QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLVI, NUMBER 2
JUNE 1988, PAGES 381-392

ON ENERGIES FOR NONLINEAR VISCOELASTIC MATERIALS OF SINGLE-INTEGRAL TYPE*

By

MORTON E. GURTIN AND WILLIAM J. HRUSA

Carnegie Mellon University, Pittsburgh, Pennsylvania

1. Introduction. Viscoelastic materials are described by constitutive relations giving the stress when the temporal history of the deformation gradient is known. For one-dimensional motions, such as those involving simple tension, this relationship has the form

$$\sigma(t) = \mathscr{S}(\varepsilon^t), \tag{1.1}$$

where σ and ε are scalar[1] measures of stress and strain, t is the time, ε^t, with values

$$\varepsilon^t(\tau) = \varepsilon(t-\tau), \quad 0 \le \tau < \infty, \tag{1.2}$$

is the history up to time t of the strain, and $\mathscr{S}$ is a (real-valued) functional defined on an appropriate set of strain histories.

For a large class of specific models the functional $\mathscr{S}$ has the form

$$\begin{aligned}\mathscr{S}(\varepsilon^t) &= S(\varepsilon^t(0)) + \int_0^\infty s(\tau, \varepsilon^t(0), \varepsilon^t(\tau))\,d\tau \\ &= S(\varepsilon(t)) + \int_0^\infty s(\tau, \varepsilon(t), \varepsilon(t-\tau))\,d\tau.\end{aligned} \tag{1.3}$$

Underlying this class of models is the assumption that contributions to the stress superpose additively in the delay time τ, an assumption motivated by Boltzmann's superposition principle for linear viscoelasticity. We shall refer to functionals of the form (1.3) as *single-integral laws*; for convenience, we normalize s so that $s(\tau, \varepsilon, 0) = 0$ for all $\tau > 0$ and $\varepsilon \in \mathbf{R}$.

An important feature of viscoelasticity—one that is especially relevant to experimental and theoretical studies in wave propagation (cf. [17])—is the *interaction* between nonlinearity and dissipation, in particular, between nonlinearity in the instantaneous response and dissipation due to memory. Single-integral laws, even though relatively simple, capture the essence of this interaction. Moreover, because of their simplicity, such laws are conducive to the characterization of real materials, and, in addition, lead to initial/boundary-value problems whose analysis is comparatively clean.

---

*Received July 22, 1987.

[1] We use a one-dimensional formulation for purposes of exposition; the extension to tensorial stress-strain relations is the subject of [20].

It is our purpose here to construct energies for stress-strain laws of single-integral type. Roughly speaking, a functional $\mathscr{V}$ is an *energy* for $\mathscr{S}$ if the corresponding constitutive relation,

$$\psi(t) = \mathscr{V}(\varepsilon^t), \tag{1.4}$$

is consistent with the **Clausius–Planck inequality**,[2]

$$\dot{\psi}(t) \leq \sigma(t)\dot{\varepsilon}(t), \tag{1.5}$$

in all constitutive processes. We shall limit our attention to energies of single-integral form:

$$\mathscr{V}(\varepsilon^t) = V(\varepsilon(t)) + \int_0^\infty v(\tau, \varepsilon(t), \varepsilon(t-\tau))\, d\tau; \tag{1.6}$$

in order to avoid trivial ambiguities we require that $V(0) = 0$ and $v(\tau, \varepsilon, 0) = 0$ for all $\tau > 0$ and $\varepsilon \in \mathbf{R}$.

As our main result we show that if $\mathscr{S}$ is a single-integral law, then a necessary and sufficient condition for the existence of an energy $\mathscr{V}$ of single-integral form is that[3]

$$\int_p^\varepsilon \{s_{,1}(\tau, \alpha, \alpha) - s_{,1}(\tau, \alpha, p)\}\, d\alpha \geq 0 \tag{1.7}$$

for all $\varepsilon, p \in \mathbf{R}$, and that, granted (1.7), the energy $\mathscr{V}$ is unique; in fact, the corresponding response functions V and v are given by the explicit relations

$$\begin{aligned} V(\varepsilon) &= \int_0^\varepsilon S(\alpha)\, d\alpha, \\ v(\tau, \varepsilon, p) &= \int_p^\varepsilon s(\tau, \alpha, p)\, d\alpha + \int_0^p s(\tau, \alpha, \alpha)\, d\alpha. \end{aligned} \tag{1.8}$$

An important feature of energies consistent with the Clausius–Planck inequality is that they lead to (physically meaningful) *Liapunov functionals* for the underlying initial/boundary-value problems. To see this consider a one-dimensional viscoelastic body which is described by the constitutive equation (1.1) and which occupies an interval B of $\mathbf{R}$ in a (fixed) reference configuration. In any motion of this body the displacement field $u(x, t)$ and the strain $\varepsilon(x, t)$ must be consistent with the field equations

$$\begin{aligned} \varepsilon(x, t) &= u_x(x, t), \\ \sigma(x, t) &= \mathscr{S}(\varepsilon^t(x, \cdot)), \\ u_{tt}(x, t) &= \sigma_x(x, t). \end{aligned} \tag{1.9}$$

In writing (1.9) we have tacitly assumed that the density is uniform (and scaled to unity) and that the body force is zero. Suppose that we are given a sufficiently regular solution of (1.9) for $x \in B$ and t in some interval I, and let

$$\psi(x, t) = \mathscr{V}(\varepsilon^t(x, \cdot)). \tag{1.10}$$

---

[2]The Clausius–Duhem inequality—when restricted to behavior which is isothermal or adiabatic—reduces to the Clausius–Planck inequality; in the former case ψ is the free energy, in the latter case the internal energy.

[3]For functions $f(a_1, a_2, a_3)$ we employ the notation $f_{,i} = \frac{\partial f}{\partial a_i}$.

Then we may use the field equations $(1.9)_{1,3}$ and the Clausius–Planck inequality (1.5) to show that, for all times $t \in I$,

$$\frac{d}{dt}\int_B \left\{\frac{1}{2}u_t(x,t)^2 + \psi(x,t)\right\} dx - \int_{\partial B} \sigma(x,t)u_t(x,t)\,dx \le 0, \tag{1.11}$$

where the integral over ∂B means evaluation at the endpoints of B. Thus if the product σu_t vanishes on ∂B, then

$$\mathscr{E}(u^t) = \int_B \left\{\frac{1}{2}u_t(x,t)^2 + \mathscr{V}(u_x^t(x,\cdot))\right\} dx \tag{1.12}$$

decreases along solutions and hence is a Liapunov functional for (1.9).[4]

An energy consistent with $\mathscr{S}$ can also be used to construct a Liapunov functional for the equation

$$\ddot{z}(t) + \mathscr{S}(z^t) = 0, \tag{1.13}$$

which models the motion of a mass suspended by a massless viscoelastic filament. (See Section 6 of [4].) If $\mathscr{V}$ is an energy for $\mathscr{S}$, then

$$\mathscr{E}(z^t) = \frac{1}{2}\dot{z}(t)^2 + \mathscr{V}(z^t) \tag{1.14}$$

is a Liapunov functional for (1.13). If $\mathscr{S}$ and $\mathscr{V}$ are single-integral laws, then, under reasonable assumptions, the Liapunov functional $\mathscr{E}$ in (1.14) can be used to establish global existence and asymptotic stability for (1.13).[5]

This paper is motivated by ideas of Coleman, Day, and Owen. For a very general class of materials with memory, Coleman [3] gives necessary and sufficient conditions for the associated constitutive relations to be compatible with the second law of thermodynamics in the form of the Clausius–Duhem inequality. Coleman's theory presupposes the existence of entropy; subsequent works of Day [7, 9] and of Coleman and Owen [5] establish the existence of entropy as a consequence of a more primitive form of the second law.

We consider here a purely mechanical theory and for that reason adopt the second law in the form of the Clausius–Planck inequality (1.5). Our constitutive relations are much more specific than those studied in [3, 5, 7, 9]; as a consequence we are able to give an explicit formula for the energy. For special cases of the function s in (1.3), energies and associated Liapunov functionals have been constructed by others.[6] Our results *completely characterize the energy* when both stress and energy are given by single integral laws.

2. Existence of an energy. We consider constitutive relations,

$$\sigma(t) = \mathscr{S}(\varepsilon^t), \tag{2.1}$$

[4]The type of bounds that can be obtained from a Liapunov functional of the form (1.11) generally are not sufficient to continue a smooth solution of (1.9) globally in time; estimates for spatial derivatives of higher order are needed (cf., e.g., [18]).

[5]There is a large literature concerning the use of Liapunov functionals to study the global existence and asymptotic behavior of solutions of functional differential equations such as (1.13) (cf., e.g., [2, 4, 11, 12]).

[6]There is a large literature (cf., e.g., [2, 4, 6, 14, 15, 16, 19]).

giving the current stress $\sigma(t)$ in terms of the strain history

$$\varepsilon^t(\tau) = \varepsilon(t-\tau), \quad 0 \le \tau < \infty, \tag{2.2}$$

for functionals $\mathscr{S}$ of the form

$$\mathscr{S}(\varepsilon^t) = S(\varepsilon^t(0)) + \int_0^\infty s(\tau, \varepsilon^t(0), \varepsilon^t(\tau))\, d\tau. \tag{2.3}$$

The strain histories we consider are smooth and bounded. Specifically, we reserve the term **history** for functions in $C^1([0,\infty))$ that are bounded and have a bounded derivative.

Functionals of the form (2.3) will be called single-integral laws. More precisely, let $\mathscr{S}$ be defined by

$$\mathscr{S}(h) = S(h(0)) + \int_0^\infty s(\tau, h(0), h(\tau))\, d\tau \tag{2.4}$$

for every history h. We say that $\mathscr{S}$ is a **single-integral law**, or that $\mathscr{S}$ is of single-integral form, if its **response pair** (S, s) satisfies the following two assumptions:

(H_1) $S \in C^1(\mathbf{R})$, $s \in C^1((0,\infty) \times \mathbf{R} \times \mathbf{R})$ with

$$s(\tau, \varepsilon, 0) = 0 \text{ for all } \varepsilon \in \mathbf{R}, \tau > 0; \tag{2.5}$$

(H_2) given any compact set $C \subset \mathbf{R}^2$, there are functions f and ℓ on $(0,\infty)$, with $f \in L^1(0,\infty)$, $\ell \in L^1(\delta,\infty)$ for all $\delta > 0$, and

$$\delta \int_\delta^\infty \ell(\tau)\, d\tau \to 0 \text{ as } \delta \to 0, \tag{2.6}$$

such that

$$\begin{aligned} &|s_{,2}(\tau,\varepsilon,p)|,\ |s_{,3}(\tau,\varepsilon,p)| \le f(\tau), \\ &|s_{,1}(\tau,\varepsilon,p) - s_{,1}(\tau,\varepsilon,q)| \le \ell(\tau)|p-q| \end{aligned} \tag{2.7}$$

for all $(\varepsilon, p), (\varepsilon, q) \in C$ and $\tau > 0$.

Remarks.

1. It follows from (2.5) and (2.7) that for any compact set $C \subset \mathbf{R}^2$,

$$|s(\tau,\varepsilon,p)| \le f(\tau)|p|, \qquad |s_{,1}(\tau,\varepsilon,p)| \le \ell(\tau)|p| \tag{2.8}$$

for all $(\varepsilon, p) \in C$ and $\tau > 0$, where f and ℓ are the functions in (H_2). As a consequence of (2.8),

$$s(\tau,\varepsilon,p) \to 0 \quad \text{as } \tau \to \infty \tag{2.9}$$

uniformly for (ε, p) in compact sets.

2. The normalization (2.5) ensures that for a given single-integral law $\mathscr{S}$ the response pair (S, s) is unique. In the literature, other normalizations (e.g., $s(\tau,\varepsilon,\varepsilon) \equiv 0$) are often used (see the note added in proof). For a functional of the form (2.4) (under suitable assumptions on s) the normalization (2.5) can be achieved by introducing the modified response functions

$$\tilde{S}(\varepsilon) = S(\varepsilon) + \int_0^\infty s(\tau,\varepsilon,0)\, d\tau, \qquad \tilde{s}(\tau,\varepsilon,p) = s(\tau,\varepsilon,p) - s(\tau,\varepsilon,0). \tag{2.10}$$

3. It is important to note that the response function $s(\tau, \varepsilon, p)$ is not required to be continuous at $\tau = 0$: our theory allows for singular kernels such as those studied by Hrusa and Renardy [13].

By a **strain path** we mean a function ε on $\mathbf{R}$ with ε^t a history at each t.

Let $\mathscr{S}$ and $\mathscr{V}$ be functionals defined on the space of histories. We say that $\mathscr{V}$ is an **energy** for $\mathscr{S}$ if, given any strain path ε, the functions

$$\sigma(t) = \mathscr{S}(\varepsilon^t), \qquad \psi(t) = \mathscr{V}(\varepsilon^t) \tag{2.11}$$

are such that ψ is differentiable and

$$\dot{\psi} \leq \sigma\dot{\varepsilon} \tag{2.12}$$

on $\mathbf{R}$. To avoid a trivial ambiguity, we add the additional requirement:

$$\mathscr{V}(0) = 0. \tag{2.13}$$

Our main result is the following theorem.

Theorem on the existence of an energy. Let $\mathscr{S}$ be a single-integral law with (S, s) the corresponding response pair. Then a necessary and sufficient condition that $\mathscr{S}$ have an energy $\mathscr{V}$ of single-integral form is that s satisfy

$$\int_p^\varepsilon \{s_{,1}(\tau, \alpha, \alpha) - s_{,1}(\tau, \alpha, p)\}\, d\alpha \geq 0 \tag{2.14}$$

for all $p, \varepsilon \in \mathbf{R}$. Granted (2.14), the energy is unique; in fact, the response pair (V, v) for $\mathscr{V}$ is given by

$$\begin{aligned} V(\varepsilon) &= \int_0^\varepsilon S(\alpha)\, d\alpha, \\ v(\tau, \varepsilon, p) &= \int_0^p s(\tau, \alpha, \alpha)\, d\alpha + \int_p^\varepsilon s(\tau, \alpha, p)\, d\alpha. \end{aligned} \tag{2.15}$$

Remarks.

1. The first of (2.15) has an obvious interpretation: V is the energy of an elastic material with response function S. To interpret $(2.15)_2$ note that $v(\tau, \varepsilon, p)$ gives the contribution at time t when the strain history at time $t - \tau$ has value p and the current strain is ε. By $(2.15)_2$, $v(\tau, \varepsilon, p)$ is the sum of two terms: the first represents an equilibrium energy for a strain path which at all times has value p; the second represents an instantaneous energy for a strain path which at all past times had value p, but which suddenly jumps to ε at the current time.

2. The necessity of (2.14) and the uniqueness of the corresponding energy apply only within the class of energies of single-integral form. Indeed, for a linear single-integral law with response function s obeying (2.14), Breuer and Onat [1] exhibit several energies of the form

$$\psi(t) = \int_{-\infty}^t \int_{-\infty}^t K(t - \tau, t - \xi)\dot{\varepsilon}(\tau)\dot{\varepsilon}(\xi)\, d\tau\, d\xi. \tag{2.16}$$

There are also examples of linear single-integral laws which do not satisfy (2.14), but which do have energies; such energies, of course, cannot be expressed in single-integral form.[7]

An interesting consequence of this theorem concerns the relaxation function which governs the stress-strain relation (1.1) when the strain history is small. With this in mind, let $\mathscr{S}$ be a single-integral law and equip the space of histories with the supremum norm $\|\cdot\|_\infty$. Then, because of (H_1) and (H_2), $\mathscr{S}$ has a Fréchet derivative at the zero history given by

$$S'(0)h(0) + \int_0^\infty s_{,3}(\tau, 0, 0)h(\tau)\, d\tau \tag{2.17}$$

for all histories h. We define the **relaxation function** G for $\mathscr{S}$ to be the function G on $[0, \infty)$ satisfying

$$G'(\tau) = s_{,3}(\tau, 0, 0), \quad G(0) = S'(0). \tag{2.18}$$

The stress-strain law (2.1) then has the asymptotic form

$$\sigma(t) = G(0)\varepsilon(t) + \int_0^\infty G'(\tau)\varepsilon(t-\tau)\, d\tau + o(\|\varepsilon^t\|_\infty). \tag{2.19}$$

Assume that $\mathscr{S}$ has an energy of single-integral form. If we consider the left side of (2.14) as a function $f(\varepsilon, p)$, then $f(\varepsilon, p)$ has a minimum at $\varepsilon = p = 0$, so that if s is sufficiently smooth, then $f_{,22}(0, 0) \geq 0$; thus

$$s_{,31}(\tau, 0, 0) \geq 0 \tag{2.20}$$

and, by $(2.16)_1$, the theorem has the following corollary.

COROLLARY. If a single-integral law $\mathscr{S}$ has an energy of single-integral form, then the relaxation function for $\mathscr{S}$ is convex.

This result depends crucially on the assumption that the energy be of single-integral form; convexity does not necessarily follow for energies of other types.

3. Proof of the theorem.

We begin our proof by establishing necessary and sufficient conditions for the existence of an energy of single-integral form.

LEMMA. Let $\mathscr{S}$ and $\mathscr{V}$ be single-integral laws. Then a necessary and sufficient condition that $\mathscr{V}$ be an energy for $\mathscr{S}$ is that the response pairs (S, s) for $\mathscr{S}$ and (V, v) for $\mathscr{V}$ satisfy $V(0) = 0$ and

$$\begin{gathered} S(\varepsilon) = V'(\varepsilon), \qquad s(\tau, \varepsilon, p) = v_{,2}(\tau, \varepsilon, p), \\ v_{,1}(\tau, \varepsilon, p) \leq v_{,1}(\tau, \varepsilon, \varepsilon) \end{gathered} \tag{3.1}$$

for all $\tau > 0$ and $\varepsilon, p \in \mathbf{R}$.

Remark. One of the principal conclusions of Coleman's theory [3] is that if $\mathscr{V}$ is an energy, h_1 is a constant history, and h_2 is a history with $h_2(0) = h_1$, then $\mathscr{V}(h_1) \leq \mathscr{V}(h_2)$; in other words, of all histories having the same current value, the constant history has the least energy. Within the present context this result can be

[7] An example can be obtained by combining Theorem 6.2 of Gurtin and Herrera [10] with Theorem 2 of Day [8].

obtained by integrating $(3.1)_3$ from τ to ∞ and using the counterpart of (2.9) for v to conclude that

$$v(\tau, \varepsilon, \varepsilon) \le v(\tau, \varepsilon, p) \tag{3.2}$$

for all $\tau > 0$ and $\varepsilon, p \in \mathbf{R}$.

Proof of the lemma. Note first that, since both $\mathscr{S}$ and $\mathscr{V}$ are single-integral laws, (H_1) implies that

$$s(\tau, \varepsilon, 0) = v(\tau, \varepsilon, 0) = 0 \text{ for all } \varepsilon \in \mathbf{R}, \quad \tau > 0, \tag{3.3}$$

and this with (2.13) yields $V(0) = 0$ when $\mathscr{V}$ is an energy for $\mathscr{S}$.

Let us agree to use the term **process** for a triplet $(\varepsilon, \sigma, \psi)$ with ε a strain path and σ, ψ defined on $\mathbf{R}$ by (2.11). For any process,

$$\dot{\psi}(t) - \sigma(t)\dot{\varepsilon}(t) = P(t)\dot{\varepsilon}(t) + Q(t) \tag{3.4}$$

with

$$\begin{aligned} P(t) &= V'(\varepsilon(t)) - S(\varepsilon(t)) \\ &\quad + \int_0^\infty \{v_{,2}(\tau, \varepsilon(t), \varepsilon(t-\tau)) - s(\tau, \varepsilon(t), \varepsilon(t-\tau))\}\, d\tau, \\ Q(t) &= \int_0^\infty v_{,3}(\tau, \varepsilon(t), \varepsilon(t-\tau))\dot{\varepsilon}(t-\tau)\, d\tau. \end{aligned} \tag{3.5}$$

Using the counterpart of (2.9) for v, we write

$$Q(t) = M_\delta(t) + N_\delta(t), \tag{3.6}$$

for each $\delta > 0$, where

$$M_\delta(t) = \int_0^\delta v_{,3}(\tau, \varepsilon(t), \varepsilon(t-\tau))\dot{\varepsilon}(t-\tau)\, d\tau, \tag{3.7}$$

and

$$\begin{aligned} N_\delta(t) &= \int_\delta^\infty \{v_{,1}(\tau, \varepsilon(t), \varepsilon(t-\tau)) - (d/d\tau)v(\tau, \varepsilon(t), \varepsilon(t-\tau)\}\, d\tau \\ &= \int_\delta^\infty \{v_{,1}(\tau, \varepsilon(t), \varepsilon(t-\tau)) - v_{,1}(\tau, \varepsilon(t), \varepsilon(t-\delta))\}\, d\tau. \end{aligned} \tag{3.8}$$

Assume that $\mathscr{V}$ is an energy for $\mathscr{S}$; then, by (2.12) and (3.4),

$$P(t)\dot{\varepsilon}(t) + Q(t) \le 0. \tag{3.9}$$

Given any strain path ε and any $\beta \in \mathbf{R}$ we can construct a sequence ε_n of strain paths such that (with obvious notation) $\dot{\varepsilon}_n(t) \to \beta$, $P_n(t) \to P(t)$, and $Q_n(t) \to Q(t)$ as $n \to \infty$. Thus,

$$P(t) = 0, \qquad Q(t) \le 0. \tag{3.10}$$

Let $\alpha, p \in \mathbf{R}$ be given and take $t = 0$. Then, by a limiting argument, it is clear that $P(0) = 0$ with ε given by

$$\varepsilon(\tau) = \begin{cases} 0 & \text{for } -\infty < \tau < r, \\ p & \text{for } r \le \tau < 0, \\ \alpha & \text{at } \tau = 0 \end{cases} \tag{3.11}$$

($r < 0$ arbitrary). If we take $p = 0$ and appeal to (3.3), we arrive at $(3.1)_1$. On the other hand, substituting (3.11) (with p arbitrary) into the expression $P(0) = 0$, using $(3.1)_1$, and then differentiating the resulting relation with respect to r leads to $(3.1)_2$.

To derive the remaining relation $(3.1)_3$, note that $M_\delta(t) + N_\delta(t) \leq 0$, and that $M_\delta(t) = 0$ for a process with $\dot{\varepsilon} = 0$ on $[t-\delta, t]$; hence a limiting argument applied to (3.8) yields the conclusion that the expression $N_\delta(0) \leq 0$ must hold with

$$\varepsilon(\tau) = \begin{cases} \alpha & \text{for } -\infty < \tau < r \text{ and } c \leq \tau \leq 0, \\ p & \text{for } r \leq \tau < c \end{cases} \tag{3.12}$$

whenever $r < c < -\delta < 0$. Modulo this constraint, r, c, and δ are arbitrary, and this yields $(3.1)_3$. This establishes the necessity of (3.1).

To establish sufficiency assume that (3.1) holds and choose a process. Then, in view of (3.1), (3.4), and (3.5),

$$\dot{\psi}(t) - \sigma(t)\dot{\varepsilon}(t) = Q(t). \tag{3.13}$$

By (3.8),

$$N_\delta(t) = Z_\delta(t) + W_\delta(t), \tag{3.14}$$

with

$$\begin{aligned} Z_\delta(t) &= \int_\delta^\infty \{v_{,1}(\tau, \varepsilon(t), \varepsilon(t-\tau)) - v_{,1}(\tau, \varepsilon(t), \varepsilon(t))\}\, d\tau, \\ W_\delta(t) &= \int_\delta^\infty \{v_{,1}(\tau, \varepsilon(t), \varepsilon(t)) - v_{,1}(\tau, \varepsilon(t), \varepsilon(t-\delta))\}\, d\tau. \end{aligned} \tag{3.15}$$

Fix t. Then there is a function ℓ, as specified in (H_2), such that, for all sufficiently small $\delta > 0$,

$$W_\delta(t) \leq |\varepsilon(t) - \varepsilon(t-\delta)| \int_\delta^\infty \ell(\tau)\, d\tau. \tag{3.16}$$

But ε is C^1; thus there is a constant K such that $|\varepsilon(t) - \varepsilon(t-\delta)| \leq K\delta$ for all sufficiently small $\delta > 0$, and, in view of (2.6), $W_\delta(t) \to 0$ as $\delta \to 0$. Further, $(3.1)_3$ yields $Z_\delta(t) \leq 0$, while (3.7) implies that $M_\delta(t) \to 0$ as $\delta \to 0$. It therefore follows from (3.6) that $Q(t) \leq 0$, which, by virtue of (3.13), is the desired conclusion. □

Proof of the theorem. Assume first that $\mathcal{S}$ has an energy $\mathcal{V}$ of single-integral form. Then, by the lemma, the corresponding response pair must be consistent with (3.1). By $(3.1)_2$ and (3.3), v has the form

$$v(\tau, \varepsilon, p) = \int_0^\varepsilon s(\tau, \alpha, p)\, d\alpha + f(\tau, p), \quad f(\tau, 0) = 0. \tag{3.17}$$

Further, in view of (3.2), $v_{,3}(\tau, p, p) = 0$, which, when applied to (3.17), yields

$$\begin{aligned} f(\tau, p) &= -\int_0^p \int_0^\lambda s_{,3}(\tau, \alpha, \lambda)\, d\alpha\, d\lambda \\ &= -\int_0^p \int_\alpha^p s_{,3}(\tau, \alpha, \lambda)\, d\lambda\, d\alpha \\ &= -\int_0^p \{s(\tau, \alpha, p) - s(\tau, \alpha, \alpha)\}\, d\alpha; \end{aligned} \tag{3.18}$$

hence (3.17) reduces to (2.15), and, because of $(3.1)_3$, (2.15) implies (2.14).

Thus (2.14) is necessary for the existence of an energy of single-integral form. Moreover, we have shown that if such an energy exists, then v must necessarily have the form $(2.15)_1$. Also, by the lemma, V must necessarily be given by $(2.15)_2$. Thus the energy is uniquely determined.

Conversely, assume that (2.14) is satisfied and define V and v through (2.15). Then the fact that $\mathscr{S}$ is a single-integral law implies that (V, v) is the response pair of a single-integral law $\mathscr{V}$. Moreover, (2.15) trivially implies $(3.1)_{1,2}$, while (2.14) and $(2.15)_2$ imply $(3.1)_3$; hence we may infer from the lemma that $\mathscr{V}$ is an energy for $\mathscr{S}$. □

4. Special models. For materials defined by the stress-strain law

$$\sigma(t) = S(\varepsilon(t)) + \int_0^\infty a'(\tau) s_0(\varepsilon(t), \varepsilon(t-\tau))\, d\tau \tag{4.1}$$

our results take a particularly concise form. We consider (4.1) with $S \in C^1(\mathbf{R})$, $s_0 \in C^1(\mathbf{R}^2)$, $a \in C^2(0,\infty)$, $a' \in L^1(0,\infty)$, $a'' \in L^1(1,\infty)$, a'' consistent with (2.6) with $\ell = |a''|$, $s_0(\varepsilon, 0) = 0$ for all $\varepsilon \in \mathbf{R}$, and $a(\tau)s_0(\varepsilon, p)$ not identically zero. Further, to avoid a trivial ambiguity in sign, we assume that $a''(\tau) > 0$ for at least one $\tau \in (0,\infty)$.

Then (4.1) generates a single-integral law, and, by the theorem, necessary and sufficient for the existence of an energy of single-integral form is that

$$a \text{ be convex} \tag{4.2}$$

and

$$\int_p^\varepsilon \{s_0(\alpha,\alpha) - s_0(\alpha, p)\}\, d\alpha \geq 0 \tag{4.3}$$

for all $\varepsilon, p \in \mathbf{R}$. When this is satisfied the corresponding energy will have response pair (V, v) defined by

$$\begin{gathered} V(\varepsilon) = \int_0^\varepsilon S(\lambda)\, d\lambda, \qquad v(\tau, \varepsilon, p) = a'(\tau) v_0(\varepsilon, p), \\ v_0(\varepsilon, p) = \int_p^\varepsilon s_0(\alpha, p)\, d\alpha + \int_0^p s_0(\alpha,\alpha)\, d\alpha. \end{gathered} \tag{4.4}$$

We now give the particular form our results take for several special models.[8]

In the case when (4.1) is linear, viz.

$$\sigma(t) = \beta\varepsilon(t) + \int_0^\infty a'(\tau)\varepsilon(t-\tau)\, d\tau, \tag{4.5}$$

we have $s_0(\varepsilon, p) = p$ and it is easy to see that (4.3) is satisfied. Thus there is an energy of single-integral form if and only if (4.2) holds; granted (4.2), the corresponding energy is given by

$$\begin{aligned} \psi(t) &= \frac{1}{2}\beta\varepsilon(t)^2 + \int_0^\infty a'(\tau)\left\{\frac{1}{2}\varepsilon(t-\tau)^2 + \varepsilon(t-\tau)[\varepsilon(t) - \varepsilon(t-\tau)]\right\} d\tau \\ &= \frac{1}{2}[\beta + a(\infty) - a(0)]\varepsilon(t)^2 - \frac{1}{2}\int_0^\infty a'(\tau)[\varepsilon(t) - \varepsilon(t-\tau)]^2\, d\tau. \end{aligned} \tag{4.6}$$

[8]These models have a large literature (cf., e.g., [18] and the references therein).

An analog of this energy is used by Dafermos [6] to establish asymptotic stability in three-dimensional linear viscoelasticity.

A popular nonlinear stress-strain law is

$$\sigma(t) = S(\varepsilon(t)) + \int_0^\infty a'(\tau) f(\varepsilon(t-\tau))\, d\tau \tag{4.7}$$

with $f \in C^1(\mathbf{R})$ and $f(0) = 0$. Under (4.7), the condition (4.3) is equivalent to the requirement that

$$f \text{ be monotone increasing;} \tag{4.8}$$

granted (4.2) and (4.8), the corresponding energy is given by

$$\psi(t) = V(\varepsilon(t)) + \int_0^\infty a'(\tau)\{F(\varepsilon(t-\tau)) + f(\varepsilon(t-\tau))[\varepsilon(t) - \varepsilon(t-\tau)]\}\, d\tau, \tag{4.9}$$

where

$$F(p) = \int_0^p f(\lambda)\, d\lambda. \tag{4.10}$$

Another important case of (4.1) is

$$\sigma(t) = S(\varepsilon(t)) + \int_0^\infty a'(\tau)\{f(\varepsilon(t)) - f(\varepsilon(t) - \varepsilon(t-\tau))\}\, d\tau, \tag{4.11}$$

where $f \in C^1(\mathbf{R})$ with $f(0) = 0$. For the constitutive assumption (4.11), the condition (4.3) holds if and only if

$$F(p) \geq 0 \tag{4.12}$$

for all $p \in \mathbf{R}$, where F is defined by (4.10); granted (4.2) and (4.12), the corresponding energy is given by

$$\psi(t) = V(\varepsilon(t)) + \int_0^\infty a'(\tau)\{F(\varepsilon(t)) - F(\varepsilon(t) - \varepsilon(t-\tau))\}\, d\tau. \tag{4.13}$$

We close with a simple example of a stress-strain law of the form (4.1) that has an energy of single-integral form as well as an energy of a different type. The constitutive relation

$$\sigma(t) = \beta\varepsilon(t) - \gamma \int_0^\infty e^{-\mu\tau} \varepsilon(t-\tau)\, d\tau \tag{4.14}$$

with $\beta, \gamma, \mu > 0$ describes a linear Maxwell material with one relaxation mode. It follows from the discussion of (4.5) that there is a unique energy of single-integral form:

$$\begin{aligned} \psi_1(t) &= \frac{1}{2}\beta\varepsilon(t)^2 - \gamma \int_0^\infty e^{-\mu\tau} \left\{ \frac{1}{2}\varepsilon(t-\tau)^2 + \varepsilon(t-\tau)[\varepsilon(t) - \varepsilon(t-\tau)] \right\} d\tau \\ &= \frac{1}{2}\left(\beta - \frac{\gamma}{\mu}\right)\varepsilon(t)^2 + \frac{\gamma}{2}\int_0^\infty e^{-\mu\tau}[\varepsilon(t) - \varepsilon(t-\tau)]^2\, d\tau. \end{aligned} \tag{4.15}$$

One can check by direction calculation that

$$\psi_2(t) = \frac{1}{2}\left(\beta - \frac{\gamma}{\mu}\right)\varepsilon(t)^2 + \frac{\gamma}{2\mu}\left[\varepsilon(t) - \mu \int_0^\infty e^{-\mu\tau}\varepsilon(t-\tau)\, d\tau\right]^2 \tag{4.16}$$

also provides an energy for (4.14). The energies ψ_1 and ψ_2 are not identical; it is easy to produce a strain path ε for which $\psi_1(0) \neq \psi_2(0)$. For linear Maxwell materials, energies of the form (4.16) have been studied by many authors. The case of several relaxation modes is discussed in detail by Breuer and Onat [1]; see Coleman and Mizel [4] for the case of a continuous relaxation-spectrum.

Note added in proof. For many purposes it is more convenient to normalize the response functions s and v so that

$$s(\tau, \varepsilon, \varepsilon) = v(\tau, \varepsilon, \varepsilon) = 0 \qquad (*)$$

for all $\tau > 0$ and $\varepsilon \in \mathbf{R}$. If $(*)$ is adopted in place of $s(\tau, \varepsilon, 0) = v(\tau, \varepsilon, 0) = 0$, then there is an energy of single-integral form if and only if

$$\int_p^\varepsilon s_{,1}(\tau, \alpha, p)\, d\alpha \le 0 \qquad (**)$$

for all $p, \varepsilon \in \mathbf{R}$; granted $(**)$ the energy is unique and its response pair (V, v) is given by

$$\begin{aligned} V(\varepsilon) &= \int_0^\varepsilon S(\alpha)\, d\alpha, \\ v(\tau, \varepsilon, p) &= \int_p^\varepsilon s(\tau, \alpha, p)\, d\alpha. \end{aligned} \qquad (***)$$

Acknowledgment. We gratefully acknowledge the Air Force Office of Scientific Research, the Army Research Office, and the National Science Foundation for support of this research.

References

[1] S. Breuer and E. T. Onat, *On the determination of the free energy in linear viscoelastic solids*, Z. Angew. Math. Phys. **15**, 184–191 (1964)
[2] T. A. Burton, *Volterra Integral and Differential Equations*, Academic Press, 1983
[3] B. D. Coleman, *Thermodynamics of materials with memory*, Arch. Rat. Mech. Anal. **17**, 1–46 (1964)
[4] B. D. Coleman and V. J. Mizel, *On the stability of solutions of functional-differential equations*, Arch. Rat. Mech. Anal. **30**, 173–196 (1968)
[5] B. D. Coleman and D. R. Owen, *On the thermodynamics of materials with memory*, Arch. Rat. Mech. Anal. **36**, 245–269 (1970)
[6] C. M. Dafermos, *An abstract Volterra equation with applications to linear viscoelasticity*, J. Differential Equations **7**, 554–569 (1970)
[7] W. A. Day, *A theory of thermodynamics for materials with memory*, Arch. Rat. Mech. Anal. **34**, 85–96 (1969)
[8] W. A. Day, *Restrictions on relaxation functions in linear viscoelasticity*, Quart. J. Mech. Appl. Math. **24**, 487–497 (1971)
[9] W. A. Day, *The Thermodynamics of Simple Materials with Fading Memory*, Springer, 1972
[10] M. E. Gurtin and I. Herrera, *On dissipation inequalities and linear viscoelasticity*, Quart. Appl. Math. **23**, 235–245 (1965)
[11] J. K. Hale, *Sufficient conditions for stability and instability of autonomous functional-differential equations*, J. Differential Equations **1**, 452–482 (1965)
[12] J. K. Hale, *Theory of Functional Differential Equations*, Springer, 1977
[13] W. J. Hrusa and M. Renardy, *On a class of quasilinear partial integrodifferential equations with singular kernels*, J. Differential Equations **64**, 195–220 (1986)
[14] J. J. Levin, *A nonlinear Volterra equation not of convolution type*, J. Differential Equations **4**, 176–186 (1968)
[15] J. J. Levin and J. A. Nohel, *Perturbations of a nonlinear Volterra equation*, Michigan Math. J. **12**, 431–447 (1965)

[16] V. J. Mizel and V. Trutzer, *Stochastic hereditary equations: existence and asymptotic stability*, J. Integral Equations 7, 1–72 (1984)

[17] J. W. Nunziato, E. K. Walsh, K. W. Schuler, and L. M. Barker, *Wave propagation in nonlinear viscoelastic solids*, in S. Flugge (ed.), *Handbuch der Physik* VI a/4, Springer, 1–108 (1974)

[18] M. Renardy, W. J. Hrusa, and J. A. Nohel, *Mathematical Problems in Viscoelasticity*, Longman Scientific and Technical, Essex, England and John Wiley, New York, 1987

[19] V. Volterra, *Sur la théorie mathématique des phénomènes héréditaires*, J. Math. Pures Appl. 7, 249–298 (1928)

[20] M. E. Gurtin and W. J. Hrusa, *On the thermodynamics of viscoelastic materials of single-integral type*, Forthcoming