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THE SHAPE OF THE STRONGEST COLUMN IS ARBITRARILY CLOSE TO THE SHAPE OF THE WEAKEST COLUMN*

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Abstract. We reconsider the problem of determining the shape of the strongest column having a given length l and volume V. Previous results [13,7] have given optimal shapes for which the cross section vanishes at certain points. Although these results are mathematically correct, Theorem 1 below explains what is wrong with these anomalous shapes.

1. Introduction. An interesting problem with a long history is to find the shape of a slender column of a given length and volume which will give the largest possible buckling load [1,2,3,5,6,7,8,10,11,12,13,14]. Mathematically, this amounts to maximizing an eigenvalue of a certain Sturm–Liouville system to obtain an isoperimetric inequality. Some solutions of the problem have been given, but those extremal shapes have points where the cross section vanishes. It is remarkable that the strongest column should have such points. These shapes have led to confusion, controversy, and several attempts to resolve the anomaly. The case in which the column is clamped at each end seems to be especially troublesome since the solution has two points in its interior where the cross section vanishes. With such boundary conditions, some shapes may yield eigenvalues of multiplicity two. Now it is clear that the optimal shapes given previously [13, 7] are mathematically correct (see also [1]). However, in this work we will show, in Theorem 1 below, that these treatments of the problem are incomplete in that, very simply, *there is no solution* within the class of shapes for which the mathematical eigenvalue problem gives an adequate description of the physical buckling problem. Mathematically speaking, the eigenvalue problems involved are not well-posed and the supremum of the buckling load is not attained for any reasonable shape. This is always the case in that it happens for clamped, pinned, or free end points and for any combination of these. *The anomalous behavior of*

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the strongest column problem is not at all related to the kinds of boundary conditions or to the multiplicity of the eigenvalues involved. It is due to the singularities of the coefficient function in the differential equation.

We will then show how to correct this difficulty by using a constraint of the form $A(x) \geq h > 0$ on the area of the cross section. Such problems have been considered by E. R. Barnes [5]. Using this constraint, we will give a direct proof, in Theorem 2 below, that there does exist a well-behaved function $A^*(x)$ which yields the maximum of the buckling load. A similar proof will work for any reasonable combination of boundary conditions.

2. On the nonexistence of the maximizing shape. Consider the case where the column is pinned at each end. The critical buckling load is determined by the first eigenvalue, denoted by $\lambda_1(A)$, of the Sturm–Liouville system

$$y'' + \lambda \frac{1}{A^2(x)} y = 0, \quad y(0) = y(l) = 0. \tag{1}$$

In order for such an eigenvalue problem to be a good mathematical model of the physical buckling problem, it must be well-posed. To be precise about this concept, let $\mathscr{C}$ be the class of all shape functions $A(x)$ which satisfy the following four conditions:

1. $\lambda_1(A)$ exists;
2. $A(x) \geq 0$;
3. $\int_0^l A(x)\,dx = V$;
4. the eigenvalue functional $\lambda_1(\cdot)$ is continuous at A. That is, if B is any function which satisfies conditions 1, 2, and 3, then $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$, such that if $\sup_{0\leq x\leq l} |A(x) - B(x)| < \delta_\varepsilon$, then $|\lambda_1(A) - \lambda_1(B)| < \varepsilon$.

The problem (1) is well-posed for all $A \in \mathscr{C}$.

Keller [7] used the condition that $\lambda(A)$ be a maximum to derive the condition $y^2 = A^3$. Substituting this relation into (1) and solving the resulting equation gives the classical solution of the strongest column problem, which we denote by $A^*(x)$. It is defined by the following equations:

$$A^*(x) = \frac{4V}{3l} \sin^2 \theta(x), \qquad y(x) = (A^*)^{3/2}(x), \qquad 2\theta(x) - \sin 2\theta(x) = 2\pi x/l. \tag{2}$$

When we define the function $\rho^*(x) = (A^*)^{-2}(x)$, we see that $\rho^*(x) = y^{-4/3}$. Therefore, ρ^* has singularities at the nodes of y and, in fact, $\int_0^l \rho^*\,dx = \infty$. This has profound and unfortunate implications for the associated eigenvalue problem, and none of the usual eigenvalue theory will be valid for such functions. One of the repercussions of this singularity is that a very small change in the classical shape $A^*(x)$ can yield a shape function for which the eigenvalue problem is well-posed but for which the buckling load is arbitrarily small. More precisely, the following theorem holds.

THEOREM 1. Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be given. Let $A^*(x)$ be the classical shape function for the pinned column defined by (2). Then there exists a shape function $B(x) \in \mathscr{C}$ for which $|A^*(x) - B(x)| < \varepsilon_1$ but for which $\lambda_1(B) < \varepsilon_2$. Thus $A^* \notin \mathscr{C}$.

Proof. Given the classical function (2) and small positive numbers δ and η, define the functions $A_\delta^*(x)$ and $B(x)$ by

$$A_\delta^*(x) = \begin{cases} A^*(\delta - x) & \text{if } 0 \le x \le \delta, \\ A^*(x) & \text{if } \delta < x \le l, \end{cases} \qquad B(x) = \frac{\eta}{l} + \left(1 - \frac{\eta}{V}\right) A_\delta^*(x).$$

Now $B(x)$ is piecewise continuous and bounded below by the positive constant η/l. It follows that $B^{-2}(x)$ is bounded above so that $B(x) \in \mathscr{C}$. Given ε_1, we will first select δ and η so small that for all x, $|A^*(x) - B(x)| < \varepsilon_1$.

Next, we will show how to make $\lambda_1(B) < \varepsilon_2$. The extremal condition used to obtain the classical solution is that $A^* = y^{2/3}$. Therefore, the function $(A^*)^{-2}(x)$ behaves like $x^{-4/3}$ as $x \to 0$, and $(A_\delta^*)^{-2}(x)$ behaves like $(\delta - x)^{-4/3}$ as $x \to \delta$. This is a nonintegrable singularity so that if $\delta > 0$ is fixed, then

$$\lim_{\eta \to 0} \int_0^l B^{-2}(x)\,dx = \int_0^l (A_\delta^*)^{-2}(x)\,dx = \infty.$$

Next, we will substitute the trial function $u = x(l - x)$ into the variational characterization of the first eigenvalue to obtain

$$\lambda_1(B) = \min_u \frac{\int_0^l u'^2\,dx}{\int_0^l u^2 B^{-2}(x)\,dx} \le \frac{l^3}{6\int_0^l x^2(l-x)^2 B^{-2}(x)\,dx}.$$

The singularity of $B^{-2}(x)$ is at $x = \delta > 0$, and the zero of the trial function u is at $x = 0$. It follows that if we hold δ fixed and let $\eta \to 0$, then $\int_0^l x^2(l-x)^2 B^{-2}(x)\,dx \to \infty$. Therefore, $\lambda_1(B)$ can be made less than any given ε_2 and the theorem follows.

3. A condition for the existence of the maximizing shape. Consider the eigenvalue problem

$$y'' + \lambda \rho(x) y = 0, \quad y(0) = y(l) = 0, \tag{3}$$

with the coefficient function $\rho(x)$ restricted to the class $\mathscr{K}(a, b, M)$, defined by the conditions

$$\int_0^l \rho(x)\,dx = M, \quad \rho(x) \ge 0, \quad a \le \rho(x) \le b.$$

Define a metric $d_0(\rho_1, \rho_2)$ on $\mathscr{K}(a, b, M)$ by

$$d_0(\rho_1, \rho_2) = \max_{0 \le x \le l} \left| \int_0^x (\rho_1(t) - \rho_2(t))\,dt \right|.$$

Although the result was not expressed in the topological language used here, Krein [9, pages 165–167] proved that the nth eigenvalue of (3), denoted by $\lambda_n(\rho)$, is a continuous function on the compact metric space $\mathscr{K}(a, b, M)$. Therefore, the maximizing function exists. We will also be concerned with the class $\mathscr{C}(h, H, V)$ of shape functions $A(x)$, defined by the conditions $A \in \mathscr{C}$ and $h \le A(x) \le H$. Using the change of notation $\rho(x) = A^{-2}(x)$, $b = h^{-2}$, and $a = H^{-2}$, we see a correspondence between the two classes. Assuming that $h > 0$, it is easy to prove that if $n \to \infty$, then $d_0(A_n, A^*) \to 0$, if and only if $d_0(A_n^{-2}, (A^*)^{-2}) \to 0$. Furthermore, since there are no singularities in this case, it follows that any function in $\mathscr{C}(h, H, V)$ yields a well-posed eigenvalue problem. Thus we obtain the following theorem.

THEOREM 2. Let $\lambda_1(A)$ denote the first eigenvalue of system (2). For any constants $0 < h < H$, there exists a function $A^*(x) \in \mathscr{C}(h, H, V)$ which maximizes $\lambda_1(A)$.

4. On the choice of a metric. In addition to the metric $d_0(\cdot, \cdot)$ defined above, consider two more metrics defined as follows:

$$d_1(\rho_1, \rho_2) = \max_{0 \le x \le l} |\rho_1(x) - \rho_2(x)|,$$
$$d_2(\rho_1, \rho_2) = \max_{0 \le x \le l} |\rho_1(x) - \rho_2(x)| + \max_{0 \le x \le l} |\rho_1'(x) - \rho_2'(x)|.$$

We have shown that if $A^*(x)$ vanishes at some point, then the eigenvalue $\lambda_1(A)$ will not be continuous at $A^*(x)$ with respect to the metric $d_1(\cdot, \cdot)$. However, one might argue that other variational problems, such as the brachistochrone, exhibit the same kind of behavior noted here but that the cycloid solution curve in that case is valid.[2] The cycloid solution curve of the brachistochrone problem has other curves which are arbitrarily close to it (in the sense of the metric $d_1(\cdot, \cdot)$) but for which the time of travel is arbitrarily large. However, the brachistochrone problem involves moving particles and directions, and the most natural metric to use there seems to be $d_2(\cdot, \cdot)$ and not $d_1(\cdot, \cdot)$. This makes the travel time a continuous function with respect to the metric.

The choice of a metric to use for the column is quite arbitrary. However, the metric $d_1(\cdot, \cdot)$, which we have used, seems the most natural in this case since directions or surface areas are not involved. The metric $d_2(\cdot, \cdot)$ is too strong and the metric $d_0(\cdot, \cdot)$ is too weak. Note also that if $A(x)$ is bounded away from zero, as it should be, then $\lambda_1(\cdot)$ is continuous at $A(x)$ with respect to the metric $d_1(\cdot, \cdot)$.

5. Final remarks. The proofs of Theorems 1 and 2 can be modified to give analogous results for other kinds of boundary conditions, including clamped or free ends and various combinations of these.

These results also extend easily to the case where $\rho(x) = (A)^{-m}(x)$ with $m \ge 1$. See also the work of D. C. Barnes [4], which involves other kinds of functions.

If the coefficient function $A^*(x)$ vanishes at some point, the eigenvalue problem (1) will not be well-posed. It appears that similar ill-conditioning phenomena may occur if the boundary conditions are perturbed. This may cause the zero of the eigenfunction to move away from the singularity of the coefficient function, giving a large integral in the denominator of the Rayleigh quotient and a small buckling load. It also seems reasonable to expect this to happen based only on the physics of the situation.

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