF')}{k(F - kF')}, \tag{7}$$

where a is a constant of integration. The k in (1) will be treated as positive. Since y is real, (6) demands that

$$k^2 - F^2 \geq 0, \tag{8}$$

or

$$\sigma k^2 - k + 1 \leq 0. \tag{9}$$

That means

$$k_{\min} \le k \le k_{\max}, \tag{10}$$

where

$$k_{\min} = \frac{1-(1-4\sigma)^{1/2}}{2\sigma}, \tag{11}$$

$$k_{\max} = \frac{1+(1-4\sigma)^{1/2}}{2\sigma}. \tag{12}$$

Equations (11) and (12) show that there are no waves if $\sigma \ge \frac{1}{4}$. This condition can be written as $U/c_{\min} \le 1$, by virtue of (3).

Equation (7) contains the factor

$$h(k) = k - FF' = -\tfrac{3}{2}\sigma k^2 - k - \tfrac{1}{2}. \tag{13}$$

This can be written as

$$h(k) = -\tfrac{3}{2}\sigma(k-k_1)(k-k_2), \tag{14}$$

where

$$k_1 = \frac{1}{3\sigma}[1-(1-3\sigma)^{1/2}], \tag{15}$$

$$k_2 = \frac{1}{3\sigma}[1+(1-3\sigma)^{1/2}]. \tag{16}$$

Thus

$$h(k) > 0 \qquad \text{if} \quad k_1 < k < k_2,$$
$$h(k) < 0 \qquad \text{if} \quad k \le k_1 \text{ or } k \ge k_2.$$

The sign of $h(k)$ affects the sign of x through (7).

Another quantity of which the sign is important is

$$q(k) = F - kF' = \frac{1}{2(k+\sigma k^3)^{1/2}}(1-\sigma k^2). \tag{17}$$

It is evident that

$$q(k) \ge 0 \qquad \text{if} \quad k < k_3 = \sigma^{-1/2},$$
$$q(k) < 0 \qquad \text{if} \quad k > k_3 = \sigma^{-1/2}.$$

The sign of $q(k)$ affects the sign of y through (6), and the sign of x through (7).

It can be shown (the demonstration is omitted here), that

$$k_1 \le k_{\min} \le k_3 \le k_2 \le k_{\max}. \tag{18}$$

Hence k_1 has no significance, since a k less than $k_{\min}$ results in no waves.

Finally, the k-values for the cusps, denoted by k_{c1} and k_{c2}, are determined from

$$\frac{d^2\eta}{d\xi^2} = 0, \tag{19}$$

or

$$\frac{d}{d\xi}\frac{k\,dk/d\xi - \xi}{(k^2-\xi^2)^{1/2}} = 0.$$

With (1), this becomes, after a brief calculation,

$$\frac{d}{d\xi}\frac{2h(k)\xi}{(1+3\sigma k^2)(k^2-\xi^2)^{1/2}} = 0. \tag{20}$$

This can only be solved numerically once σ is given, and we shall list the values of k_{c1} and k_{c2} in Table 1 for various values of σ.

TABLE 1. Important values of k and values of γ and ϕ.

$\frac{U}{c_{min}}$	k	Waves of Sets 2 and 3				
		$a = 1$	$a = 2$	$a = 3$	$a = 4$	$a = 5$
10	k_{st}			1.000025		
	k_{c1}			1.500253		
	k_{c2}			78.41496		
	k_{en}	196.3524	192.7049	189.6653	186.0177	182.3702
6	k_{st}			1.000193		
	k_{c1}			1.501962		
	k_{c2}			28.06051		
	k_{en}	70.02272	67.82574	65.84847	63.65150	61.23482
4	k_{st}			1.000978		
	k_{c1}			1.510122		
	k_{c2}			12.30653		
	k_{en}	30.71992	29.34138	27.96284	26.38736	24.71341
2	k_{st}			1.016133		
	k_{c1}			1.808496		
	k_{c2}			2.535961		
	k_{en}	7.590197	7.180394	6.715951	6.224187	5.623143
1.8	k_{st}			1.025022		
	k_{en}	6.152701	5.798128	5.416279	4.979881	4.461658
1.5	k_{st}			1.054960		
	k_{en}	4.276073	4.304920	3.759316	3.466488	3.087534
1.1	k_{st}			1.279578		
	k_{en}	2.248937	2.089277	1.941023	1.809875	1.695823

$\frac{U}{c_{min}}$	k	Waves of Set 1 $a = 1$	$a = 2$	$a = 3$	$a = 4$	$a = 5$	γ	ϕ
10	k_{st}			39999			6000	5.71°
	k_{c1}							
	k_{c2}							
	k_{en}	4816.684	7045.430	8716.984	10070.15	11104.93		
6	k_{st}			5183			750	9.59°
	k_{c1}							
	k_{c2}							
	k_{en}	613.7659	899.9819	1114.644	1278.196	1421.304		
4	k_{st}			1023			150	14.47°
	k_{c1}							
	k_{c2}							
	k_{en}	127.1359	182.6319	224.2538	255.9658	283.7139		
2	k_{st}			62.98387			10	29.98°
	k_{c1}							
	k_{c2}							
	k_{en}	11.07910	13.27845	15.37935	16.57748	17.89709		
1.8	k_{st}			40.96538			10	
	k_{en}	8.962947	10.68722	11.99766	13.10193	14.06678		
1.5	k_{st}			19.19504			10	
	k_{en}	6.057674	7.027547	7.732909	8.291320	8.732171		
1.1	k_{st}			4.576822			1	
	k_{en}	2.588232	2.752151	2.907442	3.054106	3.187829		

3. Procedure of Computation. There are three sets of waves created by the moving disturbance:

1. Set 1. Capillary waves *ahead* of the disturbance.
2. Set 2. Largely gravity waves behind the disturbance.
3. Set 3. Capillary waves *behind* (!) the disturbance.

The first set is the well-known fish-line waves. The starting k-value for this set, denoted by k_{st} in Table 1, is just k_{max}. There are no cusps in the phase lines for this set, and for each phase line the k-values decrease from k_{max}, at which $y = 0$ and x is negative, to k_2, at which $x = 0$, and then to k_3 at which x is positive and both x and y are infinite. In plotting the phase lines, one cannot reach k_3, of course, and we have stopped at an ending k, denoted by k_{en}, and given numerically for all values of c_{min}/U and for Sets 1 and 3.

Set 2 corresponds to Kelvin waves, except the effect of surface tension has been taken into account. For this set one starts with a k_{st} equal to k_{min}, at the centerline $y = 0$ and a value of x given by (7), once a is given. One then proceeds along a transverse phase line to the first cusp on either side of the centerline, where k is k_{c1}. Then one increases k toward k_{c2}, in the process tracing out the phase line corresponding to the divergent part of the Kelvin waves, except that one does not reach the origin (where the disturbance is) but reaches the second cusp instead, at which k is k_{c2}.

Then one traces another divergent wave, of Set 3, as one increases k from k_{c2} to k_3, in the process tracing out the phase line that is almost straight, and diverges from the second cusp toward infinity (with positive x), asymptotically making an angle ϕ with the centerline. The angle ϕ is recorded in Table 1 for all cases, and is, incidentally, the same for Set 1 and Set 3. That is, the asymptotes of phase lines for Set 1 at infinity also make the same angle ϕ with the centerline. The angle ϕ is determined analytically by putting $k = k_3 = \sigma^{-1/2}$ in

$$\frac{dy}{dx} = \tan\phi = -\frac{F}{(k^2 - F^2)^{1/2}}, \tag{21}$$

with F defined by (2). Equation (21) is equation (6) of [4]. It is immediately clear that, as σ decreases to zero, k_3 approaches infinity and ϕ approaches zero—a result of great significance to remote sensing.

It is clear from the foregoing description that

$$k_3 < k \leq k_{\max}$$

for Set 1,

$$k_{\min} \leq k \leq k_{c2}$$

for Set 2, and

$$k_{c2} \leq k < k_3$$

for Set 3. For Set 2 (Kelvin waves with surface tension taken into account)

$$k_{\min} \leq k \leq k_{c1}$$

for transverse waves, and

$$k_{c1} \leq k \leq k_{c2}$$

for divergent waves.

4. Results. The results for various values of σ, represented by $U/c_{\min}$, are given in Table 1 and the figures. In Table 1, the results for Sets 2 and 3 are presented together, and those for Set 1 are presented separately. For $U/c_{\min}$ equal to or less than 1, there are no waves, as mentioned in Sec. 2.

The various k-values given and the angle ϕ in Table 1 have been explained in Sec. 3. Since the wavelength for Set 2 at the centerline is enormously greater than the wavelength for Set 1 at the centerline, it is inconvenient to present the entire wave pattern in a single figure with the same length scale. For this reason, for all cases except $U/c_{\min} = 1.1$ (see Figure 1), the phase lines for Set 1 are presented separately from those for Sets 2 and 3. The ratio of the length scale for Sets 2 and 3 to that for Set 1 is denoted by γ, and given in Table 1. In reading the figures, then, one must, for all cases except $U/c_{\min} = 1.1$, imagine the parts b to be magnified γ times in one's mind. It is also helpful to keep in mind that along the centerline ($y = 0$) consecutive phase lines for Set 1 and Set 2 are spaced at one wavelength apart, and

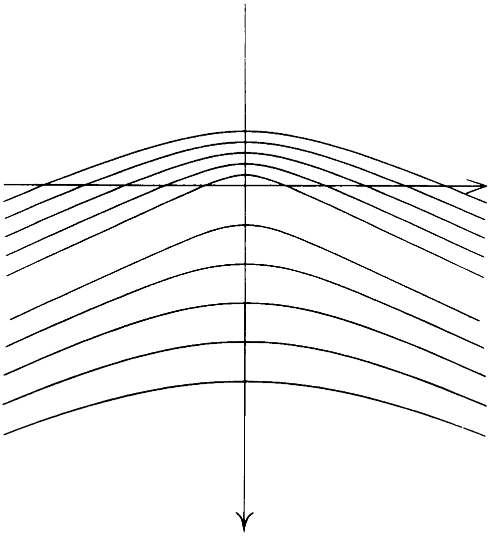

FIG. 1. Pattern of gravity-capillary waves at $U/c_{\min} = 1.1$.

that the wavelength for Set 1 (predominantly capillary waves) is very much smaller than that for Set 2 (predominantly gravity waves).

From Table 1, one sees that for transverse waves of Set 2 the k-values are of order 1. These are predominantly gravity waves. For divergent waves of Set 2, k increases from k_{c1} to k_{c2}, and k_{c2} may be considerably greater than k_{c1} if $U/c_{\min}$ is greater than 4. Hence the divergent waves of Set 2, which have their counterpart in Kelvin waves (for which only gravity is taken into account), become more and more capillary waves as k_{c2} is approached for $U/c_{\min}$ greater than 4 (which is not a sharp boundary, and is cited here only because it is one value of $U/c_{\min}$ chosen in Table 1 which seems to divide large values of k_{c2}/k_{c1} from modest ones of order 1).

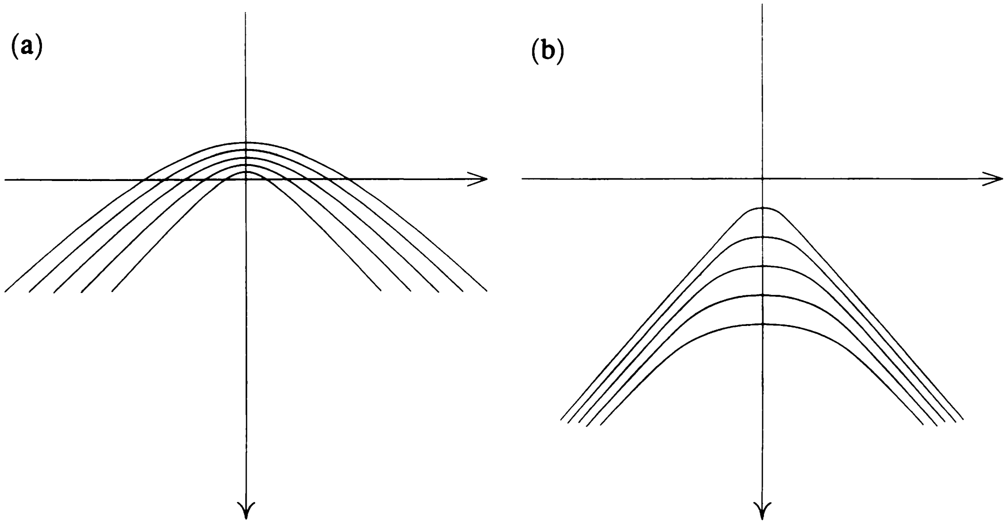

FIG. 2. Pattern of gravity-capillary waves at $U/c_{\min} = 1.5$.
(a) Set 1, (b) Sets 2 and 3 merged.

For $U/c_{\min} \geq 4$, the waves of Set 3 are predominantly capillary waves. For smaller values of $U/c_{\min}$ (see, e.g., Figure 2), one can only say that as the phase lines depart more and more from the center (where $y = 0$), the waves they represent become more and more capillary waves.

As noted before, the waves of Set 1 are predominantly capillary waves, and it is emphasized again that as the phase lines approach $y = \pm\infty$, they become increasingly straight lines that make the same angle ϕ with the centerline as the asymptotes of the phase lines of Set 3, as k_3 is approached (or as y approaches $\pm\infty$).

The most important part of the results is that there are capillary waves with wave fronts making a smaller and smaller angle ϕ with the centerline as $U/c_{\min}$ is increased. This has been observed in photographs obtained by remote sensing, and has been a point of keen interest in naval circles.

From Table 1, one sees for the case $U/c_{\min} = 2$ that k_{c2} is not equal to k_{c1}. Therefore the phase lines for Sets 2 and 3 in Fig. 4b should show a loop as in Fig. 5b. The loop is too small to be seen, and Fig. 4b instead shows a discontinuity in slope at a point near where the loop should be. This loop disappears when $U/c_{\min} = 1.8$ (which may be taken as the limiting value of $U/c_{\min}$ below which there is no loop), which may be compared with the tentative value 2 of Lamb ([1], p. 471, footnote 1). When the loop disappears, the slope at the juncture of the transverse waves of Set 2 and the divergent waves of Set 3 should be continuous. Figure 3b (for $U/c_{\min} = 1.8$) shows a slight but detectable discontinuity at that juncture. That discontinuity should not be there, and is a consequence of the finite-difference calculation when Δk, the increment of k, is not small enough for the neighborhood of the juncture.

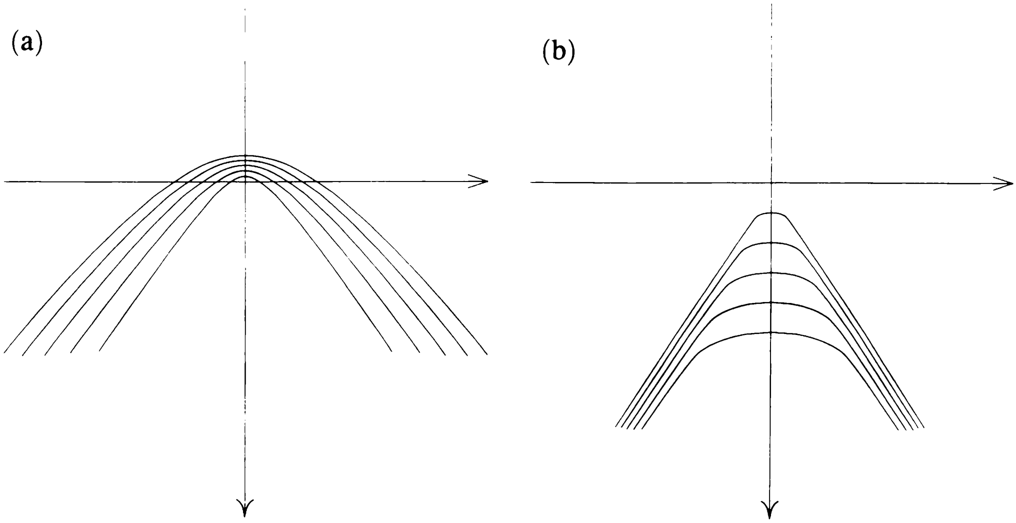

FIG. 3. Pattern of gravity-capillary waves at $U/c_{\min} = 1.8$.
(a) Set 1, (b) Sets 2 and 3 merged.

Finally, we note that the square roots in (2) and (6) involve ambiguities in sign. The square root in (6) can be positive or negative, so that both positive and negative y-values are allowable. The sign of the square root in (2), which has a consequence on the sign of x given by (7), is chosen so that waves of Set 1 start in front of the disturbance before they wrap around it to positive values of x, and that waves of Sets 2 and 3 are *behind* the disturbance, so that x is always positive for waves of these sets.

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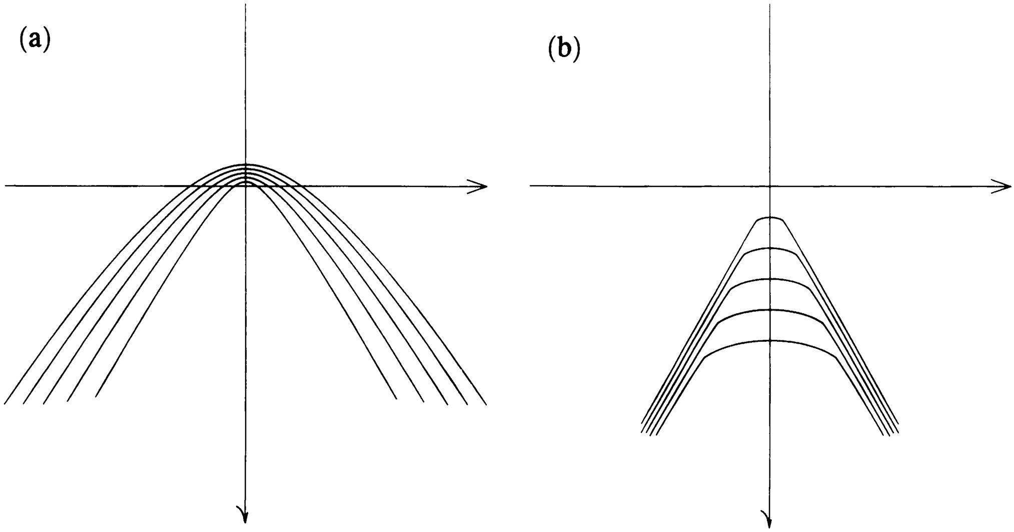

FIG. 4. Pattern of gravity-capillary waves at $U/c_{\min} = 2$.
(a) Set 1, (b) Sets 2 and 3 (loop with cusps invisible).

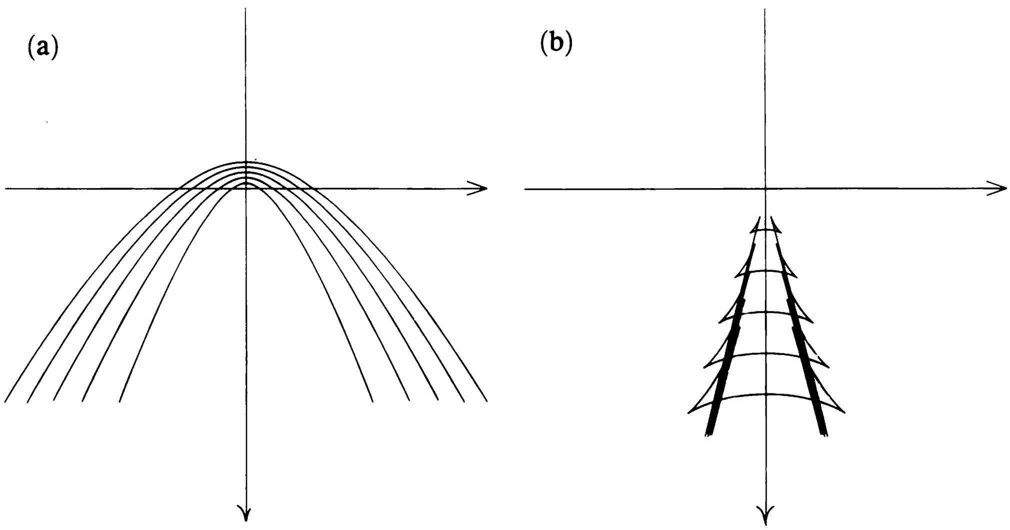

FIG. 5. Pattern of gravity-capillary waves at $U/c_{\min} = 4$.
(a) Set 1, (b) Sets 2 and 3.

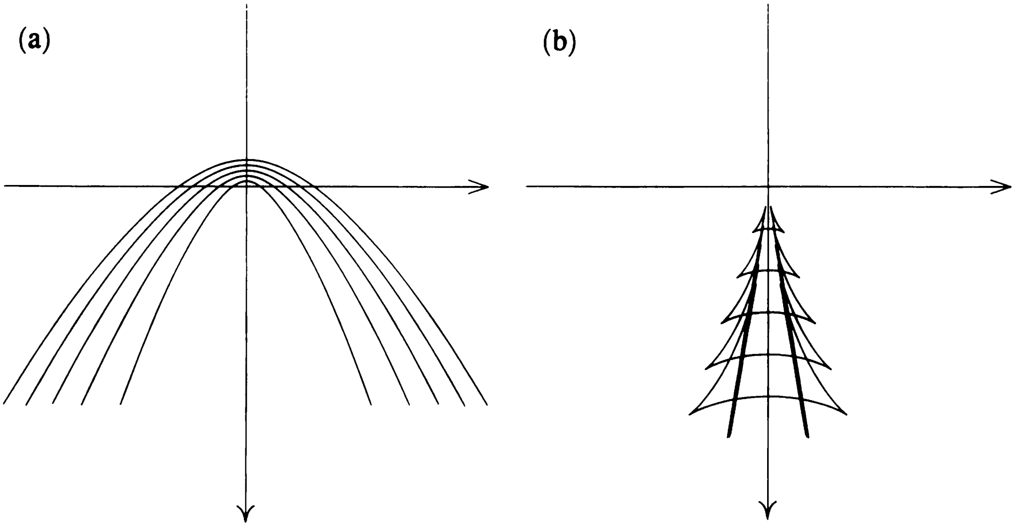

FIG. 6. Pattern of gravity-capillary waves at $U/c_{\min} = 6$.
(a) Set 1, (b) Sets 2 and 3.

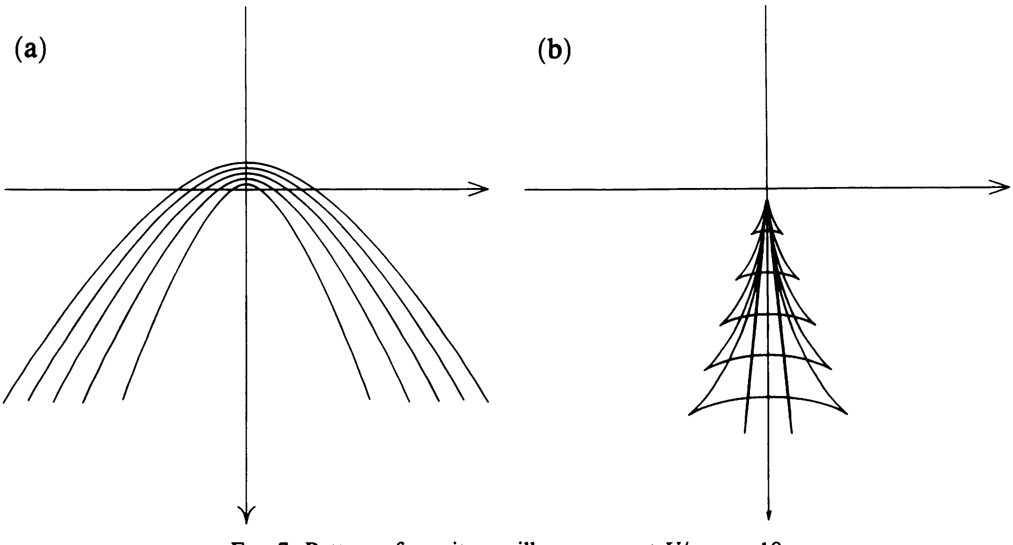

FIG. 7. Pattern of gravity-capillary waves at $U/c_{\min} = 10$.
(a) Set 1, (b) Sets 2 and 3.

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