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PATTERNS OF SHIP WAVES*

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Abstract. Patterns of water waves created by a moving disturbance representing a moving body, floating or submerged, can be found by applying (1) the principle of stationary phase, (2) the principle that the phase lines are normal to the wave-number vector, and (3) the perception that the local phase velocity of the waves must be equal to the component of the velocity of the disturbance normal to the phase line. The three equations thus obtained are solved, and formulas for the phase lines are derived, which depend explicitly on the dispersion equation, and on that equation only. These formulas are applied to deep-water surface waves, surface waves in water of finite depth, internal waves, and capillary waves in thin sheets to obtain the wave patterns sufficiently far from the moving disturbance.

Finally, the patterns of the surface waves in deep water created by a moving body are determined, with the nonuniformity of the mean velocity of the fluid in the wake taken into account. The vorticity in the direction along the phase lines is shown to be small, so that the wave motion can still be assumed irrotational in a first approximation. The wave patterns differ from the Kelvin-wave pattern, as a result of the nonuniformity of fluid velocity in the wake.

1. Introduction. The pattern of gravity waves created by a moving disturbance in deep water was determined by Lord Kelvin (Sir W. Thomson, 1887) fully a century ago, by applying his principle of stationary phase to the well-known Cauchy–Poisson solution (see Lamb, 1945, pp. 429–434) for an instantaneous concentrated force (a concentrated impulse). But Kelvin's application of his own principle of stationary phase did not result in explicit formulas giving the wave pattern created by a moving disturbance, once the dispersion equation expressing the wave velocity in terms of the wave number is known, whatever the kind of wave—gravity wave in deep water or water of finite waves, internal waves, or capillary-gravity waves. These explicit formulas were given by Yih (1985). In this paper Yih's formulas will be presented and applied to gravity waves created by a moving disturbance, which can be regarded as representing a ship, floating or submerged. Results for gravity waves in deep water or water of finite depth and for internal waves will be presented graphically. Finally, the effects of nonuniformity of fluid velocity in the wake of the ship will be

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considered, and gravity-wave patterns will be shown, with the principal effect of this nonuniformity taken into account.

2. Formulas for phase lines in a ship-wave pattern. For simplicity, we can regard the moving disturbance to be a moving pressure distribution on the free surface, although this particularity does not affect the establishment of the formulas for determining the phase lines. This fact indicates that whatever the details of the disturbance may be, the wave pattern obtained will be the same, if the region under consideration is sufficiently far from the disturbance.

Let U be the speed of the disturbance, moving horizontally to the left. The direction opposite to the velocity of the disturbance is taken to be the direction of increasing x. The y-axis is normal to this direction but is also horizontal. The z-axis is directed vertically upward. The wave-number components in the directions of increasing x and y will be denoted by ξ and η respectively, and

$$k^2 = \xi^2 + \eta^2. \tag{1}$$

For our purpose it is not necessary to give the formulas for the velocity distribution in the wave motion (or the velocity potential when it exists), or for the displacement of the free surface or interface. Such formulas are given for Kelvin's waves in Whitham (1972, p. 448, Eq. 13.56), for surface waves in water of finite depth and for gravity-capillary waves in Havelock (1908), and for internal waves in two superposed fluid layers by Hudimac (1961), Carrier and Baski (1963), and Yih (1985, Eqs. 30 and 31), among others. What is important for our purpose is that in these formulas there is always the exponential factor

$$\exp i(\xi x + \eta y) \tag{2}$$

in a double integral with respect to ξ and η.

The application of Kelvin's principle of stationary phase at any point (x, y) requires,[1] because of the factor (2),

$$\frac{y}{x} = -\frac{d\xi}{d\eta}. \tag{3}$$

The normal to any curve of constant phase has the slope η/ξ. The slope of the tangent to any curve of constant phase must then have the value $-\xi/\eta$, so that for that curve

$$\frac{dy}{dx} = -\frac{\xi}{\eta}. \tag{4}$$

The requirement that the wave velocity at any point must be equal to the component of the velocity of the disturbance normal to the wave front (or the phase line) is expressed by

$$\frac{U\xi}{k} = c(k), \tag{5}$$

[1]The Fourier integration is in the (ξ, η)-plane. The main contribution to the integral comes from the neighborhood of a ξ-η curve where a factor in the denominator vanishes. This vanishing is represented by (5). The integration in this neighborhood involves essentially an integration across this curve (giving what amounts to residues) and an integration along that curve subsequently. It is to this latter integration that the method of stationary phase applies. After this application ξ, η, and k are considered (slowly varying) functions of x and y. This understanding should be kept in mind from (3) onward.

where $c(k)$ denotes the wave velocity, dependent on k, the geometry (for instance, depth of water), and the physical parameters relevant to the problem, such as the gravitational acceleration g and the surface tension T. We are concerned in this paper with gravity waves mainly. We shall use U as the velocity scale, and a finite depth d as the length scale. If such a depth is not available we shall use U^2/g as the length scale. Then ξ, η, k, x, and y are dimensionless. The dimensionless form of (5) is now

$$\xi = F(k), \tag{6}$$

where $F(k) = kc(k)$, and the function $F(k)$ may depend on other parameters, such as the Froude number, as well as k, because $c(k)$ does.

Since (6) is an equation between ξ and k, it is convenient to write (3) and (4) in terms of ξ and k explicitly. A brief calculation gives, upon use of (1),

$$\frac{y}{x} = -\frac{1}{d\eta/d\xi} = \frac{(k^2 - \xi^2)^{1/2}}{k(dk/d\xi) - \xi}, \tag{7}$$

$$\frac{dy}{dx} = -\frac{\xi}{(k^2 - \xi^2)^{1/2}}. \tag{8}$$

Finally, upon using (6), we have

$$\frac{y}{x} = -\frac{F'(k^2 - F^2)^{1/2}}{k - FF'}, \tag{9}$$

$$\frac{dy}{dx} = -\frac{F}{(k^2 - F^2)^{1/2}}. \tag{10}$$

We can write

$$y = -f(k)F'(k^2 - F^2)^{1/2}, \tag{11}$$

$$x = f(k)(k - FF'), \tag{12}$$

which satisfy (9), and endeavor to determine $f(k)$. From (11) and (12) we obtain

$$dy = -\frac{1}{(k^2 - F^2)^{1/2}}[fF'(k - FF') + (fF')'(k^2 - F^2)], \tag{13}$$

$$dx = f(k - FF')' + f'(k - FF'). \tag{14}$$

These, together with (10), give

$$fF(k - FF')' + f'F(k - FF') = fF'(k - FF') + (fF')'(k^2 - F^2). \tag{15}$$

It is satisfying that this equation can always be explicitly integrated, for it can be rewritten as

$$fF(k - FF')' + (fF)'(k - FF') = 2fF'(k - FF') + (fF')'(k^2 - F^2), \tag{16}$$

and this can be immediately integrated to (a = constant of integration)

$$fF(k - FF') = fF'(k^2 - F^2) + a, \tag{17}$$

or

$$f = \frac{a}{k(F - kF')}. \tag{18}$$

Thus,

$$y = -\frac{aF'}{k(F - kF')}(k^2 - F^2)^{1/2}, \tag{19}$$

$$x = \frac{a(k - FF')}{k(F - kF')}. \tag{20}$$

These are the parametric equations for the curves of constant phase, when the eikonal equation is in the form of (6), *and it is remarkable that they can be explicitly given in terms of* $F(k)$. The function $F(k)$ can easily be obtained for any kind of waves and any geometry in a two-dimensional wave motion (or one-dimensional propagation). Once it is given, (19) and (20) can be applied to find the wave pattern directly. Thus they are very useful and convenient.

2.1. *Determination of the critical angle* ϕ_c. The critical angle ϕ_c is determined by

$$\frac{d^2\eta}{d\xi^2} = 0, \qquad \text{or} \qquad \frac{d}{d\xi}\frac{k\,dk/d\xi - \xi}{(k^2 - \xi^2)^{1/2}} = 0. \tag{21}$$

When (6) is the eikonal equation, this becomes

$$\frac{d}{dk}\left[\frac{k - FF'}{F'(k^2 - F^2)^{1/2}}\right] = 0. \tag{22}$$

Given F, this can be solved for k. With k known and equal to k_c (say), ξ is known from (6), η is known, and x and y are known, so that

$$\phi_c = \tan^{-1}\left.\frac{y}{x}\right|_{k_c}. \tag{23}$$

If there is more than one root of (22), take the greatest of the values of ϕ corresponding to these roots to be ϕ_c. This determines the vertex angle ($2\phi_c$) of the wedge in which waves can be found. Equation (22), too, is an important result. One expects to find waves only in some wedge where

$$-\phi_c < \phi < \phi_c.$$

3. Applications.

3.1. *Gravity waves in deep water, or Kelvin waves.* For Kelvin waves the dimensionless eikonal equation is (the length scale for this case is U^2/g)

$$\xi^2 = k. \tag{24}$$

Then

$$F(k) = k^{1/2} \quad \text{and} \quad F'(k) = \frac{1}{2}k^{-1/2}, \tag{25}$$

$$y = -\frac{a}{k^2}(k^2 - k)^{1/2}, \tag{26}$$

$$x = \frac{a(2k - 1)}{k^{3/2}}. \tag{27}$$

Following Lamb (1945, p. 433), we define θ as the angle of inclination of the normal to any curve of constant phase, so that, along any curve of constant phase

$$\frac{dy}{dx} = -\frac{\xi}{\eta} = -\cot\theta. \tag{28}$$

Because of (24),

$$k^2 - k = k^2 - \xi^2 = \eta^2,$$

so that

$$y = -\frac{a\eta}{k^2} = -\frac{a}{\xi^3}\frac{\eta}{\xi}. \tag{29}$$

Now (28) can be written as

$$\pm\cot\theta = \frac{\xi}{(k^2 - \xi^2)^{1/2}} = \frac{\xi}{(\xi^4 - \xi^2)^{1/2}} = \frac{1}{(\xi^2 - 1)^{1/2}} \tag{30}$$

so that

$$\xi = \sec\theta, \tag{31}$$

and (29) becomes

$$y = -a\cos^3\theta\tan\theta = -a\sin\theta\cos^2\theta. \tag{32}$$

Similarly, (27) becomes

$$x = \frac{a(2k-1)}{k^{3/2}} = \frac{a(2\xi^2 - 1)}{\xi^3} = a\cos\theta(2 - \cos^2\theta),$$

or

$$x = a\cos\theta(1 + \sin^2\theta). \tag{33}$$

Equations (32) and (33) are exactly the equations (Lamb, 1945, p. 434) for the Kelvin curves of constant phase for surface waves. The wave pattern is shown in Fig. 1, upon taking a equal to 1, 2, 3, etc. Since on the centerline $k = 1$, because the length scale is U^2/g and on the centerline $U = c$, the increment in a, indicating a wavelength on the centerline, corresponds to $2\pi U^2/g$ when converted to dimensional length. (Recall wave number $= 2\pi$/wavelength.) The cusps occur at $\xi = (3/2)^{1/2}$, giving a ϕ_c of 19°28′. The flow near the cusps requires better resolution. This was provided by Ursell (1960).

3.2. *Gravity waves in water of finite depth.* Let the depth of the water be d. Then, as is well known, in *dimensional* terms the phase velocity $c(k)$ is given by

$$c^2 = \frac{g}{k}\tanh kd.$$

In dimensionless terms, with d as the length scale and U the velocity scale, this becomes

$$c^2 = N_F^{-2}k^{-1}\tanh k, \tag{34}$$

where N_F is the Froude number defined by

$$N_F^2 = \frac{U^2}{gd}. \tag{35}$$

Then (6) takes the form

$$\xi = N_F^{-1}(k\tanh k)^{1/2}, \tag{36}$$

the right-hand side of which is $F(k)$. Using this $F(k)$ in (19) and (20), we obtain the wave patterns shown in Figs. 2 for

$$N_F = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.5,\ 2.0,\ \text{and } 5.0,$$

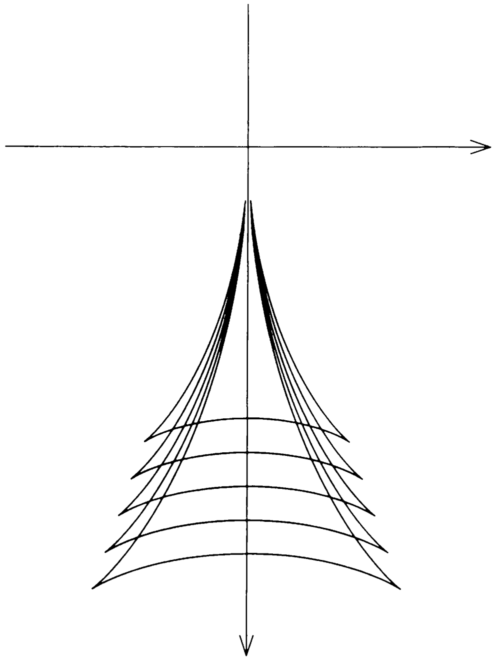

FIG. 1. Kelvin-wave pattern.

respectively. On the centerline, $\xi = k = k_0$, and (36) gives

$$N_F^2 = \frac{1}{k_0} \tanh k_0. \tag{37}$$

The wavelength of transverse waves is $2\pi d/k_0$ on the centerline. Given the Froude number N_F and the depth d, k_0 is determined from (37) and the wavelength on the centerline is known. The patterns in the figures should be read with this in mind. The fact that the patterns have been obtained by assigning integral values to a in (19) and (20) is of little importance since the scale used in plotting the figures is arbitrary.

The most important feature of Fig. 2 is that only when the Froude number N_F is less than 1 are there transverse waves. This is because the wave velocity c is bounded by $(gd)^{1/2}$, so that when N_F exceeds 1 no waves, however long, can be stationary at the centerline, so that no transverse waves can exist. When N_F exceeds 1, the phase lines at large distances from the disturbance appear to approach asymptotes.

The k_c (k at the cusps), defined by (23) when $N_F < 1$, and the ϕ_c giving the half angle of the wedge in which waves exist, depend on the value of N_F. We note that when transverse waves exist, k increases from k_0 on the centerline to k_c at the cusp, and then increases monotonically along the phase line of a divergent wave. When transverse waves do not exist, k increases along such a line as $|y|$ decreases; i.e., as the centerline is approached. The same is true for internal waves.

3.3. *Internal waves.* Let us consider gravity waves in two superposed fluids. The upper fluid has density ρ and depth h, and the lower fluid has density $\rho'(> \rho)$ and infinite depth. Let the velocity potential perturbations (from the mean flow of

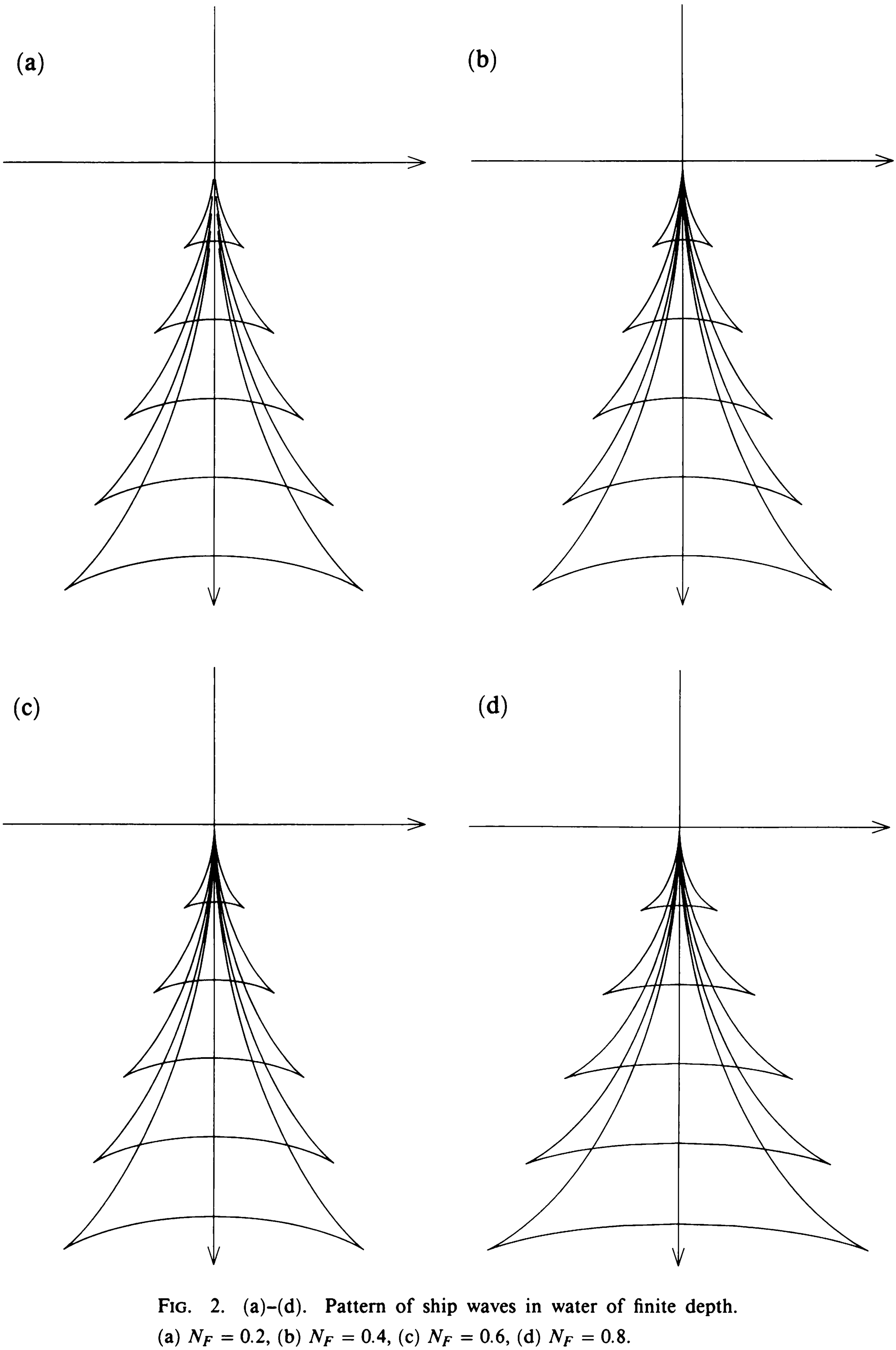

FIG. 2. (a)–(d). Pattern of ship waves in water of finite depth.
(a) $N_F = 0.2$, (b) $N_F = 0.4$, (c) $N_F = 0.6$, (d) $N_F = 0.8$.

velocity c) for the two layers be ϕ and ϕ', respectively. The displacement of the free surface is denoted by ζ and that of the interface by ζ'. All quantities will be dimensional until otherwise stated. The phase velocity c will be determined from a calculation for stationary waves, that is, for a coordinate system moving with the waves to the left. Then the kinematic condition at the free surface is (the origin of z being at the interface)

$$c\zeta_x = \phi_z \quad \text{at } z = h, \tag{38}$$

and the dynamic condition there is the Bernoulli equation

$$g\zeta + c\phi_x = 0 \quad \text{at } z = h. \tag{39}$$

Combination of (38) and (39) gives

$$s\phi_{xx} = -\phi_z \quad \text{at } z = h \tag{40}$$

for the free surface, where

$$s = \frac{c^2}{g}. \tag{41}$$

At the interface, the kinematic conditions are

$$c\zeta'_x = \phi_z \quad \text{and} \quad c\zeta'_x = \phi'_z \quad \text{at } z = 0, \tag{42}$$

and the dynamic condition is

$$\rho g\zeta' + \rho c\phi_x = \rho' g\zeta' + \rho' c\phi'_x \quad \text{at } z = 0. \tag{43}$$

Defining

$$\beta = \frac{\rho' - \rho}{\rho} \tag{44}$$

and substituting (42) into (43), we obtain

$$\frac{\beta}{s}\phi_z = -(\beta + 1)\phi'_{xx} + \phi_{xx} \quad \text{at } z = 0. \tag{45}$$

Equations (42) can be combined into

$$\phi_z = \phi'_z \quad \text{at } z = 0. \tag{46}$$

One last condition is

$$\phi' = 0 \quad \text{at } z = -\infty. \tag{47}$$

Taking

$$\phi = (Ae^{kz} + Be^{-kz})\cos kx, \qquad \phi' = Ce^{kz}\cos kx, \tag{48}$$

which satisfies (47), since k is assumed positive, we use (40), (45), and (46) to eliminate the constants A, B, and C, and obtain from a straightforward calculation

$$(ks - 1)[ks(2 + \beta + \beta e^{-2kh}) - \beta(1 - e^{-2kh})] = 0. \tag{49}$$

The root

$$s = \frac{c^2}{g} = \frac{1}{k} \tag{50}$$

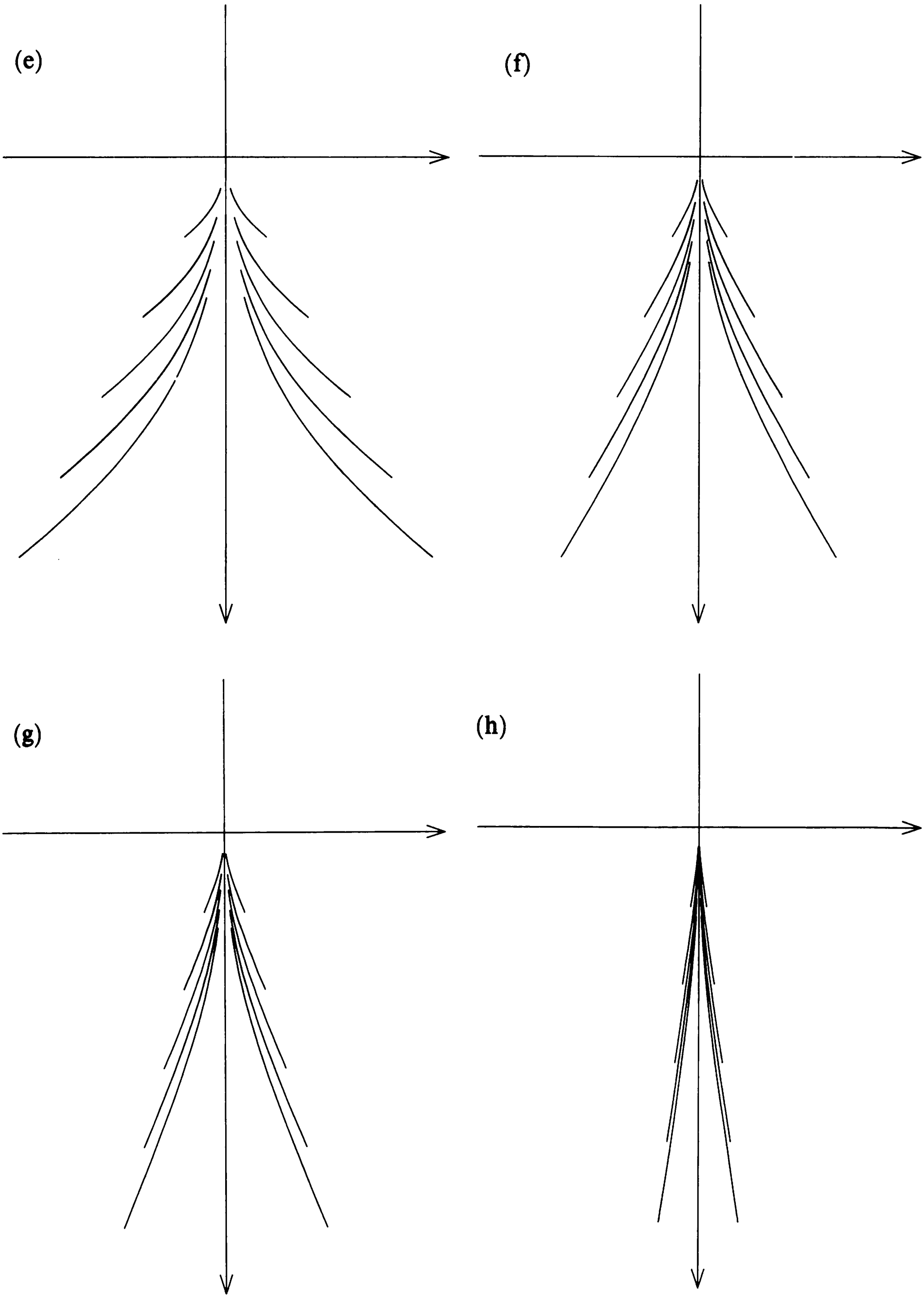

FIG. 2. (e)–(h). Pattern of ship waves in water of finite depth. (e) $N_F = 1$, (f) $N_F = 1.5$, (g) $N_F = 2$, (h) $N_F = 5$.

corresponds to irrotational wave motion of the entire fluid, with no vortex sheet even at the interface, as is well known (Yih 1960, and 1980, pp. 60–62). This wave motion has the Kelvin-wave pattern presented in subsection 3.2. The other root of (49) is

$$c^2 = \frac{\beta g}{k} \frac{e^{2kh} - 1}{(2+\beta)e^{2kh} + \beta}, \tag{51}$$

and corresponds to predominantly internal waves. Substituting (51) into (5), using h as the length scale, we obtain the dimensionless equation

$$\xi = N_F^{-1} \left(\frac{\beta \gamma k}{\alpha} \right)^{1/2}, \tag{52}$$

$$\alpha(k) = (2+\beta)e^{2k} + \beta, \tag{53}$$

$$\gamma(k) = e^{2k} - 1, \tag{54}$$

and N_F is the Froude number now defined by

$$N_F^2 = \frac{U^2}{gh}.$$

We have done the calculations for the internal-wave patterns for $\beta = 0.04$. The patterns are shown in Fig. 3 for

$$N_F = 0.01,\ 0.05,\ 0.1,\ 0.15,\ 0.2,\ 0.5,\ 1, \text{ and } 2,$$

respectively. Again transverse waves exist only if N_F is sufficiently small. The critical N_F can be calculated from (52) upon putting ξ equal to k, letting k approach zero (to get the longest wavelength possible), and taking the limit. Doing so, we obtain the critical Froude number

$$(N_F)_c = \frac{\beta}{1+\beta}. \tag{55}$$

When $N_F > (N_F)_c$, no transverse waves are possible. For $\beta = 0.04$, $(N_F)_c$ is approximately 0.2. That is why in Fig. 3(e) no transverse waves appear.

As for surface waves in water of finite depth, the angle ϕ_c which the line of cusps makes with the centerline, defined by (23), depends on N_F. This dependence is evident upon examination of Figs. 3(c) and 3(d). From Fig. 3(e) onward to Fig. 3(h), the wave region narrows as the phase lines, which appear to approach asymptotes, make smaller and smaller angles with the centerline.

3.4. *Capillary waves on a thin sheet.* For a fluid sheet of thickness $2h$, there are two modes of capillary waves. For the one mode, the sheet deforms as a whole antisymmetrically, with hardly any change in thickness. In this mode the waves are nondispersive. For the other mode, the sheet deforms symmetrically, and the dimensional dispersive equation is, upon neglect of gravity effects and on the assumption that the wavelength anywhere is much greater than h,

$$c^2 = \frac{Th}{\rho} k^2,$$

where T is the surface tension, ρ is density of the fluid, and c and k are the dimensional phase velocity and wave number, respectively. (See Whitham, 1972, p.

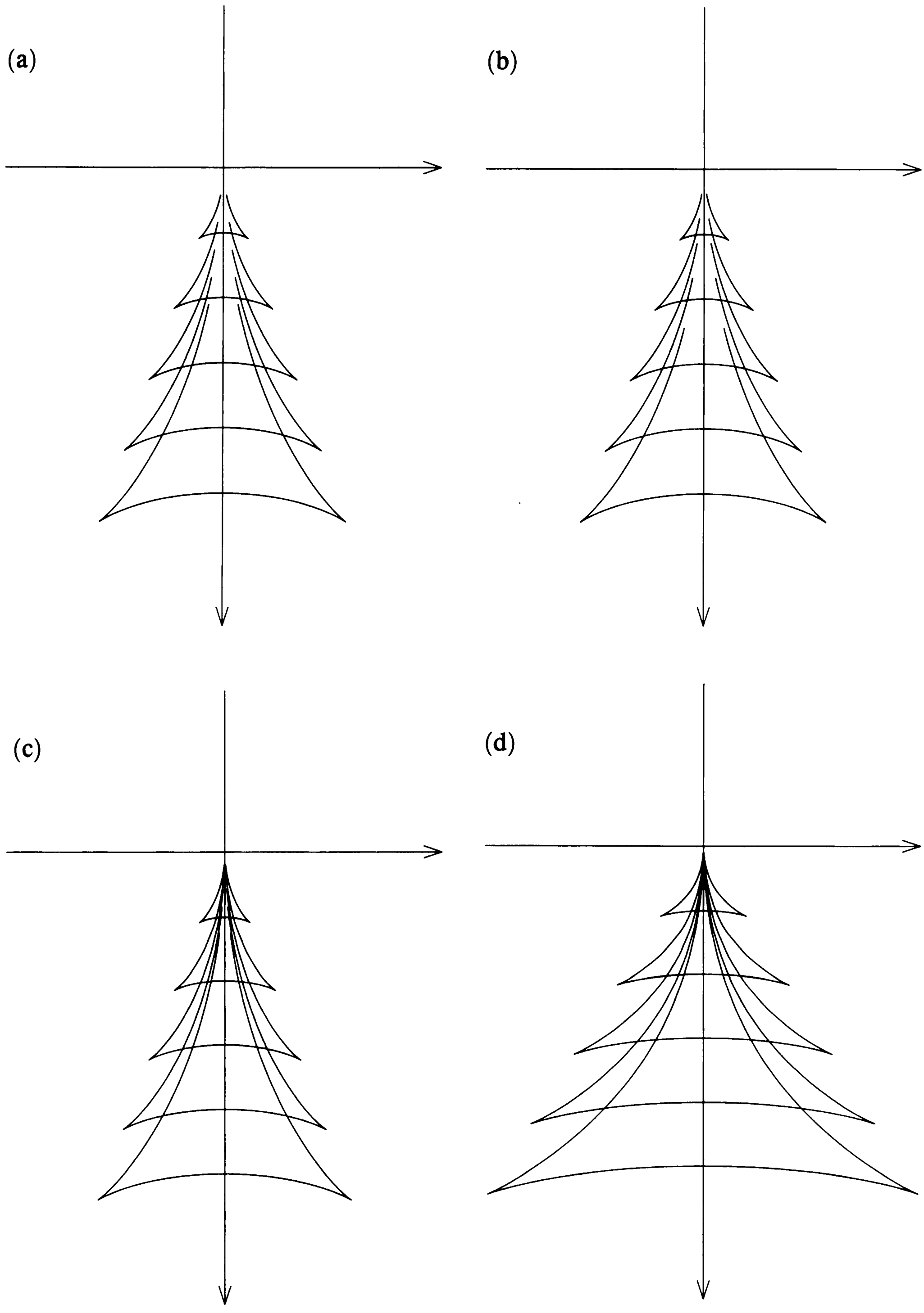

FIG. 3. (a)–(d). Pattern of internal waves created by a moving body.
(a) $N_F = 0.01$, (b) $N_F = 0.05$, (c) $N_F = 0.1$, (d) $N_F = 0.15$.

405.) We shall use h as the length scale and $(\rho h^3/T)^{1/2}$ as the time scale. Then the dimensionless dispersion equation becomes

$$c = k, \qquad \text{or} \quad F(k) = ck = k^2.$$

Putting this into (11) and (12), we have

$$y = -\frac{2a}{k}(1-k^2)^{1/2},$$
$$x = -\frac{a(1-2k^2)}{k^2},$$

in which the square root may be positive or negative, while k is restricted to values less than 1.

It is a very simple matter to show that the x and y given above satisfy a parabolic relation

$$\frac{x}{a} = 1 - \left(\frac{y}{a}\right)^2. \tag{56}$$

The source of the disturbance is at $x = 0 = y$, at which a must be taken to be zero. Along the centerline ($y = 0$) *in front* of the disturbance, the dimensionless k is 1. That is to say, the dimensionless wavelength λ (in units of h) is 2π. Hence to show the wave pattern one must take successively

$$a = 2\pi,\ 4\pi,\ 6\pi,\ \text{etc.}$$

Whitham (1972, pp. 415–416) gave a more indirect derivation of the pattern of capillary waves in thin sheets, and obtained "roughly parabolic crests" in his Fig. 12.7. From our derivation here it is clear that the crests are not merely roughly parabolic, but exactly parabolic.

We note that Whitham's Fig. 12.7 (Whitham 1972, p. 416) is for the disturbance moving to the left. Our formula (56) is for the disturbance moving to the right. Otherwise our pattern for capillary waves is the same as his.

4. Gravity waves in the wake of a moving body. The wake behind a moving body is an important effect of viscosity, for it is ultimately related to the boundary layer on the body. Again let U^2/g be the length scale and use x and r as dimensionless coordinates, r being the radial distance from the x-axis. The velocity distribution in the wake can take a variety of forms, depending on the shape of the body and the Reynolds number, as is well known. Using U again as the velocity scale, the dimensionless velocity can be represented, without serious error, by a class of profiles as follows (α and β are not the same as in subsection 3.3):

$$u = 1 - \alpha x^{-2/3}\exp(-\beta r^2 x^{-2/3}). \tag{57}$$

The momentum flux across any section of constant x below the free surface is, for the mean flow given by (57),

$$\frac{\pi\alpha}{2\beta}\rho U^6 g^{-2}, \tag{58}$$

and this is equal to the viscous drag force $C_D A\rho U^2$, if A is a cross section of the body and C_D the coefficient of drag. This equality is based on the neglect of the

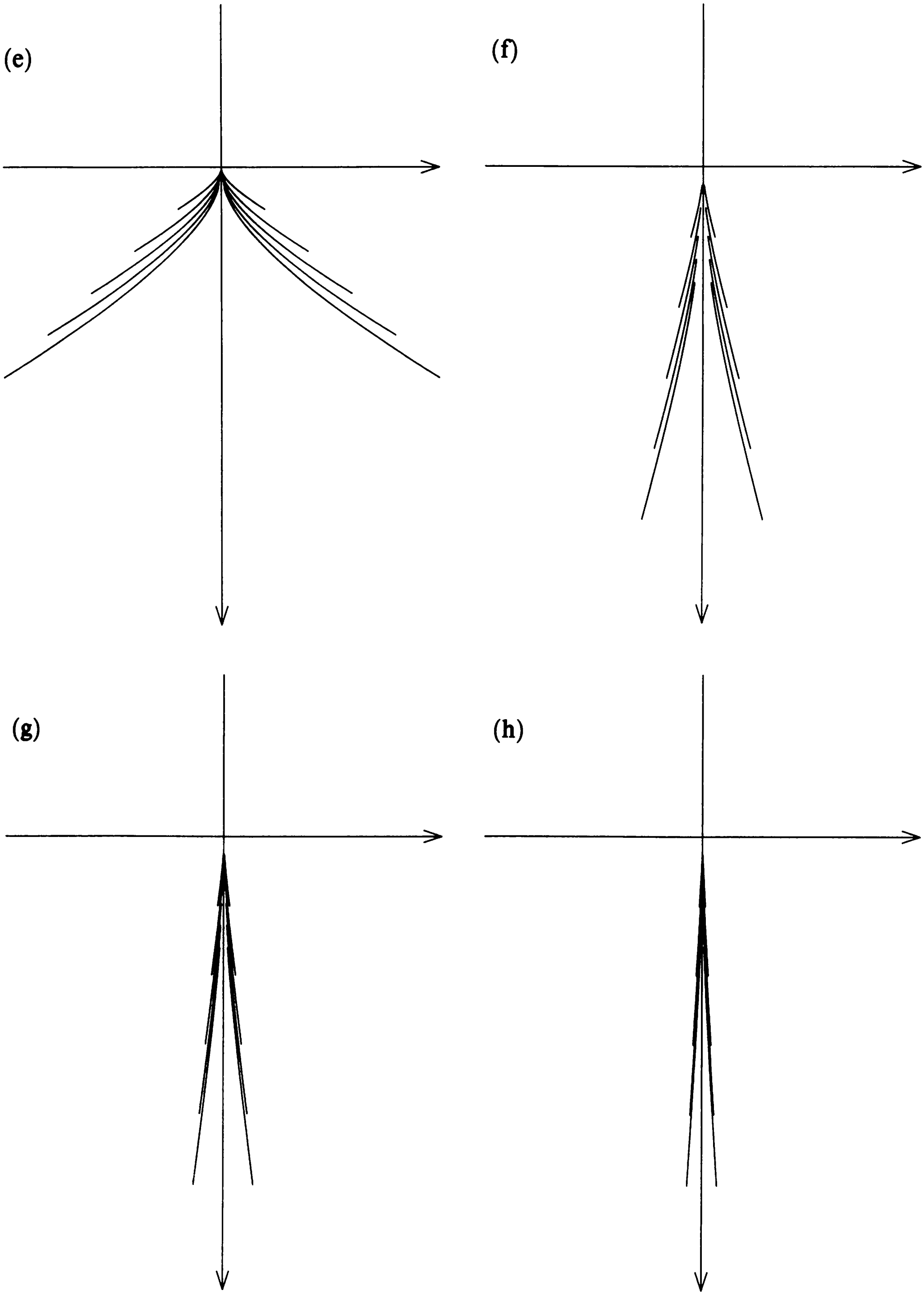

FIG. 3. (e)–(h). Pattern of internal waves created by a moving body.
(e) $N_F = 0.2$, (f) $N_F = 0.5$, (g) $N_F = 1$, (h) $N_F = 2$.

longitudinal stress across any section of constant x, and gives

$$\frac{\alpha}{\beta} = \frac{2}{\pi} C_D F_A^{-4}, \tag{59}$$

where F_A is a Froude number based on a linear dimension $A^{1/2}$, defined by

$$F_A^4 = \frac{U^4}{g^2 A}. \tag{60}$$

In our calculation we have taken

$$C_D = 0.05, \tag{61}$$

$$\frac{\alpha}{\beta} = \frac{1}{20}. \tag{62}$$

This corresponds to

$$F_A = \left(\frac{2}{\pi}\right)^{1/4} = 0.89, \tag{63}$$

not an unreasonable number. Other values of C_D and F_A can be taken, resulting in other values of α/β.

On the free surface (57) becomes

$$u = 1 - \alpha x^{-2/3} \exp(-\beta y^2 x^{-2/3}). \tag{64}$$

We shall assume this velocity to be prevailing in the fluid, and furthermore shall assume the wave motion to be irrotational. This requires some justification and explanation. Assuming (64) throughout the fluid amounts to assuming that the wave motion does not penetrate the fluid significantly to distances below the free surface where u is very different from that given by (64). This is true only if x is large. As to the irrotationality of the wave motion, one justifies it on two grounds.

(i) The length scale of the variation of u for large values of x is large, and therefore much greater than the wavelengths in the wave region.

(ii) All the vorticity lines for the flow (57) are circles in planes normal to the x-axis. Near the free surface they are nearly vertical, whereas the local wave motion, assumed irrotational in the plane containing the wave-number vector and a vertical line, can become rotational only if the local wave motion bends the vertical vorticity lines of the mean flow in the direction of the phase lines. But any bending of the vorticity lines by the local wave motion would be in the direction normal to the phase lines. Thus the irrotationality of the local wave motion is hardly affected.

We can use equations (3) and (4) again, but must modify (6) to

$$u\xi = F(k). \tag{65}$$

The calculation is one of step-by-step computation, as k is increased from k_0 (now not constant) on the centerline toward k_c (at the cusp) and beyond. The k_0 for the transverse waves on the centerline is determined from (64).

We have performed the calculations for

$$\alpha = 1/8, 1/4, 1/2, 3/4,$$
$$\beta = 2.5, 5, 10, 15.$$

The results are shown in Fig. 4, in which (a) is for $(\alpha, \beta) = (1/8, 2.5)$, etc. In starting the lines of the same phase, we start with $x = 8$ on the centerline, calculate the wavelength of the transverse wave there, and mark off the next point on the centerline of the same phase, and use that point to start the calculation for the next line of the same phase, and so on. From Fig. 4 it can be seen that near the centerline the transverse waves bent back toward positive x more pronouncedly than in the Kelvin-wave pattern, but become straight sooner. The curvature of the transverse waves at the centerline is larger when there is a wake.

For the Kelvin-wave pattern,

$$\xi = \xi_c = (3/2)^{1/2}$$

at the cusp, and $\phi_c = 19°28'$. In our case, since u in (65) is variable, it is not obvious that ϕ_c has the same value. However, our calculations seem to give the same value. An explanation of this on analytical grounds goes as follows. It can easily be verified that for *any fixed* value of u the maximum value of $d\eta/d\xi$ occurs at $\xi = (3/2)^{1/2}$, and gives $\phi_c = 19°28'$. Now fix the x and y in u, and perform the calculation for the phase lines as if u were constant. A cusp is encountered at some point (x_c, y_c). If these are not the same as the fixed x and y, the cusp has no real significance. But if they happen to be the same, then, since the calculation in a neighborhood of that point is now valid, it must be a cusp in the wave pattern. This can be seen in the following way. Start from any cusp point (x_c, y_c), calculate $u(x_c, y_c)$, fix u at this value, then use the equations of Section 2 to trace out the two branches of the phase line originating from the cusp. Let us call this line A. Then starting from the same cusp, use the actual u in the step-by-step numerical computation in this section to trace out the two branches of the phase line originating from the cusp. Let us call this line B. Lines A and B are not the same, of course, but they have the same cusp and are nearly coincident near the cusp.

5. Range of k values for the figures. For the benefit of those who wish to reproduce the wave patterns given in the figures of this paper, we supply the ranges of the values of the dimensionless k for each of the figures, except Figures 4(a)–4(d), in Table 1.

TABLE 1

Values for $k_{\min}$ and k at cusp for the figures.

Figure	1	2a	2b	2c	2d	2e-2h
$k_{\min}$	1	25	6.24995	2.7554	1.37458	1.5
k at cusp	$(3/2)^{1/2}$	37.432	9.39995	4.0574	1.74508	

Figure	3a	3b	3c	3d	3e-3h
$k_{\min}$	196.0784	7.84313	1.91763	0.6113	0.3
k at cusp	294.3604	11.74913	2.72963	0.7933	

For Figures 4, the minimum k for each of the phase lines is for the transverse wave at the centerline, and this minimum k is determined by the equality of the local wave velocity of the transverse waves at the centerline and the local speed of the fluid in the wake. Because of the nonhomogeneity of fluid velocity in the wake (even apart from

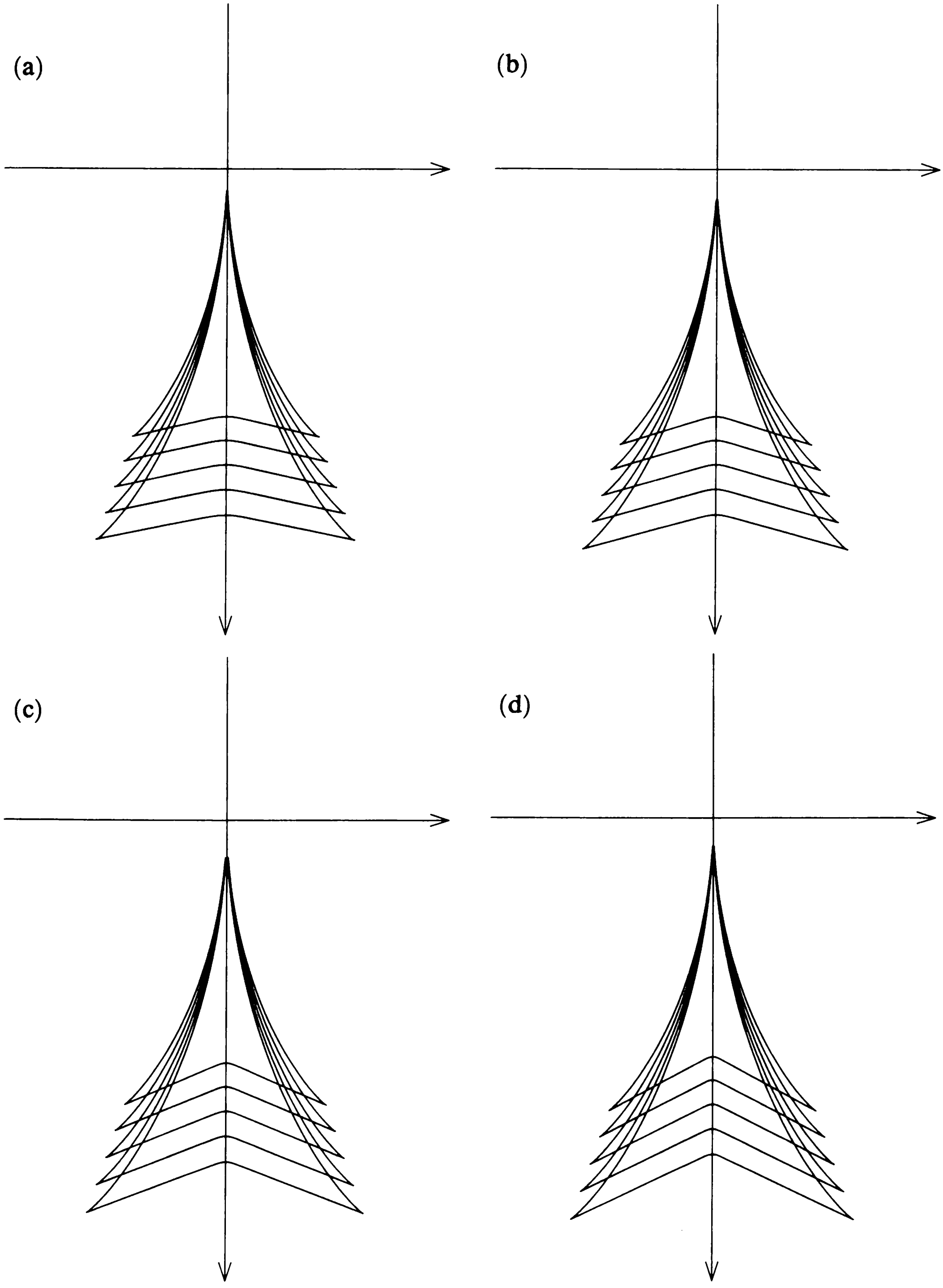

FIG. 4. (a)–(d). Pattern of gravity waves in the wake of a ship. (a) $\alpha = 1/8, \beta = 2.5$, (b) $\alpha = 1/4, \beta = 5$, (c) $\alpha = 1/2, \beta = 10$, (d) $\alpha = 3/4, \beta = 15$. The phase difference between consecutive phase lines is approximately 1 radian.

wave effects), the spacing of crests at the centerline cannot be determined simply, and can only be approximated by computing the locations of many intermediate phase lines. For this reason only phase lines, which are not necessarily crests, are shown in Figure 4, to give a general idea of the wave pattern. The maximum k for all the figures, including Figure 4, is infinity. We have simply stopped at a large enough k when the position of the disturbance has become too near to draw phase lines distinctly. We believe this description will be sufficient to guide anyone wishing to reproduce our figures.

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