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FINITE GROUPS OF GRAVITY WAVES*

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Summary. Groups of gravity waves of finite length created in deep, originally quiescent water by an oscillating or moving surface pressure are constructed by superposition of the Cauchy–Poisson solution. This construction gives substance and reassurance to the concept of "wave packets" progressing with group velocity associated with the individual waves in the packets, a concept so important to water-wave research. The effects of viscosity are taken into account, thereby not only justifying the extensively used but completely artificial damping factor initiated by Lamb (1916, see Lamb 1945, p. 413), but also showing the hitherto largely unexplored spatial damping of waves.

1. Introduction. The phenomenon of an isolated water-wave group and its velocity of advance were observed by Russel (1844, see Lamb 1945, p. 380). The first derivation of the group velocity of dispersive waves seems to have been given by Stokes (1876, see Lamb 1945, p. 381), although the significance of the group velocity as the velocity of propagation of the wave number was already at least implicit in the solution of Cauchy (1815, see Lamb 1945, p. 384, and p. 17 for reference to original work) and Poisson (1816, see Lamb 1945, p. 384) for waves created by an initial concentrated disturbance, long before Russel's observations and Stokes' work.

In Stokes' derivation, two wave trains of slightly different wave numbers and correspondingly slightly different frequencies are superposed, and the result is a train of waves of the mean wave number and the mean frequency bounded by an envelope with an amplitude sinusoidally and slowly varying with time and distance. The velocity of the envelope is the group velocity, which for gravity waves is less than the phase velocity of the individual waves contained in the envelope.

Stokes' derivation has the great merit of simplicity. Although the requirement of two wave trains of slightly different wave numbers and frequencies seems artificial at first sight, waves of neighboring wave numbers and frequencies do arise naturally in many problems to which the Fourier analysis can be applied, and these are essential for the formation of wave groups. Wave trains with two discrete wave numbers are merely an extreme idealization. However, these wave trains necessarily entail

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infinitely many wave groups, and this fact renders Stokes' construction inadequate for explaining Russel's observations or for supporting the many and frequent statements or implications in contemporary literature concerning isolated wave groups. It is thus very desirable to construct some examples of dispersive-wave groups of finite length.

As already mentioned in the foregoing, an interpretation of the Cauchy–Poisson solution, as given in pp. 384–398 of Lamb's book (1945), shows the significance of the group velocity as the velocity with which the wave number propagates—or a packet of waves of that wave number propagates. This strongly suggests that if the forcing at the free surface has a certain frequency, or moves with a certain velocity, a group of waves with that frequency, or moving with a phase velocity equal to that velocity, will be created. The present paper is the outcome of acting on that suggestion.

The deep mystery that when a wave maker oscillates $n(\gg 1)$ times in deep water (for instance) only $n/2$ waves are created can be dispelled satisfactorily only by considering the cancellation of waves, perhaps principally near the front of the group. The mechanics of that cancellation is already contained in the solution given in this paper, but has not been pursued in detail. However, the construction herein of wave groups of finite length, in showing the existence of such groups that propagate with their appropriate group velocities, and in being the *result* of that cancellation, provides an important step toward dispelling that mystery, and gives one reassurance when one talks about isolated wave packets.

In making the solutions determinate, the simple damping factor employed by Lamb (1916, see Lamb 1945, p. 413) will first be used, but will be justified later in this paper on the basis of the Navier–Stokes equations governing the dynamics of viscous fluids. This justification is a second purpose of this paper.

Stokes' construction of infinitely many wave groups and Lamb's artifice of the exponential damping factor have the merit of simplicity, and since they both contain a measure of what is needed, have been successful in explaining things and thus very useful to workers on water waves. The simplicity and the success have long been a blessing, but the very simplicity and success have in time become a curse, since they discourage the expenditure of arduous work to put something better in their place. The time for replacing them has arrived, and this work, motivated by the demand of reason, is a tribute to Professor C. C. Lin on the occasion of his retirement.

2. The Cauchy–Poisson solution. Let x and y be Cartesian coordinates, with x measured in a horizontal direction and y measured vertically upward from the free surface when the fluid (water) is at rest, and let t be the time. Consider irrotational gravity waves created in deep water by a distribution of the velocity potential ϕ applied instantaneously at the free surface at $t = 0$, and let this initial ϕ be denoted by ϕ_0 and given by

$$\phi_0 = aF(x), \tag{1}$$

where $F(x)$ is dimensionless and a has the dimension L^2/T, i.e., the dimension of the velocity potential.

The velocity potential ϕ has to satisfy the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0. \tag{2}$$

At the free surface, where

$$y = \eta(x, t), \tag{3}$$

the kinematic condition is

$$\eta_t = \phi_y, \tag{4}$$

with ϕ_y evaluated at $y = 0$, and the dynamic condition requiring constant pressure is

$$\phi_t + g\eta = 0, \tag{5}$$

where g is the gravitational acceleration. The ϕ_y and ϕ_t in (4) and (5) are evaluated at $y = 0$ in a linear theory. Another boundary condition is

$$\phi \to 0 \quad \text{as } y \to -\infty. \tag{6}$$

Let k denote the wave number and σ the corresponding frequency (defined as 2π divided by the period). Then the solution for ϕ satisfying (1), (2), (4), (5), and (6) is

$$\phi = \frac{a}{\pi} \int_0^\infty \cos \sigma t e^{ky}\, dk \int_{-\infty}^\infty F(\alpha) \cos k(x - \alpha)\, d\alpha, \tag{7}$$

and the corresponding solution for η is

$$\eta = \frac{a}{\pi g} \int_0^\infty \sigma \sin \sigma t\, dk \int_{-\infty}^\infty F(\alpha) \cos k(x - \alpha)\, d\alpha, \tag{8}$$

provided

$$\sigma^2 = gk. \tag{9}$$

Equation (9) is the dispersion equation, which has a different form if surface-tension effects are included, or if the water depth is finite.

The famous Cauchy–Poisson solution is obtained if one lets $F(\alpha)$ in (7) and (8) be a Dirac distribution, i.e., if $F(\alpha)$ is zero everywhere except at $\alpha = 0$, in such a way that

$$\frac{1}{L} \int_{-\varepsilon}^{\varepsilon} F(\alpha)\, d\alpha = 1$$

for any ε however small, if L is the length scale. We can work with (7) and (8) without using the Cauchy–Poisson solution. But we note in passing that the Cauchy–Poisson solution demonstrates beautifully how wave packets of various wave numbers disperse, each packet of a given wave number propagating with the group velocity for that wave number. So this kinematic significance of the group velocity was already implied in the work of Cauchy in 1815 and Poisson in 1816 (see Lamb 1945, p. 17 and p. 384 for dates of the references) long before this significance was reaffirmed and emphasized by others in the second half of the twentieth century.

Before we go on to use (7) and (8) to construct single wave groups, we note that (9) gives two values for σ for each real positive value of k, one positive and one negative. Taking the positive root gives the same ϕ and η as taking the negative root, as can be seen from (7) and (8). If we take both the positive and the negative roots of σ and add the results in each of (7) and (8), we merely get double the values of ϕ_0, ϕ, and η, with no other effects. Consequently we need not consider the negative root, and henceforth consider σ to be positive. Remembering this will remove a lot of ambiguities later.

The physical meaning of (1) has seldom been made clear. Lamb referred to it as an impulse. But since the pressure on the free surface is given by (for $y = 0$)

$$p = -\rho(\phi_t + g\eta), \tag{10}$$

and the right-hand side is zero by virtue of (7) and (8), provided the integrals are convergent, the pressure on the free surface is always zero (or constant if we did not drop the constant in the Bernoulli equation above). So the use of the term impulse is rather confusing. The correct way of thinking is to regard (7) and (8) as valid only for $t \geq 0$, and to assume both ϕ and η to be zero for $t \leq 0$. Then ϕ_t is infinite at $t = 0$, and integration of ϕ_t between $t = -\varepsilon$ and $t = 0$ gives ϕ at $t = 0$ equal to

$$\phi = \frac{a}{\pi}\int_0^\infty e^{ky}\,dk \int_{-\infty}^\infty F(\alpha)\cos k(x-\alpha)\,d\alpha,$$

while integrating η in the same interval gives nothing. Hence the integration of p in the same interval at $y = 0$ gives $-\rho\phi_0$. This is what has been considered the impulse.

3. Wave groups produced by an oscillating pressure distribution. Using (7) and (8) as building blocks, one can obtain by time-wise superposition a more general solution as follows:

$$\phi = \frac{a}{\pi}\int_0^\infty \left[\int_0^t \omega\cos\sigma(t-\tau)\sin\omega\tau e^{-\mu(t-\tau)}\,d\tau \int_{-\infty}^\infty F(\alpha)\cos k(x-\alpha)\,d\alpha\right] e^{ky}\,dk, \tag{11}$$

$$\eta = \frac{a}{\pi g}\int_0^\infty \left[\int_0^t \omega\sigma\sin\sigma(t-\tau)\sin\omega\tau e^{-\mu(t-\tau)}\,d\tau \int_{-\infty}^\infty F(\alpha)\cos k(x-\alpha)\,d\alpha\right] dk. \tag{12}$$

In (11) and (12), we have added the exponential damping factor $\exp[-\mu(t-\tau)]$, where μ is a damping coefficient much smaller than ω, and not the dynamic viscosity, to make the results determinate. This device may be called the device of fading memory, and was used by Lamb (1916, see Lamb 1945, p. 413). The only difference in its usage here is that we consider the damping to start from $t = \tau$, at which an element of ϕ_0 acts. Later in this paper, we shall provide a rational foundation for the use of some damping factor, though not exactly the same as the one above, by invoking the solution for water waves in a viscous fluid.

The solution for ϕ given by (11) certainly satisfies the Laplace equation and (6). Equations (11) and (12) also satisfy (4) at the free surface, provided (9) is satisfied.

A look at the remaining condition, the dynamic condition at the free surface, is most revealing. The linearized Bernoulli equation at the free surface is

$$p = -\rho(\phi_t + g\eta). \tag{13}$$

A calculation with (11) and (12) gives

$$p = -\frac{\rho a}{\pi}\int_0^\infty \left[\omega \sin \omega t \int_{-\infty}^{\infty} F(\alpha)\cos k(x-\alpha)\,d\alpha\right] dk, \tag{14}$$

after y is equated to zero after the calculation. Note that differentiation of (11) with respect to t within the integral signs (for τ) contributes a term that exactly cancels $g\eta$ on the right-hand side of (13), on account of (9). Since

$$\frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} F(\alpha)\cos k(x-\alpha)\,d\alpha\right] dk = F(x),$$

equation (14) is

$$p = -\rho a\omega \sin \omega t F(x). \tag{15}$$

Hence (11) and (12) are solutions for wave motion created by the pressure distribution (15), from $t = 0$ up to time t, at the free surface, in water otherwise at rest.

It is important to note that (14) results from the variable upper limit of the τ-integral in (11). This explains why, in spite of (9), which is derived from the condition of constant or zero pressure at the free surface, an x-dependent pressure distribution is nonetheless obtained. This fact is related to the interpretation of (7) as giving an impulse at $t = 0$. But interpreting (11) as corresponding to a pressure distribution (15) makes things much easier to grasp, since pressure is far more familiar a quantity than impulse. Without (11) and (12), the formulation of the problem of wave generation by a pressure distribution is cumbersome at best.

Now let

$$F(x) = 1 \quad \text{in } -b \le x \le b, \tag{16}$$

and zero elsewhere. The visible quantity η then can be calculated from (12), and is

$$\eta = \frac{a\omega}{\pi g}\int_0^\infty \sigma I_1 I_2\,dk, \tag{17}$$

where

$$\begin{aligned} I_1 &= \int_0^t \sin\sigma(t-\tau)\sin\omega\tau e^{-\mu(t-\tau)}\,d\tau \\ &= \frac{1}{2}e^{-\mu t}\int_0^t \{\cos[\sigma t-(\sigma+\omega)\tau]-\cos[\sigma t-(\sigma-\omega)\tau]\}e^{\mu\tau}\,d\tau, \end{aligned} \tag{18}$$

and

$$I_2 = \int_{-b}^{b} \cos k(x-\alpha)\,d\alpha = -\frac{1}{k}[\sin k(x-b)-\sin k(x+b)]. \tag{19}$$

Multiplying I_1 to I_2, expressing the product in terms of sine functions, and treating these as the imaginary parts of exponential functions, one can carry out the integration with respect to τ, and obtain

$$\eta = -\frac{a\omega e^{-\mu t}}{4\pi g}\int_0^\infty \frac{\sigma I}{k}\,dk, \tag{20}$$

where

$$I = f(x_1) - f(x_2), \tag{21}$$

$$
\begin{aligned}
f(x_1) = \operatorname{Im}\Bigg[& \frac{1}{i(\sigma+\omega)+\mu}\{\exp i(kx_1+\omega t-i\mu t)-\exp i(kx_1-\sigma t)\} \\
& + \frac{1}{-i(\sigma+\omega)+\mu}\{\exp i(kx_1-\omega t-i\mu t)-\exp i(kx_1+\sigma t)\} \\
& - \frac{1}{i(\sigma-\omega)+\mu}\{\exp i(kx_1-\omega t-i\mu t)-\exp i(kx_1-\sigma t)\} \\
& - \frac{1}{-i(\sigma-\omega)+\mu}\{\exp i(kx_1+\omega t-i\mu t)-\exp i(kx_1+\sigma t)\}\Bigg], \quad (22)
\end{aligned}
$$

and

$$x_1 = x - b, \qquad x_2 = x + b. \tag{23}$$

In (22), Im means "the coefficient of i" in the expression that follows it.

Recalling that we need only consider positive values of σ, we see that the first two members within the brackets of (22) will make no contributions to waves in the flow, that the third member will give waves propagating to the right, and the fourth member will give waves propagating to the left. Similarly the term containing x_2 in (20) will contain a term giving waves going to the left and one going to the right. Obviously the solution for η will be symmetric with respect to $x = 0$. We need therefore consider only waves propagating to the right. Doing that, and writing

$$dk = 2\sigma\, d\sigma/g,$$

we see from (18) and (19) that the part of η corresponding to waves propagating to the right is contained in

$$-\frac{a\omega}{2\pi g}\operatorname{Im}(J_1 + J_2), \tag{24}$$

where

$$J_1 = \int_0^\infty \frac{1}{\sigma-(\omega+i\mu)}[\exp i(kx_1-\omega t)-\exp i(kx_2-\omega t)]\,d\sigma \tag{25}$$

and

$$J_2 = -\int_0^\infty \frac{e^{-\mu t}}{\sigma-(\omega+i\mu)}[\exp i(kx_1-\sigma t)-\exp i(kx_2-\sigma t)]\,d\sigma. \tag{26}$$

The integral J_1 is obtained upon taking the contour in Figure 1, for positive x_1 or x_2, and the contour in Figure 2, for negative x_1 or x_2. In Figures 1 and 2, $\sigma = \sigma_r + i\sigma_i$, and the radius of the circular portion is very large.

The angle of inclination of the slanted portion of the contour in these figures can have any value between zero and $\pi/2$. It has been chosen to be $\pi/8$ because when viscous effects are taken into account later, it will be seen that the angle must not exceed $\pi/6$. The contribution to J_1 from the circular part is zero. The contribution from the slanted lines can be shown not to contain any discrete wave component in the following way. Let J_{1s} be the integral J_1, but with lower and upper limits changed to

$$\sigma_\infty = \lim_{|\sigma|\to\infty} |\sigma| e^{i\pi/8} \text{ and zero,}$$

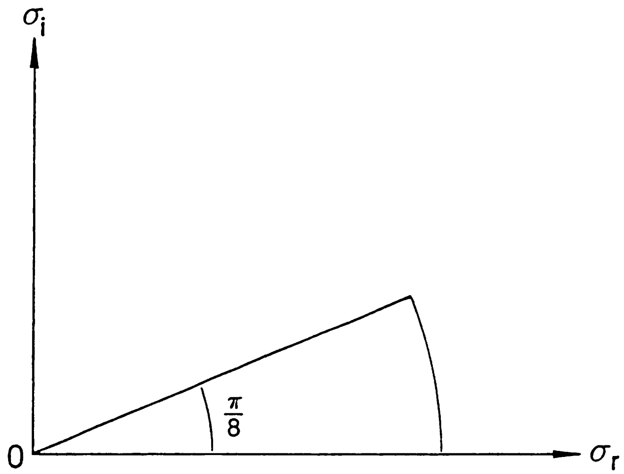

FIG. 1. Contour for evaluating the integral in (25), for positive x_1 or x_2. $\sigma = \sigma_r + i\sigma_i$.

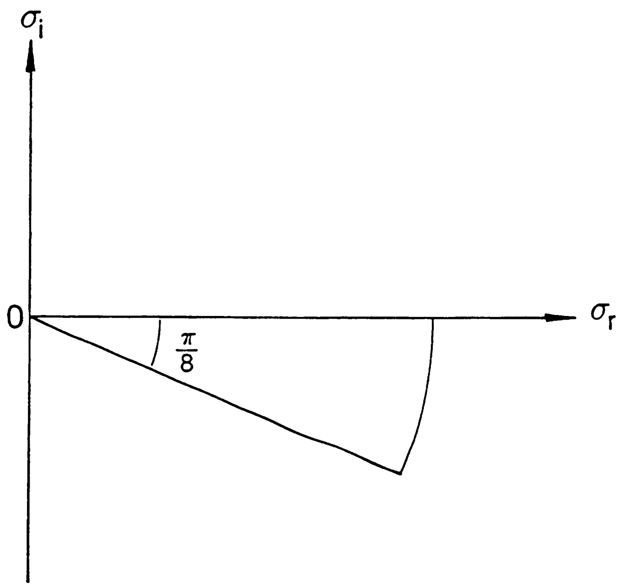

FIG. 2. Contour for evaluating the integral in (25), for negative x_1 or x_2. $\sigma = \sigma_r + i\sigma_i$.

respectively, for Figure 1. Multiply J_{1s} by

$$\exp i(k'x - \sigma' t), \quad \text{with } k'g = \sigma'^2,$$

and integrate with respect to x over the entire x-axis, from minus to plus infinity. The integrals with respect to x and σ are both convergent along the slanted line in Figure 1, and the result is not singular in any way. If J_{1s} contained a discrete Fourier component, the result would be infinite for some value of k'. A similar argument applies to the slanted line in Figure 2. Thus, the wavy part of J_1 is, with $k_e = \omega^2/g$,

$$2\pi i[\exp\{i(k_e x_1 - \omega t) - 2\mu\omega g^{-1} x_1\} - \exp\{i(k_e x_2 - \omega t) - 2\mu\omega g^{-1} x_2\}] \quad (27)$$

for positive x_1, zero for negative x_2, and

$$-2\pi i \exp\{i(k_e x_2 - \omega t) - 2\mu\omega g^{-1} x_2\} \quad (28)$$

for

$$-b < x < b.$$

The integral J_2 requires more care. Since $k = \sigma^2/g$,

$$\exp i(kx_1 - \sigma t) = \exp[i(\sigma_r^2 - \sigma_i^2)g^{-1}x_1 - i\sigma_r t - \sigma_i(2\sigma_r g^{-1} x_1 - t)], \quad (29)$$

and similarly when x_2 replaces x_1. For a given x_1 and a given t, and for the first term in J_2, if

$$x_1 - \frac{gt}{2\sigma_r} \quad (30)$$

is positive we use a circular contour above the σ_r-axis, followed by a slanted line, as shown in Figure 3. At the value of σ_r, denoted by $\hat{\sigma}_r$, that makes (30) vanish, the contour follows a vertical path from P to its image point Q below the σ_r-axis. Then the lower slanted line is followed all the way to the origin, whereby the circuit is completed. Use Figure 4 if $\hat{\sigma}_r$ is reached before $\pi/8$.

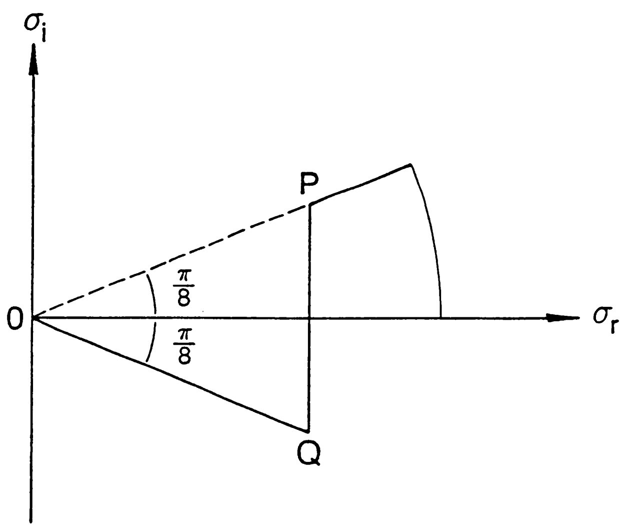

FIG. 3. Contour for evaluating the integral in (28), if $\pi/8$ is reached before $\hat{\sigma}_r$. $\sigma = \sigma_r + i\sigma_i$.

Hence, if $\hat{\sigma}_r > \omega$, the pole $\omega + i\mu$ is not within the circuit, and the residue of the contour integral of the first term in the integral of (26) is zero. Otherwise it is

$$2\pi i E(x_1), \quad (31)$$

where

$$E(x) = \exp[i(k_e x - \omega t) - 2\mu\omega g^{-1} x]; \quad (32)$$

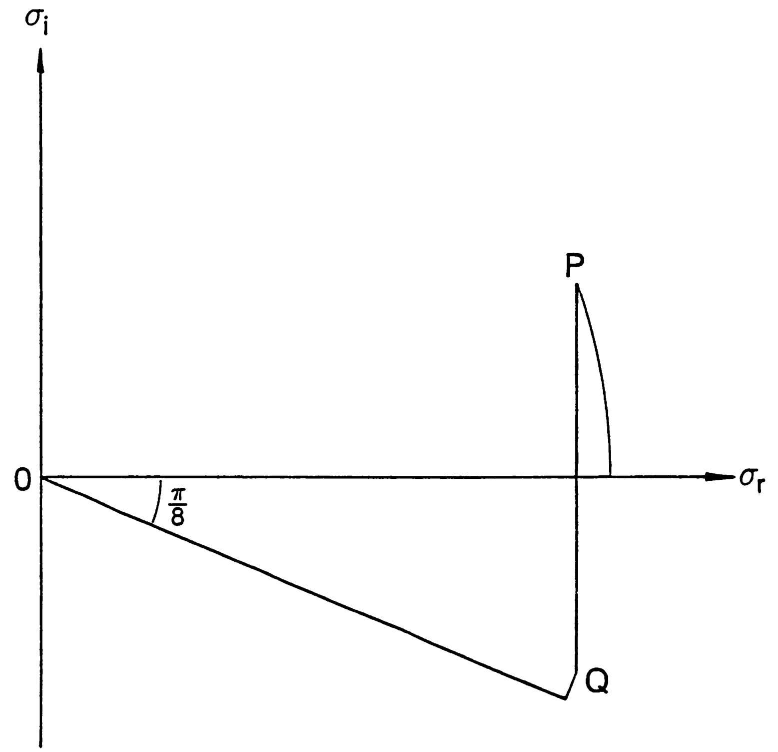

FIG. 4. Contour for evaluating the integral in (28), if $\hat{\sigma}_r$ is reached before $\pi/8$. $\sigma = \sigma_r + i\sigma_i$.

similarly for the second integral in (26). Hence we have the wavy part of J_2 equal to

$$-2\pi i[E(x_1) - E(x_2)] \tag{33}$$

if

$$x_1 - \tilde{c}_g t > 0, \tag{34}$$

where

$$\tilde{c}_g = \frac{\omega}{2k_e} = \frac{g}{2\omega} \tag{35}$$

is the group velocity for gravity waves in deep water with frequency ω. On the other hand, the wavy part of J_2 is zero if

$$x_2 - \tilde{c}_g t < 0, \tag{36}$$

and is

$$2\pi i E(x_2) \tag{37}$$

if

$$-b + \tilde{c}_g t < x < b + \tilde{c}_g t. \tag{38}$$

In the evaluation of J_2 by using the contour in Figure 3, the integral over the slanted lines again contributes nothing to the wavy part of J_2. The contribution from the vertical leg in Figure 3 to the wavy part of J_2 is also zero. This can be seen by taking the part involving x_1, since the development involving x_2 follows the same arguments. That part is, in view of (29) and the definition of $\hat{\sigma}_r$,

$$J_3 = -\exp(i\hat{\sigma}_r t/2 - \mu t)\int_{\sigma_{iP}}^{\sigma_{iQ}} \frac{i}{\sigma - (\omega + i\mu)} \exp(-i\sigma_i^2 g^{-1} x_1)\, d\sigma_i, \tag{39}$$

where $\hat{\sigma}_r \neq \omega$. For a fixed x_1 and large t, it can be shown from (40) that $J_3 \sim t^{-1}$, since $\hat{\sigma}_r$ is proportional to t for fixed x_1, and $\sigma = \hat{\sigma}_r + i\sigma_i$. The same conclusion holds for the part of J_2 containing x_2. Hence the contribution of the vertical leg in the contour in Figure 3 to J_2 is only transient for any x_1.

Let the part of η that corresponds to right-going waves be denoted by η_{wr}. Going back to (24), and using the results obtained for the wavy parts of J_1 and J_2, we can evaluate η_{wr}. Let

$$G(x) = \sin(k_e x - \omega t)\exp(-2\mu\omega g^{-1} x).$$

then the final results are:

(i) if $x < -b, \eta_{wr} = 0$;

(ii) if $x_2 - \tilde{c}_g t < 0$ and $x_1 < 0 < x_2, \eta_{wr} = -\dfrac{a\omega}{g} G(x_2)$; (40)

(iii) if $x_2 - \tilde{c}_g t < 0$ and $x_1 > 0, \eta_{wr} = \dfrac{a\omega}{g}[G(x_1) - G(x_2)]$; (41)

(iv) if $b < -b + \tilde{c}_g t < x < b + \tilde{c}_g t, \eta_{wr} = \dfrac{a\omega}{g} G(x_1)$; (42)

(v) if $x_1 > \tilde{c}_g t, \eta_{wr} = 0$.

There is a slight reduction of k_e, of $O(\mu^2)$, which has been neglected.

These results may seem complicated. They become immediately easy to grasp if one restates them as follows.

Two wave trains both with wave number k_e and frequency ω, one starting from $x = -b$ and the other starting from $x = b$, progress to the right with group velocity $\tilde{c}_g$ into otherwise quiet water.

Since the flow is symmetric with respect to $x = 0$, there is a set of results for left-going waves which can be obtained by symmetry arguments from results (i) to (v). These can be restated as follows.

Two other wave trains, both with wave number k_e and frequency ω, one starting from $x = b$ and the other starting from $x = -b$, progress to the left with group velocity $\tilde{c}_g$ into otherwise quiet water.

Note that where the two wave trains both exist [see (iii)], they reinforce each other if $k_e b = \pi/2$ or $(2n + 1)\pi/2$, but tend to cancel each other if $k_e b = n\pi$. (The reinforcement or cancellation would be complete if $\mu = 0$.)

The investigation for Section 3 is now finished, and we call attention to the fact that the radiation condition of Sommerfeld has not been applied because it is not at all needed, that the wave trains are exponentially damped with respect to x_1 and x_2, that they may reinforce or cancel each other where they both exist, and that they progress into wave-free water. The artificial factor $\exp[-\mu(t - \tau)]$ will be discussed in terms of the true viscosity of the fluid in a later section.

4. Wave trains of finite length. If, in the problem treated in Section 3, the oscillating pressure is removed at $t = T$, the integral I_1 given by (18) is now replaced by

$$I_1 = \int_0^T \sin\sigma(t-\tau)\sin\omega\tau e^{-\mu(t-\tau)}\,d\tau$$
$$= \frac{1}{2}e^{-\mu t}\int_0^T \{\cos[\sigma t-(\sigma+\omega)\tau]-\cos[\sigma t-(\sigma-\omega)\tau]\}e^{\mu\tau}\,d\tau.$$

The development in Section 3 can be repeated, and one obtains the result that two wave trains with wave number k_e, frequency ω, and length $\tilde{c}_g T$, progress to the right. One of these terminates at a point which is at a distance $\tilde{c}_g(t-T)$ from $x = b$, and the other terminates at the same distance from $x = -b$. Similarly there are two left-going wave trains. The flow is symmetric with respect to $x = 0$. Again where the right-going wave trains co-exist, they may reinforce or (partially) cancel each other, similarly for the left-going wave trains.

Thus we have constructed single wave groups of finite length. Each would progress into wave-free water and leave the water behind wave-free, except for the waves of the other trains.

For inviscid fluids, the μ in the exponential factors in the final results can be put to zero, since it has served the purpose of making the flow determinate. The factors can then be dropped. The same holds true for (40)–(42).

5. Wave groups produced by a moving pressure distribution. Consider the waves created by a pressure distribution moving to the left with speed c:

$$p = \beta\rho c^2 \quad \text{in } -b < x+ct < b. \tag{43}$$

Then (11) and (12) are replaced by

$$\phi = \frac{\beta c^2}{\pi}\int_0^\infty \left[\int_0^t \cos\sigma(t-\tau)e^{-\mu(t-\tau)}\,d\tau \int_{-b-c\tau}^{b-c\tau}\cos k(x-\alpha)\,d\alpha\right]e^{ky}\,dk, \tag{44}$$

$$\eta = \frac{\beta c^2}{\pi g}\int_0^\infty \left[\sigma\int_0^t \sin\sigma(t-\tau)e^{-\mu(t-\tau)}\,d\tau \int_{-b-c\tau}^{b-c\tau}\cos k(x-\alpha)\,d\alpha\right]dk. \tag{45}$$

Proceeding as in Section 3, we have

$$\eta = -\frac{\beta c^2}{\pi}\int_0^\infty \frac{\sigma}{k} I\,dk, \tag{46}$$

where

$$I = \int_0^t \sin\sigma(t-\tau)[\sin k(x_1+c\tau)-\sin k(x_2+c\tau)]e^{-\mu(t-\tau)}\,d\tau$$
$$= \frac{1}{2}RP[H(x_1)-H(x_2)], \tag{47}$$

$$H(x_1) = \frac{1}{i(kc+\sigma)+\mu}[e^{ik(x_1+ct)} - e^{i(kx_1+\sigma t)-\mu t}]$$
$$- \frac{1}{i(kc-\sigma)+\mu}[e^{ik(x_1+ct)} - e^{i(kx_1+\sigma t)-\mu t}], \tag{48}$$

where x_1 and x_2 are given by (23).

The roots of

$$kc - \sigma - i\mu = 0 \tag{49}$$

are, since $kg = \sigma^2$,

$$\sigma = \frac{g}{2c} \pm \left(\frac{g^2}{4c^2} + \frac{ig\mu}{c}\right)^{1/2}. \tag{50}$$

One of them is in the first quadrant in the complex σ-plane, and the other in the fourth. For small μ the roots can be approximated by

$$\sigma = -i\mu - \frac{c}{g}\mu^2, \qquad \sigma = \frac{g}{c}(1 + \mu^2) + i\mu. \tag{51}$$

The roots of

$$kc + \sigma - i\mu = 0 \tag{52}$$

are obviously the negatives of those of (49). So the roots of (49) and (52) are in the first or fourth quadrant only. Furthermore, the root of (52) in the first quadrant is outside of the contour in Figure 1, since μ is assumed much smaller than g/c. This can be seen from the first root given by (50) after the signs have been changed. Thus, in evaluating I in (47) by using the contours shown in the figures, as the situation demands, it is only the root given by (50) with the positive sign that is significant in determining the wavy part of η.

The terms in (48) corresponding to waves are, after the relevant part is extracted from the second bracket,

$$\frac{ig}{(g + i2\mu c)(\sigma - gc^{-1} - i\mu)}[\exp ik(x_1 + ct) - \exp\{i(kx_1 + \sigma t) - \mu t\}],$$

if terms of $O(\mu^2)$ are neglected. Similar results hold for the terms in $H(x_2)$ corresponding to waves. The rest of the development follows closely the steps described in Section 3, and is omitted here. The final results are as follows.

(i) There are no waves ahead of the moving disturbance.

(ii) Behind the disturbance there are two overlapping wave trains, both of wave velocity c and wave number $k_e = g/c^2$, and both of length $(c_g)_e t$, where $(c_g)_e$ is the group velocity of the waves, and is equal to $c/2$. One of the trains starts at $x_2 = 0$ and the other at $x_1 = 0$. Depending on the length over which the disturbance acts, the two trains may reinforce or partially cancel each other where they overlap.

(iii) The damping factor for the train starting at $x_1 = 0$ is

$$\exp[-2\mu(x_1 + ct)/c],$$

and the damping factor for the other train is the same factor with x_2 replacing x_1.

(iv) There is a slight increase of k_e of $O(\mu^2)$ over g/c^2, as a result of the second equation in (51). This is neglected.

If the moving disturbance is a moving body, floating or submerged, the development and the results are similar. The creation of an ever-lengthening gravity-wave group (which is the sum of the two trains, with the parts outside of their common interval neglected) behind the body allows one to calculate the wave drag from the rate of increase of the wave energy behind the body, upon letting μ be zero. This is a much more direct way of seeing things than calculating the wave drag from an infinite wave train behind the body. In that case, as is well known, one has to calculate the energy flux (or rate of work done) at a section behind the body.

6. Gravity-wave trains of finite length created by a moving disturbance in deep water. If a surface pressure is applied at $t = 0$ and moves to the left with speed c, and is then removed at time T, a group of waves of length $cT/2$ will be formed, and will move to the left with the group velocity $c/2$ (if the effect of the spread of the disturbance is neglected). Only gravity waves have been considered here. Had surface-tension effects been included, one would expect two wave groups, one of the gravity type and the other of the capillary type. When the disturbance is removed, the two groups will separate, since the train of the capillary type has a greater group velocity, even though the individual waves in either group still move with the same phase velocity c.

7. Effects of viscosity. When viscous effects are taken into account, but the Reynolds number ($g^2/(\omega^3\nu)$ or $c^3/(g\nu)$, as the case may be, ν being the kinematic viscosity) is large, Lamb's solution (1945, pp. 625–627) applies, and in the solution for ϕ or η instead of the factor $\exp i(kx - \sigma t)$ one now has, with σ^2 still equal to gk,

$$\exp[i(kx - \sigma t) - 2\nu k^2 t].$$

Thus one may consider the factor $\exp(-\mu t)$ as a useful but empirical representation of the true factor $\exp(-2\nu k^2 t)$. Replacing the former by the latter, one can carry out the calculation in Section 3 or 5 as before, and the results are the same. Of course, since the new factor involves k, one has to go through the calculation to see that it will cause no new difficulties. But the contours in the figures have been chosen with the factor $\exp(-2\nu k^2 t)$ in mind, and in the following we shall show that indeed no new difficulties arise.

The factor in Section 3,

$$(\sigma - \omega - i\mu)^{-1},$$

is now replaced by

$$(\sigma - \omega - i2\nu k^2)^{-1}, \tag{53}$$

and the factors in Section 5,

$$(kc - \sigma - i\mu)^{-1} \quad \text{and} \quad (kc + \sigma - i\mu)^{-1},$$

are now replaced by

$$(kc - \sigma - i2\nu k^2)^{-1} \quad \text{and} \quad (kc + \sigma - i2\nu k^2)^{-1}. \tag{54}$$

We have to determine the poles of (53) and (54). Aside from the important one which is, for small ν, at

$$\sigma = \omega + i2\nu\omega^4 g^{-2} \tag{55}$$

approximately, the other three of (53) are at large values of $|\sigma|$, given approximately for small ν by

$$1 - i2\nu g^{-1}\sigma^3 = 0. \tag{56}$$

If we write

$$\sigma = |\sigma| e^{i\theta},$$

then (56) gives

$$\theta = -\frac{\pi}{6}, \quad \frac{\pi}{2}, \quad \frac{7\pi}{6}. \tag{57}$$

The contours in the figures avoid all three poles with these values of θ. Indeed, they were chosen with this avoidance in mind in the first place.

As to the poles of (54), one is at $\sigma = 0$. Examination of (47) and (48) with μ replaced by $2\nu k^2$ reveals that this is not really a pole, since the numerators of (48) also vanish at $\sigma = 0$. Thus $H(x_1)$ and $H(x_2)$ do not become infinite at $\sigma = 0$.

The important pole of the first factor in (54) is at

$$\sigma = \frac{g}{c} + \frac{i2\nu g^2}{c^4} \tag{58}$$

approximately, for small ν. The other two poles of that factor are at

$$\sigma = \left(\frac{gc}{2\nu}\right)^{1/2} \left[\exp\left(-\frac{i\pi}{4}\right), \exp\left(\frac{i3\pi}{4}\right)\right] \tag{59}$$

approximately, for small ν. The poles of the second factor in (54) are at values of σ which are the negatives of those given by (58) and (59). All the poles except the one given by (58) are outside of the contours chosen in the figures. Thus for Section 5 new difficulties do not arise either when the new damping factor $\exp(-2\nu k^2 t)$ is used.

Note that in Lamb's solution the stress layer at the free surface has been ignored, since the Reynolds number is assumed high, and therefore the normal stress at the free surface is simply represented by $-p$.

When μ is replaced by $2\nu k^2$ to begin with in the development in Sections 3 and 5, all the results remain valid after μ in the results is replaced by $2\nu k_e^2$, as the mathematics requires. Thus the damping factor has been replaced by one involving the wave number, as required by the Navier–Stokes equations, and the artificiality of a frequently invoked device in wave dynamics has been removed. This and the construction of gravity-wave groups of finite length constitute the dual purpose of this paper.

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Reference

[1] H. Lamb, *Hydrodynamics*, Dover, New York, 1945