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THE BOUNDARY LAYER DUE TO A MOVING HEATED LINE ON A HORIZONTAL SURFACE*

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Abstract. A line heat source lies on an adiabatic horizontal surface. The governing equations under Boussinesq approximation show nonexistence of horizontal boundary layers if the source is still. Boundary layer solutions exist only when the source is moving laterally on the bottom surface with a certain minimum speed. Perturbation solutions for weak heat input agree well with exact numerical integration. The velocity and temperature profiles show similarity. Nonexistence and nonuniqueness are found.

Introduction. The lateral free convection on a horizontal surface is important in, for example, the spreading of fires. The boundary layer due to a uniformly heated semi-infinite horizontal plate was studied by Stewartson [1], Gill et al [2], and others. Gill [2] showed, for a heated plate, a boundary layer exists only on the top surface. This paper investigates the existence and solutions of a related problem: the horizontal boundary layer due to a line heat source on an adiabatic surface. First we show that a horizontal boundary layer does not exist if the line heat source is still. Then we shall discuss the possible solutions if the line heat source is moving laterally.

The basic boundary layer equations derived through scaling by Stewartson [1], Rotem and Claassen [3] are

$$uu_x + vu_y = -\frac{1}{\rho_0}p_x + \nu u_{yy}, \tag{1}$$

$$0 = -p_y + \rho_0 g[1 - \beta(T - T_\infty)], \tag{2}$$

$$uT_x + vT_y = \frac{\nu}{\mathrm{Pr}}T_{yy}, \tag{3}$$

$$u_x + v_y = 0. \tag{4}$$

Here (u, v) are velocity components in directions (x, y) respectively (Fig. 1), p is the pressure, T is the temperature, ν is the kinematic viscosity, Pr is the Prandtl number, ρ_0 is the density far from the plate, and β is the coefficient of thermal expansion. Consistent with the Boussinesq approximation, buoyancy effects are felt only through the vertical pressure gradient. The gravitational acceleration g is defined as positive

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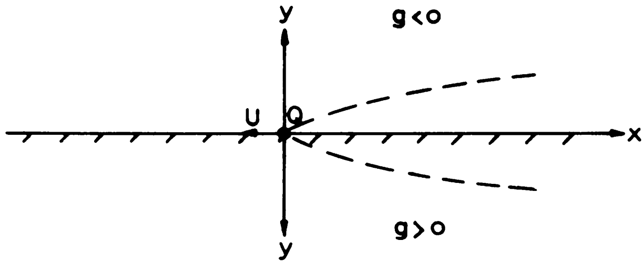

FIG. 1. The coordinate system. Heat source is at the origin on an adiabatic surface.

for the bottom surface and negative for the top surface. Without loss of generality, only the positive x-axis will be considered.

The nonmoving line heat source. In this case the fluid motion is caused solely by the lateral pressure gradient due to uneven heating. Differentiate Eq. (2) with respect to x to obtain

$$-p_{xy} = \rho_0 g\beta T_x. \tag{5}$$

Integration in y gives

$$p_x = \rho_0 g\beta \int_y^\infty T_x \, dy. \tag{6}$$

We let the heat source be at the origin and we expect $T_x < 0$. Thus, for the top surface, $g < 0$ and $p_x > 0$. Since pressure increases with x, the induced flow direction is towards the origin and a horizontal thermal boundary layer does not exist. Actually a thermal plume (vertical boundary layer) would occur along the y-axis instead.

If the line source is at the bottom of the surface, $g > 0$ and $p_x < 0$. We shall investigate further since this pressure gradient is favorable. Eliminating pressure from Eqs. (1,2) yields

$$uu_{xy} + vu_{yy} = g\beta T_x + \nu u_{yyy}. \tag{7}$$

The total heat Q across any $x = \text{const}$ surface is

$$Q = \rho C_p \int_0^\infty u(T - T_\infty) \, dy, \tag{8}$$

where C_p is the specific heat. In order that Q be constant and Eq. (7) admit similarity solutions, the only choice, excluding the constant multiples, is

$$u = \nu f'(\eta), \qquad v = -\frac{\nu}{2\sqrt{x}}(f - \eta f'), \tag{9}$$

$$T - T_\infty = \frac{\nu^2}{g\beta\sqrt{x}}\theta(\eta), \tag{10}$$

where $\eta \equiv y/\sqrt{x}$. Eq. (7) becomes

$$f'f'' + ff''' = \theta + \eta\theta' - 2f''''. \tag{11}$$

Using the fact $f'' \to 0$, $\theta \to 0$ at infinity, Eq. (11) is integrated once to give

$$ff'' = \theta\eta - 2f'''. \tag{12}$$

Now integrate from zero to ∞ to obtain

$$2f''(0) = -\int_0^\infty (f')^2 f\eta - \int_0^\infty \theta\eta\, d\eta. \tag{13}$$

On the bottom surface $g > 0$, since $T > T_\infty$; thus $\theta > 0$ from Eq. (10). Then Eq. (13) shows $f''(0) < 0$ or $\frac{\partial u}{\partial y}(0) < 0$. This is a contradiction to the $u > 0$ flow caused by the favorable pressure gradient concluded earlier. Physically the hot fluid would accumulate near the heat source at the bottom surface, instead of forming a horizontal boundary layer.

We conclude that horizontal boundary layers do not exist for a nonmoving heat source on a horizontal plate.

The moving line heat source. Let the coordinate axes be travelling with the line heat source which is moving with velocity $u = -U$. Then the plate and the fluid at infinity are moving with $u = U$ relative to the axes. A new set of similarity transformations, similar to Eqs. (9, 10), is defined:

$$u = Uf'(\eta), \qquad v = -\frac{1}{2}\sqrt{\frac{U\nu}{x}}(f - \eta f'), \tag{14}$$

$$T = T_\infty + \frac{U^2\sqrt{U}}{g\beta\sqrt{\nu}}\frac{1}{\sqrt{x}}\theta(\eta), \tag{15}$$

where

$$\eta \equiv \sqrt{\frac{U}{\nu x}}y. \tag{16}$$

Eqs. (7,3) become

$$ff'' = \theta\eta - 2f''', \tag{17}$$

$$\theta' + \frac{\mathrm{Pr}}{2}f\theta = 0. \tag{18}$$

The boundary conditions are now different:

$$f(0) = \theta'(0) = 0, \qquad f'(0) = 1, \tag{19}$$

$$f'(\infty) = 1, \qquad \theta(\infty) = 0. \tag{20}$$

The total heat flux is then

$$Q = \frac{\rho C_p U^3}{g\beta}\int_0^\infty f'\theta\, d\eta. \tag{21}$$

Since fluid motion can be caused by momentum input and is not solely due to pressure gradient, arguments similar to the nonmoving heat source cannot be used here. We shall seek solutions to Eqs. (17-20) and then discuss their applicability.

Perturbation solution for weak heat input. We expect the normalized temperature difference θ is small. Let its maximum value be $\theta(0) = \varepsilon$, $|\varepsilon| \ll 1$. Perturb about uniform flow as follows:

$$f = \eta + \varepsilon f_1 + O(\varepsilon^2), \tag{22}$$

$$\theta = \varepsilon\theta_1 + O(\varepsilon^2). \tag{23}$$

Eqs. (17-20) give

$$2f_1''' + \eta f_1'' = \eta\theta_1, \tag{24}$$

$$\theta_1' + \frac{\mathrm{Pr}}{2}\eta\theta_1 = 0, \tag{25}$$

$$f_1(0) = \theta_1'(0) = f_1'(0) = 0, \tag{26}$$

$$f_1'(\infty) = \theta_1(\infty) = 0, \tag{27}$$

$$\theta_1(0) = 1. \tag{28}$$

Eqs. (25, 27, 28) give

$$\theta_1 = \exp(-\mathrm{Pr}\,\eta^2/4). \tag{29}$$

After some work, Eqs. (24, 26, 27) yield, for $\mathrm{Pr} \neq 1$,

$$f_1 = \frac{1}{(\mathrm{Pr}-1)}\sqrt{\frac{\pi}{\mathrm{Pr}}}\left\{\frac{2}{\sqrt{\mathrm{Pr}}}\left[\zeta\,\mathrm{erfc}\,\zeta - \frac{1}{\sqrt{\pi}}(e^{-\zeta^2}-1)\right] - 2\left[\frac{\eta}{2}\,\mathrm{erfc}\left(\frac{\eta}{2}\right) - \frac{1}{\sqrt{\pi}}(e^{-\eta^2/4}-1)\right]\right\} \tag{30}$$

where $\zeta \equiv \sqrt{\mathrm{Pr}}\,\eta/2$ and erfc is the complementary error function. For $\mathrm{Pr} = 1$,

$$f_1 = e^{-\eta^2/4} - 1. \tag{31}$$

The heat input is then

$$\frac{Qg\beta}{\rho C_p U^3} = \int_0^\infty \varepsilon\theta_1\,d\eta + O(\varepsilon^2) = \varepsilon\sqrt{\frac{\pi}{\mathrm{Pr}}} + O(\varepsilon^2). \tag{32}$$

This links ε to Q. The horizontal velocity is

$$\begin{aligned}\frac{u}{U} &= 1 - \varepsilon\frac{1}{(1-\mathrm{Pr})}\sqrt{\frac{\pi}{\mathrm{Pr}}}\left(\mathrm{erfc}\,\zeta - \mathrm{erfc}\,\frac{\eta}{2}\right) + O(\varepsilon^2), \quad \mathrm{Pr} \neq 1,\\ &= 1 - \varepsilon\frac{\eta}{2}e^{-\eta^2/4} + O(\varepsilon^2), \quad \mathrm{Pr} = 1.\end{aligned} \tag{33}$$

For the top surface $g < 0$, $\theta_1 > 0$, $\varepsilon < 0$, the velocity is increased. The opposite is true for the bottom surface. The induced vertical velocity at infinity is

$$\frac{v}{U} = -\sqrt{\frac{\nu}{Ux}}\frac{1}{2}(f - \eta f')|_{\eta\to\infty} = \sqrt{\frac{\nu}{Ux}}\frac{\varepsilon}{\mathrm{Pr}(1+\sqrt{\mathrm{Pr}})} + O(\varepsilon^2). \tag{34}$$

The surface shear is

$$\tau = \rho_0\nu\left.\frac{\partial u}{\partial y}\right|_0 = \rho_0 U\sqrt{\frac{U\nu}{x}}f''(0), \tag{35}$$

where

$$f''(0) = -\varepsilon\frac{1}{\sqrt{\mathrm{Pr}}(1+\sqrt{\mathrm{Pr}})} + O(\varepsilon^2). \tag{36}$$

Numerical solutions. Numerical integration is necessary if the heat input is not small. For given Prandtl number and a guessed $\theta(0)$ and $f''(0)$, Eqs. (17-19) are integrated as an initial value problem by the fifth-order Runge-Kutta-Fehlberg algorithm. The value of $f''(0)$ is adjusted such that $f'(\infty) = 1$ is satisfied. Eq. (18) shows θ always decays monotonically to zero for $f > 0$. Eq. (17) shows, for large η,

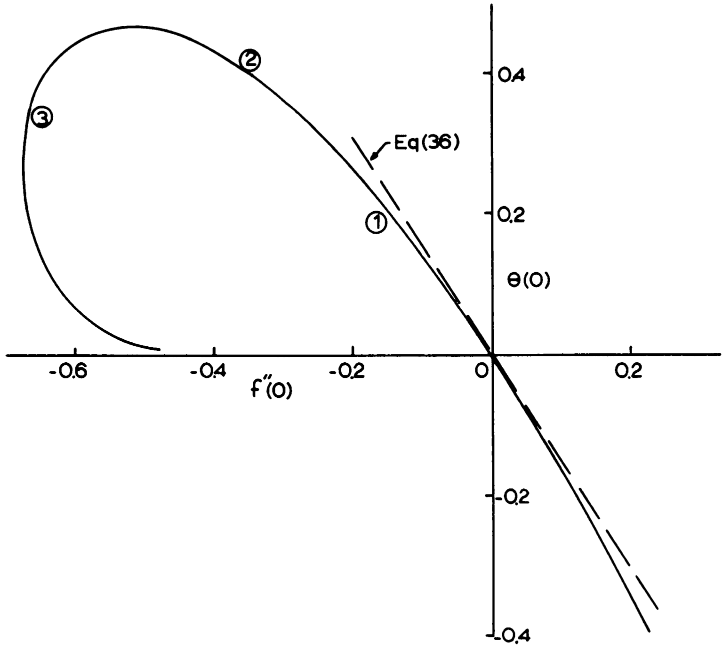

FIG. 2. Locus of initial values, Pr=0.7. Circled numbers refer to solutions depicted in Fig. 7.

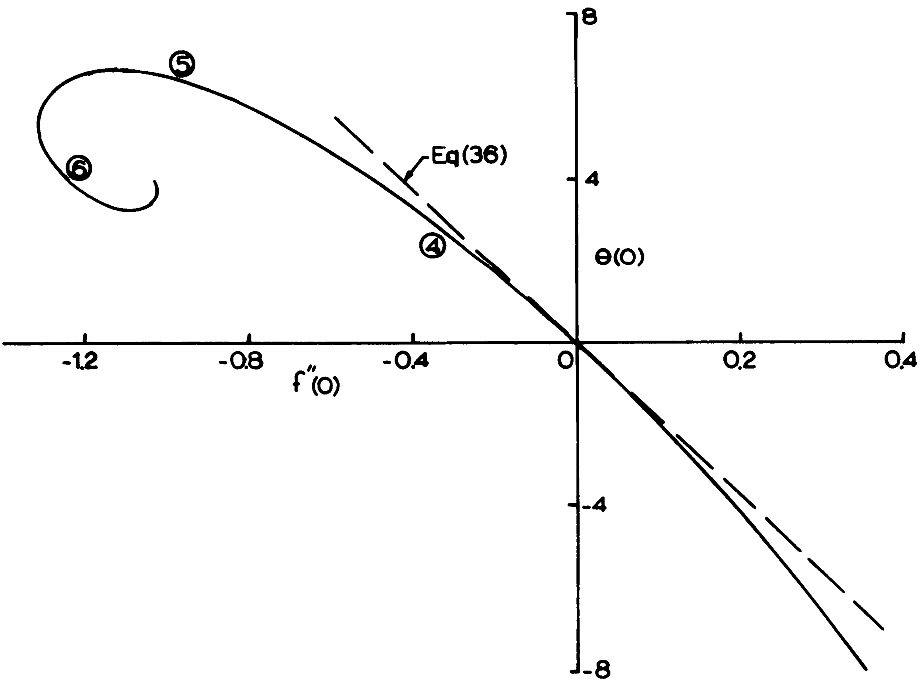

FIG. 3. Locus of initial values, Pr=7. Circled numbers refer to solutions depicted in Fig. 10.

the function f has a particular solution which decays as θ, a constant term, a linear term, and another exponential decay solution. Stability of the shooting scheme is assured since there are no exponentially increasing solutions.

The result for Pr = 0.7 (air) is shown in Fig. 2. $\theta(0) > 0$ represents the lower surface and $\theta(0) < 0$ represents the upper surface. Our perturbation results compare well with numerical integration near the origin where both $\theta(0)$ and $f''(0)$ are small. For $\theta(0)$ large and negative, $f''(0)$ is large and positive. As seen from Fig. 2, maximum $\theta(0)$ and minimum $f''(0)$ are bounded. Their significance will be discussed later. A similar locus of initial values is shown in Fig. 3 for Pr = 7 (water).

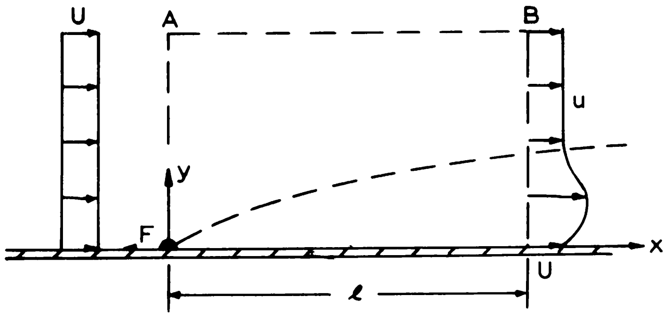

FIG. 4. Control volume.

Now a mathematical solution does not necessarily imply physical existence. Consider a segment of length ℓ of the boundary layer (Fig. 4). The points AB are well outside the boundary layer at $y \to \infty$. Since Q is constant, energy is conserved. A mass balance gives

$$\text{mass into AB} = \int_0^\infty \rho_0(u - U)\,dy. \tag{37}$$

Force balance in the x-direction gives

$$\begin{aligned} &-(\text{force on line heat source}) - (\text{drag on surface}) \\ &\quad = (\text{momentum out}) - (\text{momentum in}), \end{aligned} \tag{38}$$

or

$$\begin{aligned} -F - \int_0^\ell \rho_0 \nu \frac{\partial u}{\partial y}\bigg|_{y=0} dx &= \int_0^\infty \rho_0 u^2\Big|_{x=\ell}\, dy \\ &- \left[\int_0^\infty \rho_0 U^2\,dy + \int_0^\infty \rho_0(u|_{x=\ell} - U)U\,dy\right]. \end{aligned} \tag{39}$$

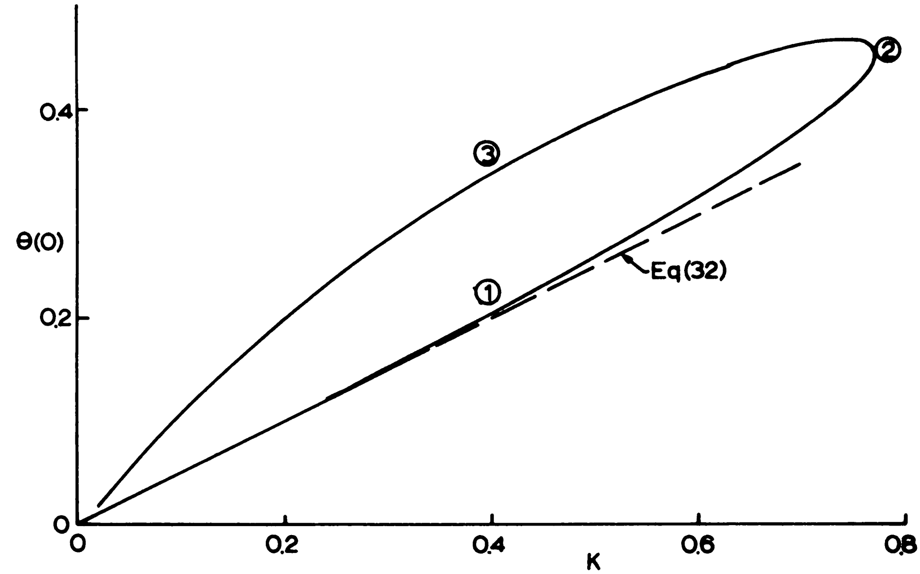

FIG. 5. Normalized maximum temperature as a function of K, Pr=0.7.

Thus

$$\begin{aligned}\frac{F}{\rho_0 U\sqrt{U\nu\ell}} &= -2f''(0) - \int_0^\infty f'(f'-1)\,d\eta \\ &= -2f''(0) - f(f'-1)|_0^\infty + \int_0^\infty ff''\,d\eta \\ &= -2f''(0) + \int_0^\infty \theta\eta\,d\eta - 2f''|_0^\infty \\ &= \int_0^\infty \theta\eta\,d\eta.\end{aligned} \tag{40}$$

Due to the second law of thermodynamics, any moving object in an otherwise still fluid must experience positive drag. Thus the force on line heat source F must be positive. From Eq. (40), $\theta > 0$; thus only the boundary layer on the bottom surface exists!

Let K be the normalized heat input $Qg\beta/(\rho C_p U^3)$, which is the governing parameter of the problem. After θ and f are solved, K can be integrated from Eq. (21). The normalized maximum temperature $\theta(0)$ for the lower surface is plotted against K in Fig. 5 for Pr $= 0.7$. We see that two solutions exist for $0 < K < 0.77$. It is not possible to determine which solution would occur in practice since a stability analysis has not been done. However, the solution with the lower temperature $\theta(0)$ is more likely to happen since this branch is asymptotic to the perturbation solution about uniform flow. For $K > 0.77$, solutions do not exist. This means the heat input Q is too large, or speed U too small, to generate a boundary layer. This result agrees with the previous conclusions where we showed the nonexistence of a boundary layer when the heat source is nonmoving ($U = 0$). For given Q sufficient lateral velocity is needed to "spread" the heated fluid into a boundary layer.

Fig. 6 shows the normalized force on the moving heat source $G \equiv F/(\rho_0 U\sqrt{U\nu\ell})$ as a function of K. The dashed line is the relation for small $\theta(0)$,

$$K = \frac{\sqrt{\pi\,\mathrm{Pr}}}{2}G + O(\varepsilon). \tag{41}$$

For given K, the lower branch shows positive slope, i.e., the force experienced by the line source increases with the heat input. The upper branch, however, has a segment of negative slope which is unlikely to happen, although we are not able to decisively exclude these solutions.

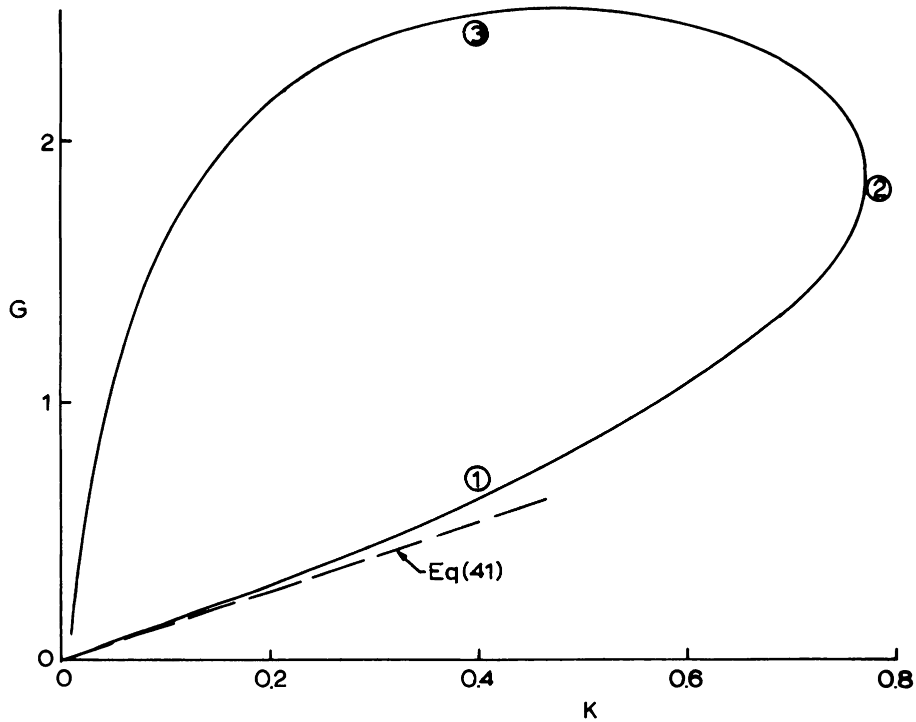

FIG. 6. Normalized force as a function of K, Pr=0.7.

The velocity and temperature profiles are shown in Fig. 7 for Pr = 0.7. State ①, on the lower branch of Figs. 5, 6, is at $K = 0.4$. State ② is at maximum $K = 0.77$. State ③ is on the upper branch also with $K = 0.4$. The temperature is maximum on the plate at $\eta = 0$, then decays to zero as η is increased. The velocity is unity on the plate, decreases in the boundary layer, then increases to one as free stream is approached. Although State ③ has the same K value as State ①, the boundary layer thickness is twice as large. Both temperature and shear are higher.

The situation is somewhat different for Pr = 7. Figure 8 shows there is one solution when $0 < K < 1.35$, three solutions for $1.35 < K < 1.5$, two solutions for $1.5 < K < 3.31$, and no solution for $K > 3.31$. Both maximum temperature $\theta(0)$ and maximum K value (above which solutions cease to exist) increase with Prandtl number. The boundary layer thickness increases along the curve passing through

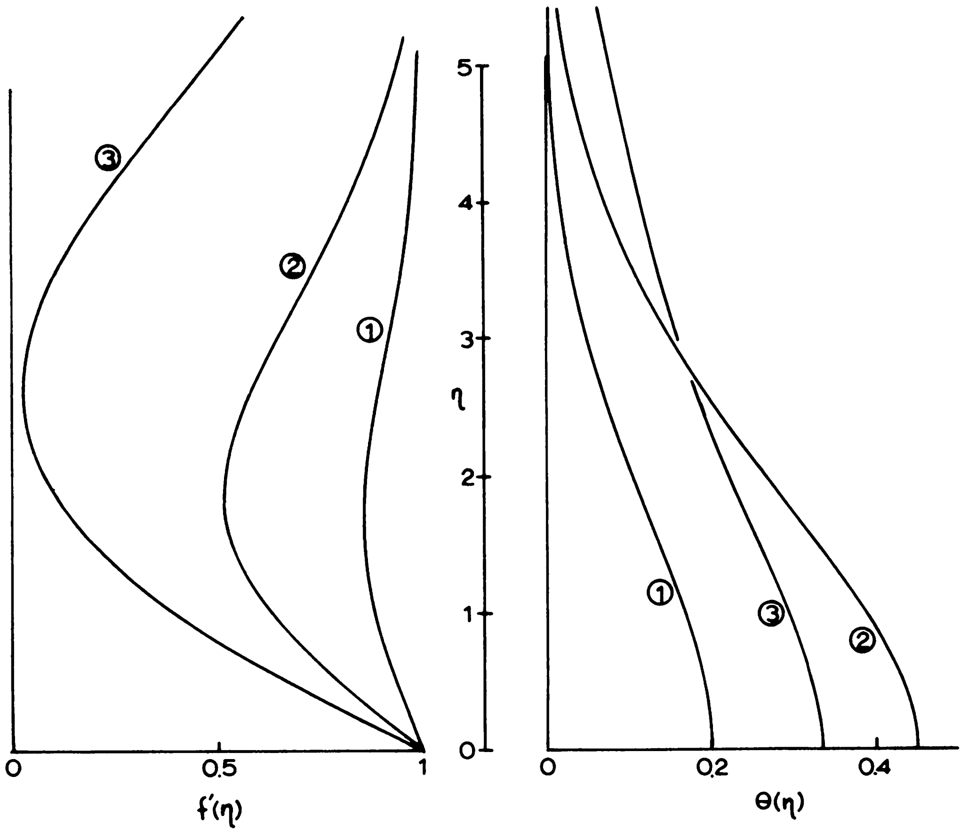

FIG. 7. Temperature and velocity profiles ① $K = 0.4$, ② $K = 0.77$, ③ $K = 0.4$, Pr=0.7.

States ④, ⑤, ⑥, ⑦. At State ⑦ the boundary layer becomes too thick ($\eta \sim 40$) for the solution to be meaningful. Figure 9 shows the normalized force and Fig. 10 shows the temperature and velocity profiles. Comparing Figs. 10 and 7, we see increased Prandtl number in general decreases the boundary layer thickness.

Discussion. The boundary layer caused by a heated plate [2] is distinctly different from that due to a heated line source studied in this paper. The boundary layer of the former exists only on the top surface while the boundary layer of the latter exists on the bottom of the surface, and only when the heat source is moving with some speed. If a cold source is substituted for a heat source then the results of the present paper apply to the top surface instead.

Equation (40) shows the force on the source is proportional to $\sqrt{\ell}$. Thus the length of the boundary layer ℓ can not be infinite. We conjecture that the boundary layer breaks down after a finite length. Similar phenomena were noted in the experiments of a heated plate [3].

In conclusion, the simple governing equations of the present problem (Eqs. 17-20) belie their complexity. We have found nonexistence, nonuniqueness, and possible instability in this interesting nonlinear problem.

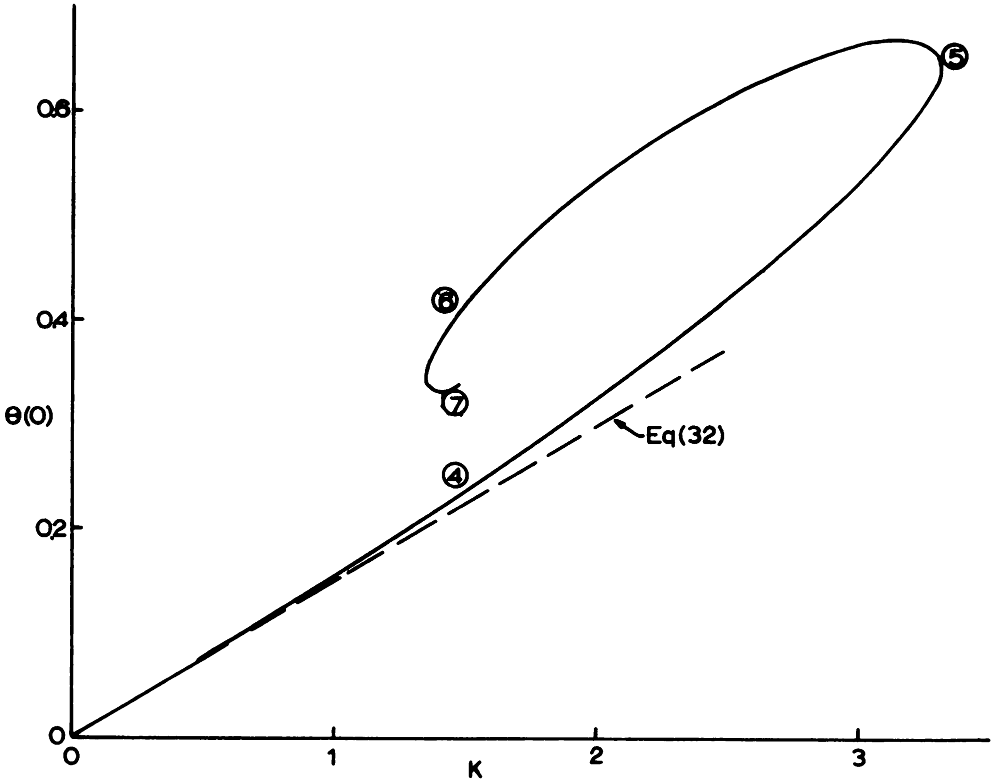

FIG. 8. Normalized maximum temperature as a function of K, Pr=7.

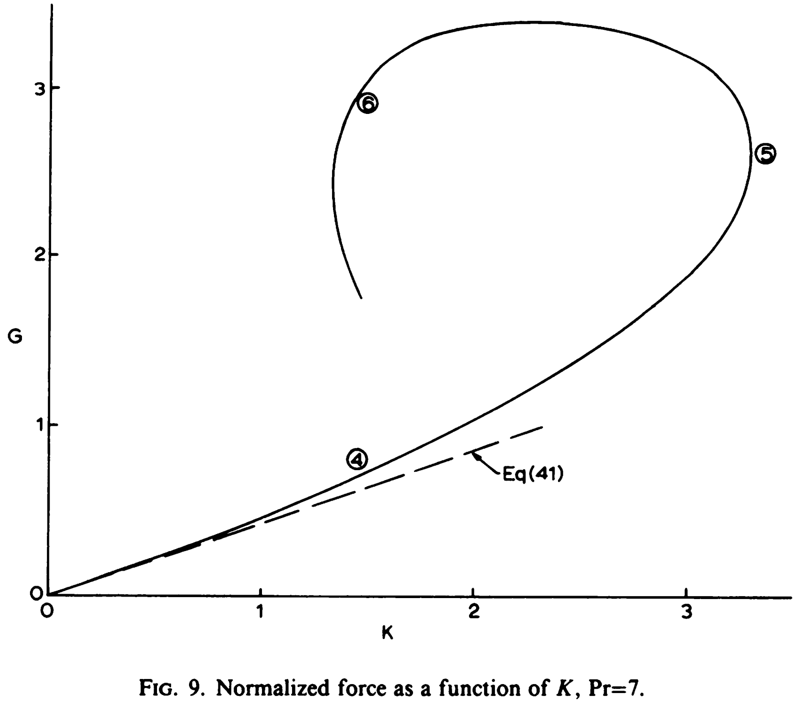

FIG. 9. Normalized force as a function of K, Pr=7.

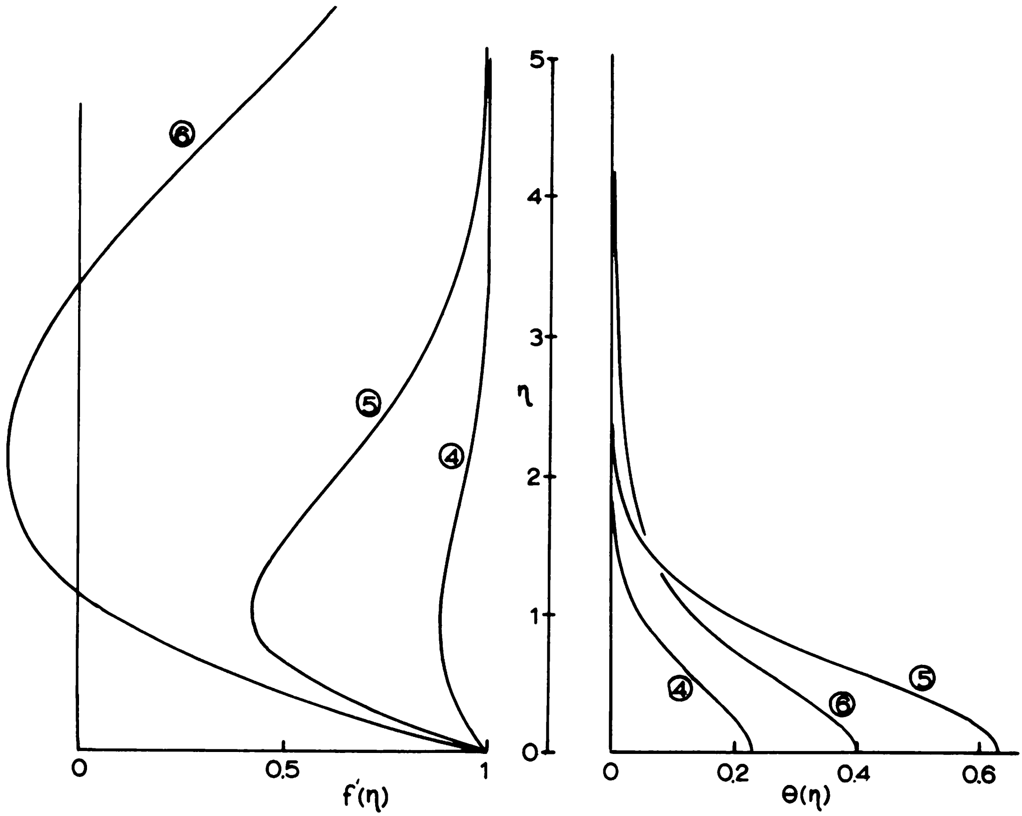

FIG. 10. Temperature and velocity profiles ④ $K = 1.46$, ⑤ $K = 3.31$, ⑥ $K = 1.46$, Pr=7.

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