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SOME IDENTITIES AND THE STRUCTURE OF $\mathbf{N}_i$ IN THE STROH FORMALISM OF ANISOTROPIC ELASTICITY*

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Abstract. The Stroh formalism of anisotropic elasticity leads to a 6×6 real matrix $\mathbf{N}$ that can be composed from three 3×3 real matrices $\mathbf{N}_i$ ($i = 1, 2, 3$). The eigenvalues and eigenvectors of $\mathbf{N}$ are all complex. New identities are derived that express certain combinations of the eigenvalues and eigenvectors in terms of the real matrices $\mathbf{N}_i$ and the three real matrices $\mathbf{H}$, $\mathbf{S}$, $\mathbf{L}$ introduced by Barnett and Lothe. It is shown that the elements of $\mathbf{N}_1$ and $\mathbf{N}_3$ have simple expressions in terms of the reduced elastic compliances. We prove that $-\mathbf{N}_3$ is positive semidefinite and, with this property, we present a direct proof that $\mathbf{L}$ is positive definite.

1. Introduction. In a fixed rectangular coordinate system x_i, let the stress σ_{ij} and strain ε_{ij} of an anisotropic elastic material be related by

$$\sigma_{ij} = C_{ijks}\varepsilon_{ks}, \tag{1.1}$$

in which repeated indices imply summation and C_{ijks} are elasticity constants having the symmetry property

$$C_{ijks} = C_{jiks} = C_{ijsk} = C_{ksij}. \tag{1.2}$$

If u_k are the displacement components, the strain-displacement and equilibrium equations are

$$\varepsilon_{ks} = \tfrac{1}{2}(u_{k,s} + u_{s,k}), \tag{1.3}$$

$$\sigma_{ij,j} = 0, \tag{1.4}$$

where a comma stands for partial differentiation.

Consider a two-dimensional deformation in which u_k ($k = 1, 2, 3$) depends on x_1 and x_2 only. The general solution has the form [1,2]

$$u_k = a_k f(z), \tag{1.5}$$

$$z = x_1 + p x_2, \tag{1.6}$$

where p and a_k are, respectively, the eigenvalue and eigenvector of the elasticity constants to be determined and f is an arbitrary function of its argument. One then obtains from (1.1) and (1.3)

$$\sigma_{ij} = (C_{ijk1} + pC_{ijk2})a_k f'(z), \tag{1.7}$$

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where a prime denotes differentiation with its argument. Equation (1.4) now leads to, in matrix notation,

$$\{\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p^2\mathbf{T}\}\mathbf{a} = \mathbf{0} \tag{1.8}$$

in which the superscript T denotes the transpose and the 3×3 matrices **Q**, **R**, **T** are

$$\left.\begin{aligned} Q_{ik} &= C_{i1k1}, \\ R_{ik} &= C_{i1k2}, \\ T_{ik} &= C_{i2k2}. \end{aligned}\right\} \tag{1.9}$$

We note that **Q** and **T** are symmetric and, subject to the positiveness of strain energy, are positive definite. Equation (1.8) provides the eigenvalues p and the associated eigenvectors **a**.

Introducing a new vector

$$\mathbf{b} = (\mathbf{R}^{\mathrm{T}} + p\mathbf{T})\mathbf{a} = -\frac{1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a}, \tag{1.10}$$

where the second equality comes from (1.8), (1.7) for $i = 1$ and 2 can be written as

$$\sigma_{i1} = -\phi_{i,2}, \qquad \sigma_{i2} = \phi_{i,1}, \tag{1.11}$$

where ϕ is the stress function given by

$$\phi = \mathbf{b}f(z). \tag{1.12}$$

The two equations in (1.10) can be rewritten in the following standard eigenvalue problem

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}, \tag{1.13}$$

in which [2, 3, 4]

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix}, \qquad \boldsymbol{\xi} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \tag{1.14}$$

$$\left.\begin{aligned} &\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, \qquad \mathbf{N}_2 = \mathbf{T}^{-1} = \mathbf{N}_2^{\mathrm{T}}, \\ &\mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q} = \mathbf{N}_3^{\mathrm{T}}. \end{aligned}\right\} \tag{1.15}$$

In view of the symmetry of **Q** and **T**, we see that $\mathbf{N}_2$ and $\mathbf{N}_3$ are symmetric. If we introduce the 6×6 matrix **J**,

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \tag{1.16}$$

where **I** is the identity matrix, we have

$$\mathbf{J}\mathbf{N} = (\mathbf{J}\mathbf{N})^{\mathrm{T}} = \mathbf{N}^{\mathrm{T}}\mathbf{J}. \tag{1.17}$$

In this paper, we will present new identities which relate p, **a**, **b**, and $\mathbf{N}_i$ $(i = 1, 2, 3)$. We will show that $\mathbf{N}_1$ and $\mathbf{N}_3$ have simple structures whose elements can be expressed in terms of the elastic compliance S_{ijks}, which is the inverse of C_{ijks}. With the property that $-\mathbf{N}_3$ is positive semidefinite in any reference coordinate which is rotated about the x_3-axis, we present a direct proof that the real matrix **L** introduced by Barnett and Lothe [5] is positive definite.

2. The orthogonality relations. There are six eigenvalues and six eigenvectors from (1.13). Since p cannot be real if the strain energy is positive [6], we have three pairs of complex conjugates for p. If P_α $(\alpha = 1, 2, \dots, 6)$ are the eigenvalues and $\boldsymbol{\xi}_\alpha$ $(\alpha = 1, 2, \dots, 6)$ are the associated eigenvectors, we let

$$\left.\begin{aligned} P_{\alpha+3} &= \overline{P}_\alpha, && \operatorname{Im} p_\alpha > 0, \\ \boldsymbol{\xi}_{\alpha+3} &= \overline{\boldsymbol{\xi}}_\alpha && (\alpha = 1, 2, 3), \end{aligned}\right\} \tag{2.1}$$

where an over bar denotes the complex conjugate. Unless stated otherwise, we assume that p_α are distinct so that $\boldsymbol{\xi}_\alpha$ span a six-dimensional space.

The left eigenvectors $\boldsymbol{\eta}_\alpha$ satisfy the equation

$$\mathbf{N}^{\mathrm{T}}\boldsymbol{\eta} = p\boldsymbol{\eta}, \tag{2.2}$$

and are bi-orthogonal to $\boldsymbol{\xi}_\alpha$. If we multiply (1.13) by $\mathbf{J}$ and use (1.17), we have

$$\mathbf{N}^{\mathrm{T}}(\mathbf{J}\boldsymbol{\xi}) = p(\mathbf{J}\boldsymbol{\xi}). \tag{2.3}$$

We may therefore take [4,7]

$$\boldsymbol{\eta} = \mathbf{J}\boldsymbol{\xi}, \tag{2.4}$$

and normalize the $\boldsymbol{\xi}_\alpha$ so that the bi-orthogonality relation takes the form

$$\boldsymbol{\xi}_\alpha^{\mathrm{T}}\mathbf{J}\boldsymbol{\xi}_\beta = \delta_{\alpha\beta}, \tag{2.5}$$

in which $\delta_{\alpha\beta}$ is the Kronecker delta.

Introducing the 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \qquad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3], \tag{2.6}$$

and the 6×6 matrix $\mathbf{U}$ by

$$\mathbf{U} = [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \overline{\boldsymbol{\xi}}_1, \overline{\boldsymbol{\xi}}_2, \overline{\boldsymbol{\xi}}_3] = \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix}, \tag{2.7}$$

the orthogonality relation (2.5) can be written as

$$\mathbf{U}^{\mathrm{T}}\mathbf{J}\mathbf{U} = \mathbf{I}. \tag{2.8}$$

Expanding the matrix product on the left using (1.16) and (2.7) leads to

$$\mathbf{A}^{\mathrm{T}}\mathbf{B} + \mathbf{B}^{\mathrm{T}}\mathbf{A} = \mathbf{I} = \overline{\mathbf{A}}^{\mathrm{T}}\overline{\mathbf{B}} + \overline{\mathbf{B}}^{\mathrm{T}}\overline{\mathbf{A}}, \tag{2.9a}$$

$$\mathbf{A}^{\mathrm{T}}\overline{\mathbf{B}} + \mathbf{B}^{\mathrm{T}}\overline{\mathbf{A}} = \mathbf{0} = \overline{\mathbf{A}}^{\mathrm{T}}\mathbf{B} + \overline{\mathbf{B}}^{\mathrm{T}}\mathbf{A}. \tag{2.9b}$$

Equation (2.8) implies that $\mathbf{U}^{\mathrm{T}}$ and $\mathbf{J}\mathbf{U}$ are the inverse of each other and hence the order of the product can be interchanged. We have

$$\mathbf{J}\mathbf{U}\mathbf{U}^{\mathrm{T}} = \mathbf{I}, \tag{2.10}$$

or

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} + \overline{\mathbf{A}}\,\overline{\mathbf{A}}^{\mathrm{T}} = \mathbf{0} = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \overline{\mathbf{B}}\,\overline{\mathbf{B}}^{\mathrm{T}}, \tag{2.11a}$$

$$\mathbf{B}\mathbf{A}^{\mathrm{T}} + \overline{\mathbf{B}}\,\overline{\mathbf{A}}^{\mathrm{T}} = \mathbf{I} = \mathbf{A}\mathbf{B}^{\mathrm{T}} + \overline{\mathbf{A}}\,\overline{\mathbf{B}}^{\mathrm{T}}. \tag{2.11b}$$

Equations (2.11a) imply that $\mathbf{AA}^{\mathrm{T}}$ and $\mathbf{BB}^{\mathrm{T}}$ are purely imaginary. Hence let

$$\mathbf{H} = 2i\mathbf{AA}^{\mathrm{T}} = \mathbf{H}^{\mathrm{T}}, \tag{2.12}$$

$$\mathbf{L} = -2i\mathbf{BB}^{\mathrm{T}} = \mathbf{L}^{\mathrm{T}}, \tag{2.13}$$

where $\mathbf{H}$ and $\mathbf{L}$ are real and symmetric. Equations (2.11b) imply that

$$\mathbf{AB}^{\mathrm{T}} = \tfrac{1}{2}(\mathbf{I} - i\mathbf{S}), \qquad \mathbf{S} = i(2\mathbf{AB}^{\mathrm{T}} - \mathbf{I}), \tag{2.14}$$

where $\mathbf{S}$ is real. (In Secs. 4 and 6, $\mathbf{S}$ is used for the elastic compliance.) The three real matrices $\mathbf{H}$, $\mathbf{L}$, $\mathbf{S}$, first introduced by Barnett and Lothe [5], can be shown to have simple structure when referred to a suitably chosen basis [8]. Moreover, $\mathbf{H}$ and $\mathbf{L}$ are positive definite, if the strain energy is positive [4].

From the relations

$$\mathbf{BA}^{-1} = (\mathbf{AB}^{\mathrm{T}})^{\mathrm{T}}(\mathbf{AA}^{\mathrm{T}})^{-1}, \qquad \mathbf{AB}^{-1} = (\mathbf{AB}^{\mathrm{T}})(\mathbf{BB}^{\mathrm{T}})^{-1},$$

and (2.12) to (2.14), we obtain

$$\mathbf{BA}^{-1} = i\mathbf{M}, \qquad \mathbf{AB}^{-1} = -i\mathbf{M}^{-1}, \tag{2.15}$$

where $\mathbf{M}$ is the impedance matrix [9]

$$\mathbf{M} = \mathbf{H}^{-1}(\mathbf{I} + i\mathbf{S}) = (\mathbf{I} - i\mathbf{S}^{\mathrm{T}})\mathbf{H}^{-1}, \tag{2.16a}$$

$$\mathbf{M}^{-1} = \mathbf{L}^{-1}(\mathbf{I} + i\mathbf{S}^{\mathrm{T}}) = (\mathbf{I} - i\mathbf{S})\mathbf{L}^{-1}. \tag{2.16b}$$

The second equalities in (2.16) come from the fact that $\mathbf{H}^{-1}\mathbf{S}$ and $\mathbf{SL}^{-1}$ are antisymmetric [10, 4, 7]. Hence $\mathbf{M}$ is a Hermitian matrix.

Before we close this section we write the general solution for $\mathbf{u}$ and ϕ obtained from (1.5) and (1.12) as

$$\mathbf{u} = \sum_{\alpha=1}^{3}\{\mathbf{a}_\alpha f_\alpha(z_\alpha) + \overline{\mathbf{a}_\alpha} f_{\alpha+3}(\overline{z}_\alpha)\}, \tag{2.17a}$$

$$\phi = \sum_{\alpha=1}^{3}\{\mathbf{b}_\alpha f_\alpha(z_\alpha) + \overline{\mathbf{b}}_\alpha f_{\alpha+3}(\overline{z}_\alpha)\}, \tag{2.17b}$$

in which $f_1, f_2, \ldots, f_6$ are arbitrary functions of their argument and

$$z_\alpha = x_1 + p_\alpha x_2. \tag{2.18}$$

If we define the 3×1 matrices $\mathbf{t}_1$ and $\mathbf{t}_2$ by

$$\mathbf{t}_1 = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix}, \qquad \mathbf{t}_2 = \begin{bmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{bmatrix}, \tag{2.19}$$

the stresses are, by (1.11),

$$\mathbf{t}_1 = -\phi_{,2}, \qquad \mathbf{t}_2 = \phi_{,1}. \tag{2.20}$$

We see that $\mathbf{t}_1$ and $\mathbf{t}_2$ are, respectively, the surface traction on the x_1 = constant plane and x_2 = constant plane.

3. Identities involving the eigenvalues p. If we write the inverse of $\mathbf{N}$ as

$$\mathbf{N}^{-1} = \begin{bmatrix} \mathbf{N}_1^{(-1)} & \mathbf{N}_2^{(-1)} \\ \mathbf{N}_3^{(-1)} & \mathbf{N}_1^{(-1)\mathrm{T}} \end{bmatrix}, \tag{3.1}$$

it can be shown that [4]

$$\left.\begin{aligned} \mathbf{N}_1^{(-1)} &= -\mathbf{Q}^{-1}\mathbf{R}, \qquad \mathbf{N}_2^{(-1)} = -\mathbf{Q}^{-1} = (\mathbf{N}_2^{(-1)})^{\mathrm{T}}, \\ \mathbf{N}_3^{(-1)} &= \mathbf{T} - \mathbf{R}^{\mathrm{T}}\mathbf{Q}^{-1}\mathbf{R} = (\mathbf{N}_3^{(-1)})^{\mathrm{T}}. \end{aligned}\right\} \tag{3.2}$$

From (1.13) we see that for any integer n, positive or negative,

$$\mathbf{N}^n \boldsymbol{\xi} = p^n \boldsymbol{\xi}. \tag{3.3}$$

Using the notation of (1.14) and (2.6), we have

$$\mathbf{N}^n \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{P}^n \\ \mathbf{B}\mathbf{P}^n \end{bmatrix}, \tag{3.4}$$

in which $\mathbf{P}$ is the diagonal matrix

$$\mathbf{P} = \mathrm{diag}[p_1, p_2, p_3]. \tag{3.5}$$

If we post-multiply (3.4) by $[\mathbf{A}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}}]$ we obtain

$$\mathbf{N}^n \begin{bmatrix} \mathbf{A}\mathbf{A}^{\mathrm{T}} & \mathbf{A}\mathbf{B}^{\mathrm{T}} \\ \mathbf{B}\mathbf{A}^{\mathrm{T}} & \mathbf{B}\mathbf{B}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{P}^n\mathbf{A}^{\mathrm{T}} & \mathbf{A}\mathbf{P}^n\mathbf{B}^{\mathrm{T}} \\ \mathbf{B}\mathbf{P}^n\mathbf{A}^{\mathrm{T}} & \mathbf{B}\mathbf{P}^n\mathbf{B}^{\mathrm{T}} \end{bmatrix}. \tag{3.6}$$

It is readily shown from (1.17) that

$$\mathbf{J}\mathbf{N}^n = (\mathbf{J}\mathbf{N}^n)^{\mathrm{T}}. \tag{3.7}$$

Therefore, if $\mathbf{N}^n$ is written as

$$\mathbf{N}^n = \begin{bmatrix} \mathbf{N}_1^{(n)} & \mathbf{N}_2^{(n)} \\ \mathbf{N}_3^{(n)} & \mathbf{N}_1^{(n)\mathrm{T}} \end{bmatrix}, \tag{3.8}$$

we have

$$\left.\begin{aligned} \mathbf{N}_2^{(n)} &= \mathbf{N}_2^{(n)\mathrm{T}}, \qquad \mathbf{N}_3^{(n)} = \mathbf{N}_3^{(n)\mathrm{T}}, \\ \mathbf{N}_1^{(0)} &= \mathbf{I}, \qquad \mathbf{N}_2^{(0)} = \mathbf{N}_3^{(0)} = \mathbf{0}. \end{aligned}\right\} \tag{3.9}$$

Using (2.12), (2.13), (2.14), and (3.8) on the left-hand side of (3.6) leads to the following identities for any integer n:

$$\left.\begin{aligned} \mathbf{A}\mathbf{P}^n\mathbf{A}^{\mathrm{T}} &= \tfrac{1}{2}(\mathbf{N}_2^{(n)}\mathbf{M} - i\mathbf{N}_1^{(n)})\mathbf{H}, \\ \mathbf{A}\mathbf{P}^n\mathbf{B}^{\mathrm{T}} &= \tfrac{1}{2}(\mathbf{N}_1^{(n)}\mathbf{M}^{-1} + i\mathbf{N}_2^{(n)})\mathbf{L}, \\ \mathbf{B}\mathbf{P}^n\mathbf{A}^{\mathrm{T}} &= \tfrac{1}{2}(\mathbf{N}_1^{(n)\mathrm{T}}\mathbf{M} - i\mathbf{N}_3^{(n)})\mathbf{H}, \\ \mathbf{B}\mathbf{P}^n\mathbf{B}^{\mathrm{T}} &= \tfrac{1}{2}(\mathbf{N}_3^{(n)}\mathbf{M}^{-1} + i\mathbf{N}_1^{(n)\mathrm{T}})\mathbf{L}, \end{aligned}\right\} \tag{3.10}$$

in which $\mathbf{M}$ is defined in (2.16).

Since the 6×6 matrix on the right-hand side of (3.6) is symmetric, the product of the two 6×6 matrices on the left-hand side must also be symmetric. This property leads to, using (3.7) and (2.12) through (2.14),

$$\begin{bmatrix} \mathbf{N}_1^{(n)} & \mathbf{N}_2^{(n)} \\ \mathbf{N}_3^{(n)} & \mathbf{N}_1^{(n)\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{N}_1^{(n)} & \mathbf{N}_2^{(n)} \\ \mathbf{N}_3^{(n)} & \mathbf{N}_1^{(n)\mathrm{T}} \end{bmatrix}. \tag{3.11}$$

This is equation (14) in [11].

If we post-multiply (3.4) by $[\mathbf{A}^{-1}, \mathbf{B}^{-1}]$ and use (2.15), we have

$$\left.\begin{aligned} \mathbf{AP}^n\mathbf{A}^{-1} &= \mathbf{N}_1^{(n)} + i\mathbf{N}_2^{(n)}\mathbf{M}, \\ \mathbf{AP}^n\mathbf{B}^{-1} &= \mathbf{N}_2^{(n)} - i\mathbf{N}_1^{(n)}\mathbf{M}^{-1}, \\ \mathbf{BP}^n\mathbf{A}^{-1} &= \mathbf{N}_3^{(n)} + i\mathbf{N}_1^{(n)\mathrm{T}}\mathbf{M}, \\ \mathbf{BP}^n\mathbf{B}^{-1} &= \mathbf{N}_1^{(n)\mathrm{T}} - i\mathbf{N}_3^{(n)}\mathbf{M}^{-1}. \end{aligned}\right\} \tag{3.12}$$

From the relations

$$\left.\begin{aligned} (\mathbf{AP}^n\mathbf{B}^{-1})(\mathbf{BP}^{-n}\mathbf{A}^{-1}) &= \mathbf{I}, \\ (\mathbf{AP}^n\mathbf{A}^{-1})(\mathbf{AP}^{-n}\mathbf{B}^{-1}) &= -i\mathbf{M}^{-1}, \\ (\mathbf{BP}^n\mathbf{A}^{-1})(\mathbf{AP}^{-n}\mathbf{A}^{-1}) &= i\mathbf{M}, \end{aligned}\right\} \tag{3.13}$$

more identities can be obtained by substituting (3.12) into (3.13).

We note that the left-hand sides of the identities in (3.10) and (3.12) involve products of complex quantities for which it is not clear what the real and imaginary parts are. The right-hand sides of the identities provide the answer to the real and imaginary parts. These identities will be useful in obtaining a real form solution to two-dimensional anisotropic elasticity problems. (See [12], for example.) Identities (3.10) for $n = 1$ have been obtained by Nishioka and Lothe [13] for the real parts and by Bacon et al [14] for the imaginary parts. For $n \neq 1$, the real parts of

$$\mathbf{AP}^{-1}\mathbf{A}^{\mathrm{T}}, \quad \mathbf{AP}^2\mathbf{A}^{\mathrm{T}}, \quad \mathbf{BP}^{-1}\mathbf{B}^{\mathrm{T}}, \text{ and } \mathbf{BP}^2\mathbf{B}^{\mathrm{T}}$$

are presented in [13], and the imaginary part of $\mathbf{AP}^2\mathbf{A}^{\mathrm{T}}$ is given in [15].

4. Positive definiteness of the reduced elastic compliance. In many applications, it is more convenient to write the stresses and strains as 6×1 column matrices $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ by introducing the notation

$$\left.\begin{aligned} \sigma_{11} &= \sigma_1, \quad \sigma_{22} = \sigma_2, \quad \sigma_{33} = \sigma_3, \\ \sigma_{23} &= \sigma_4, \quad \sigma_{31} = \sigma_5, \quad \sigma_{12} = \sigma_6, \end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned} \varepsilon_{11} &= \varepsilon_1, \quad \varepsilon_{22} = \varepsilon_2, \quad \varepsilon_{33} = \varepsilon_3, \\ 2\varepsilon_{23} &= \varepsilon_4, \quad 2\varepsilon_{31} = \varepsilon_5, \quad 2\varepsilon_{12} = \varepsilon_6. \end{aligned}\right\} \tag{4.2}$$

The stress-strain laws (1.1) can then be written as

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}, \tag{4.3}$$

in which $\mathbf{C}$ is a 6×6 symmetric matrix. The inverse of (4.3) is

$$\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma}, \tag{4.4}$$

where $\mathbf{S}$ denotes (in this section and in Sec. 6 only) the elastic compliance, which is also symmetric. The strain energy W is

$$W = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} = \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon} = \tfrac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon} = \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{S}\boldsymbol{\sigma}, \tag{4.5}$$

which tells us that $\mathbf{C}$ and $\mathbf{S}$ must be positive definite for W to be positive.

For the two-dimensional deformations considered here

$$\varepsilon_3 = 0 = S_{3j}\sigma_j. \tag{4.6}$$

Solving for σ_3 and substituting in (4.4), we have

$$\varepsilon_i = \sum_{j\neq 3} \hat{S}_{ij}\sigma_j, \qquad i \neq 3, \tag{4.7}$$

in which the symmetric matrix

$$\hat{S}_{ij} = S_{ij} - S_{i3}S_{3j}/S_{33} \tag{4.8}$$

is the reduced elastic compliance. Since

$$\hat{S}_{i3} = \hat{S}_{3i} = 0 \quad \text{for any } i, \tag{4.9}$$

there is, in fact, no need to exclude $j = 3$ in the summation of (4.7). Likewise, there is no need to exclude $i = 3$ in (4.7) because it reduces to a trivial identity when $i = 3$. The matrix $\hat{\mathbf{S}}$ considered as a 6×6 matrix has zero elements on the 3rd column and the 3rd row. If we remove the 3rd column and the 3rd row, the reduced 5×5 matrix $\hat{\mathbf{S}}$ is positive definite. To see this, we introduce the 5×1 matrices $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\varepsilon}}$ which are obtained, respectively, by deleting from $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ the third element. Equation (4.7) is equivalent to

$$\hat{\boldsymbol{\varepsilon}} = \hat{\mathbf{S}}\hat{\boldsymbol{\sigma}}. \tag{4.10}$$

In view of the fact that $\varepsilon_3 = 0$, the strain energy is

$$W = \tfrac{1}{2}\hat{\boldsymbol{\sigma}}^{\mathrm{T}}\hat{\boldsymbol{\varepsilon}} = \tfrac{1}{2}\hat{\boldsymbol{\sigma}}^{\mathrm{T}}\hat{\mathbf{S}}\hat{\boldsymbol{\sigma}}, \tag{4.11}$$

which shows that the 5×5 matrix $\hat{\mathbf{S}}$ is positive definite if W is. In particular, the 2×2 matrices

$$\begin{bmatrix} \hat{S}_{11} & \hat{S}_{15} \\ \hat{S}_{15} & \hat{S}_{55} \end{bmatrix}, \qquad \begin{bmatrix} \hat{S}_{22} & \hat{S}_{24} \\ \hat{S}_{24} & \hat{S}_{44} \end{bmatrix} \tag{4.12}$$

are positive definite. So are the following two matrices

$$\begin{bmatrix} \hat{S}_{11} & -\hat{S}_{15} \\ -\hat{S}_{15} & \hat{S}_{55} \end{bmatrix}, \qquad \begin{bmatrix} \hat{S}_{22} & -\hat{S}_{24} \\ -\hat{S}_{24} & \hat{S}_{44} \end{bmatrix} \tag{4.13}$$

because the eigenvalues of these two matrices are identical to the eigenvalues of the corresponding matrices in (4.12), which are positive and nonzero. These two matrices will be useful later on in representing the elements of $\mathbf{N}_3$ and $\mathbf{N}_3^{(-1)}$.

5. Representation of uniform strain solution. When the strains ε_{ij} are uniform, the displacements u_i are, aside from a rigid body translation, linear in the coordinates x_j. We may write

$$u_i = (\varepsilon_{ij} + \omega_{ij})x_j, \tag{5.1}$$

where ω_{ij} is an arbitrary antisymmetric constant matrix. This satisfies the strain-displacement equations (1.3). For the two-dimensional deformations considered here, $\varepsilon_{33} = 0$, and the u_i do not depend on x_3. We therefore choose

$$\omega_{ij} = \begin{bmatrix} 0 & \omega & -\varepsilon_{13} \\ -\omega & 0 & -\varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & 0 \end{bmatrix}, \tag{5.2}$$

where ω is an arbitrary constant. Equation (5.1) now has the form

$$\mathbf{u} = \boldsymbol{\varepsilon}_1 x_1 + \boldsymbol{\varepsilon}_2 x_2, \tag{5.3}$$

in which

$$\left.\begin{aligned} \boldsymbol{\varepsilon}_1 &= \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} - \omega \\ 2\varepsilon_{13} \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \frac{1}{2}\varepsilon_6 - \omega \\ \varepsilon_5 \end{bmatrix}, \\ \boldsymbol{\varepsilon}_2 &= \begin{bmatrix} \varepsilon_{12} + \omega \\ \varepsilon_{22} \\ 2\varepsilon_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\varepsilon_6 + \omega \\ \varepsilon_2 \\ \varepsilon_4 \end{bmatrix}. \end{aligned}\right\} \tag{5.4}$$

The following question arises. Inasmuch as (2.17a) is a general solution for $\mathbf{u}$, how does one choose the arbitrary functions f_α so that (2.17a) reduces to (5.3)?

The first step is to let f_α be proportional to its argument. Since $\mathbf{u}$ is real, we let

$$\mathbf{u} = \sum_{\alpha=1}^{3} (\mathbf{a}_\alpha z_\alpha q_\alpha + \bar{\mathbf{a}}_\alpha \bar{z}_\alpha \bar{q}_\alpha), \tag{5.5}$$

where q_α are arbitrary complex constants. The problem is to determine the constants q_α. Introducing the diagonal matrix

$$\mathbf{Z} = \mathrm{diag}[z_1, z_2, z_3] = x_1 \mathbf{I} + x_2 \mathbf{P}, \tag{5.6}$$

we write (5.5) in matrix notation as

$$\mathbf{u} = \mathbf{AZq} + \bar{\mathbf{A}}\bar{\mathbf{Z}}\bar{\mathbf{q}} = (\mathbf{Aq} + \bar{\mathbf{A}}\bar{\mathbf{q}})x_1 + (\mathbf{APq} + \bar{\mathbf{A}}\bar{\mathbf{P}}\bar{\mathbf{q}})x_2. \tag{5.7}$$

In view of (5.3), we have

$$\left.\begin{aligned} \mathbf{Aq} + \bar{\mathbf{A}}\bar{\mathbf{q}} &= \boldsymbol{\varepsilon}_1, \\ \mathbf{APq} + \bar{\mathbf{A}}\bar{\mathbf{P}}\bar{\mathbf{q}} &= \boldsymbol{\varepsilon}_2. \end{aligned}\right\} \tag{5.8}$$

If we let

$$\mathbf{h} = \mathbf{Aq}, \tag{5.9}$$

we obtain

$$\left.\begin{aligned} \mathbf{h} + \bar{\mathbf{h}} &= \boldsymbol{\varepsilon}_1, \\ \mathbf{APA}^{-1}\mathbf{h} + \bar{\mathbf{A}}\bar{\mathbf{P}}\bar{\mathbf{A}}^{-1}\bar{\mathbf{h}} &= \boldsymbol{\varepsilon}_2. \end{aligned}\right\} \tag{5.10}$$

The first equation of (5.10) tells us that the real part of $\mathbf{h}$ is $\boldsymbol{\varepsilon}_1/2$. The imaginary part of $\mathbf{h}$ can be determined from the second equation of (5.10). With $\mathbf{APA}^{-1}$ given by the identity in (3.12), the result is

$$\begin{aligned} 2\mathbf{h} &= \boldsymbol{\varepsilon}_1 - i\{\mathbf{HT}\boldsymbol{\varepsilon}_2 + (\mathbf{HR}^{\mathrm{T}} + \mathbf{S})\boldsymbol{\varepsilon}_1\} \\ &= 2\mathbf{A}\{\mathbf{A}^{\mathrm{T}}\mathbf{T}\boldsymbol{\varepsilon}_2 + (\mathbf{A}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}})\boldsymbol{\varepsilon}_1\}, \end{aligned} \tag{5.11}$$

where the second equality is obtained with the use of (2.12) and (2.14). It follows from (5.9) that

$$\mathbf{q} = \mathbf{A}^{\mathrm{T}}\mathbf{T}\boldsymbol{\varepsilon}_2 + (\mathbf{A}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}})\boldsymbol{\varepsilon}_1. \tag{5.12}$$

Associated with (5.5) or (5.7) is the stress function

$$\boldsymbol{\phi} = \mathbf{BZq} + \bar{\mathbf{B}}\bar{\mathbf{Z}}\bar{\mathbf{q}} = (\mathbf{Bq} + \bar{\mathbf{B}}\bar{\mathbf{q}})x_1 + (\mathbf{BPq} + \bar{\mathbf{B}}\bar{\mathbf{P}}\bar{\mathbf{q}})x_2. \tag{5.13}$$

With $\mathbf{q}$ given by (5.12) and using (2.13), (2.14), and the identities in (3.10), we have

$$\boldsymbol{\phi} = (\mathbf{R}^{\mathrm{T}}\boldsymbol{\varepsilon}_1 + \mathbf{T}\boldsymbol{\varepsilon}_2)x_1 - (\mathbf{Q}\boldsymbol{\varepsilon}_1 + \mathbf{R}\boldsymbol{\varepsilon}_2)x_2. \tag{5.14}$$

We will show in the next section that this leads to a correct stress-strain law.

6. Representation of $\mathbf{N}_1$, $\mathbf{N}_3$, $\mathbf{N}_1^{(-1)}$, and $\mathbf{N}_3^{(-1)}$. In this section we will derive an alternate expression for the 3×3 matrices $\mathbf{N}_1$, $\mathbf{N}_3$, $\mathbf{N}_1^{(-1)}$, and $\mathbf{N}_3^{(-1)}$ defined in (1.15) and (3.2). With the elastic constants C_{ijks} redefined as a 6×6 symmetric matrix $C_{\alpha\beta}$, the 3×3 matrices of (1.9) have the expressions

$$\mathbf{Q} = \begin{bmatrix} C_{11} & C_{16} & C_{15} \\ C_{61} & C_{66} & C_{65} \\ C_{51} & C_{56} & C_{55} \end{bmatrix}, \tag{6.1a}$$

$$\mathbf{R} = \begin{bmatrix} C_{16} & C_{12} & C_{14} \\ C_{66} & C_{62} & C_{64} \\ C_{56} & C_{52} & C_{54} \end{bmatrix}, \tag{6.1b}$$

$$\mathbf{T} = \begin{bmatrix} C_{66} & C_{62} & C_{64} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{42} & C_{44} \end{bmatrix}. \tag{6.1c}$$

We notice that

$$R_{2k} = T_{1k}, \qquad R_{k1} = Q_{k2}. \tag{6.2}$$

Hence

$$\begin{aligned}(\mathbf{N}_1)_{i2} &= -(\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}})_{i2} = -(\mathbf{T}^{-1})_{ik}R_{2k} \\ &= -(\mathbf{T}^{-1})_{ik}T_{1k} = -\delta_{i1}.\end{aligned} \tag{6.3}$$

This means that the second column of $\mathbf{N}_1$ has the elements $(-1, 0, 0)$. Next, using (6.2) and (6.3),

$$\begin{aligned}(\mathbf{N}_3)_{i2} &= R_{ik}(\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}})_{k2} - Q_{i2} \\ &= R_{ik}\delta_{k1} - R_{i1} = 0.\end{aligned} \tag{6.4}$$

Therefore, the second column and, in view of the symmetry of $\mathbf{N}_3$, the second row of $\mathbf{N}_3$ contain only zero elements. Similarly, one can show that

$$(\mathbf{N}_1^{(-1)})_{i1} = -\delta_{i2}, \tag{6.5}$$

$$(\mathbf{N}_3^{(-1)})_{i1} = 0. \tag{6.6}$$

With the new notation for the stresses, (2.19) is equivalent to

$$\mathbf{t}_1 = \begin{bmatrix} \sigma_1 \\ \sigma_6 \\ \sigma_5 \end{bmatrix}, \qquad \mathbf{t}_2 = \begin{bmatrix} \sigma_6 \\ \sigma_2 \\ \sigma_4 \end{bmatrix}, \tag{6.7}$$

and (4.3) in conjunction with (2.20) can be written as

$$\left.\begin{aligned} -\phi_{,2} &= \mathbf{t}_1 = \mathbf{Q}\varepsilon_1 + \mathbf{R}\varepsilon_2, \\ \phi_{,1} &= \mathbf{t}_2 = \mathbf{R}^{\mathrm{T}}\varepsilon_1 + \mathbf{T}\varepsilon_2, \end{aligned}\right\} \tag{6.8}$$

in which ε_1 and ε_2 are defined in (5.4). In view of (6.2), the arbitrary constants ω contained in ε_1 and ε_2 on the right of (6.8) cancel each other. We see that ϕ given by (5.14) satisfies (6.8). By eliminating ε_2 or ε_1 between the two equations in (6.8), we obtain

$$-\mathbf{N}_3\varepsilon_1 = \mathbf{t}_1 + \mathbf{N}_1^{\mathrm{T}}\mathbf{t}_2, \tag{6.9a}$$

$$\mathbf{N}_3^{(-1)}\varepsilon_2 = (\mathbf{N}_1^{(-1)})^{\mathrm{T}}\mathbf{t}_1 + \mathbf{t}_2. \tag{6.9b}$$

We will consider (6.9a) first. The arbitrary constant ω in the second element of ε_1 disappears after multiplication by $\mathbf{N}_3$ because the second column of $\mathbf{N}_3$ contains only zero elements. If we use (4.7) to express ε_1 in terms of σ_i on the left of (6.9a) and use (6.7) to express $\mathbf{t}_1$ and $\mathbf{t}_2$ in terms of σ_i on the right of (6.9a), we can equate the coefficients of σ_i. Using the properties (6.3) and (6.4), we find that $\mathbf{N}_1$ and $\mathbf{N}_3$ have the expressions

$$-\mathbf{N}_3 = \frac{1}{\Delta}\begin{bmatrix} \hat{S}_{55} & 0 & -\hat{S}_{15} \\ 0 & 0 & 0 \\ -\hat{S}_{15} & 0 & \hat{S}_{11} \end{bmatrix}, \tag{6.10}$$

$$-\mathbf{N}_1 = \begin{bmatrix} r_6 & 1 & s_6 \\ r_2 & 0 & s_2 \\ r_4 & 0 & s_4 \end{bmatrix}, \tag{6.11}$$

in which

$$\left.\begin{aligned} &\Delta = \hat{S}_{11}\hat{S}_{55} - (\hat{S}_{15})^2 > 0, \\ &r_i = (-\hat{S}_{55}\hat{S}_{1i} + \hat{S}_{15}\hat{S}_{5i})/\Delta \quad (i = 6, 2, 4), \\ &s_i = (\hat{S}_{15}\hat{S}_{1i} - \hat{S}_{11}\hat{S}_{5i})/\Delta \quad (i = 6, 2, 4). \end{aligned}\right\} \tag{6.12}$$

In the same manner, we obtain from (6.9b) the following expressions for $\mathbf{N}_3^{(-1)}$ and $\mathbf{N}_1^{(-1)}$.

$$\mathbf{N}_3^{(-1)} = \frac{1}{\Delta'}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \hat{S}_{44} & -\hat{S}_{24} \\ 0 & -\hat{S}_{24} & \hat{S}_{22} \end{bmatrix}, \tag{6.13}$$

$$-\mathbf{N}_1^{(-1)} = \begin{bmatrix} 0 & r'_1 & s'_1 \\ 1 & r'_6 & s'_6 \\ 0 & r'_5 & s'_5 \end{bmatrix}, \tag{6.14}$$

in which

$$\left.\begin{aligned} &\Delta' = \hat{S}_{22}\hat{S}_{44} - (\hat{S}_{24})^2 > 0, \\ &r'_i = (-\hat{S}_{44}\hat{S}_{2i} + \hat{S}_{24}\hat{S}_{4i})/\Delta' \quad (i = 1, 6, 5), \\ &s'_i = (\hat{S}_{24}\hat{S}_{2i} - \hat{S}_{22}\hat{S}_{4i})/\Delta' \quad (i = 1, 6, 5). \end{aligned}\right\} \tag{6.15}$$

From the positive definiteness of the matrices shown in (4.13), we see that Δ and Δ' are positive and $-\mathbf{N}_3$ and $\mathbf{N}_3^{(-1)}$ are positive semidefinite. To be more specific, let $\mathbf{y}$ be a 3×1 constant matrix. Then

$$\mathbf{y}^{\mathrm{T}}(-\mathbf{N}_3)\mathbf{y} \geq 0. \tag{6.16}$$

The equality holds only when $y_1 = y_3 = 0$. For $\mathbf{N}_3^{(-1)}$, the equality holds only when $y_2 = y_3 = 0$.

7. A direct proof of positive definiteness of L. The three real matrices **H**, **L**, and **S** of (2.12), (2.13), and (2.14) can be obtained by an integral formalism proposed by Barnett and Lothe [5] in which the necessity of finding the eigenvalues p_α and the eigenvectors $\mathbf{a}_\alpha$, $\mathbf{b}_\alpha$ is circumvented. For **L**, the integral is

$$\mathbf{L} = \frac{1}{\pi}\int_0^\pi -\mathbf{N}_3(\omega)\, d\omega, \tag{7.1}$$

in which

$$\mathbf{N}_3(\omega) = \mathbf{R}(\omega)\mathbf{T}^{-1}(\omega)\mathbf{R}^{\mathrm{T}}(\omega) - \mathbf{Q}(\omega), \tag{7.2}$$

and $\mathbf{Q}(\omega)$, $\mathbf{R}(\omega)$, $\mathbf{T}(\omega)$ are the generalization of **Q**, **R**, **T** of (1.9) defined as

$$\left.\begin{aligned} Q_{ik}(\omega) &= C_{ijks}n_j n_s, \\ R_{ik}(\omega) &= C_{ijks}n_j m_s, \\ T_{ik}(\omega) &= C_{ijks}m_j m_s, \end{aligned}\right\} \tag{7.3}$$

$$n_i = [\cos\omega, \sin\omega, 0], \qquad m_i = [-\sin\omega, \cos\omega, 0]. \tag{7.4}$$

When $\omega = 0$, (7.3) reduces to (1.9).

We introduce a new coordinate system x_i^*, which is obtained from x_i by rotating about the x_3-axis an angle ω. That is,

$$\mathbf{x}^* = \mathbf{\Omega}\mathbf{x}, \tag{7.5}$$

$$\mathbf{\Omega} = \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7.6}$$

The elastic constants C^*_{ijks} referred to the rotated coordinates x_i^* are

$$C^*_{ijks} = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}C_{pqrt}. \tag{7.7}$$

Noticing that

$$\Omega_{1j} = n_j, \qquad \Omega_{2j} = m_j, \tag{7.8}$$

it is readily shown that (see also [16])

$$\left.\begin{aligned} Q^*_{ik} &= C^*_{i1k1} = \Omega_{ip}\Omega_{kr}Q_{pr}(\omega), \\ R^*_{ik} &= C^*_{i1k2} = \Omega_{ip}\Omega_{kr}R_{pr}(\omega), \\ T^*_{ik} &= C^*_{i2k2} = \Omega_{ip}\Omega_{kr}T_{pr}(\omega), \end{aligned}\right\} \tag{7.9}$$

or, in matrix notation,

$$\left.\begin{aligned} \mathbf{Q}(\omega) &= \mathbf{\Omega}^{\mathrm{T}}\mathbf{Q}^*\mathbf{\Omega}, \\ \mathbf{R}(\omega) &= \mathbf{\Omega}^{\mathrm{T}}\mathbf{R}^*\mathbf{\Omega}, \\ \mathbf{T}(\omega) &= \mathbf{\Omega}^{\mathrm{T}}\mathbf{T}^*\mathbf{\Omega}. \end{aligned}\right\} \tag{7.10}$$

Therefore,

$$\mathbf{N}_3(\omega) = \mathbf{\Omega}^{\mathrm{T}}\mathbf{N}_3^*\mathbf{\Omega}, \tag{7.11}$$

where

$$\mathbf{N}_3^* = \mathbf{R}^*\mathbf{T}^{*-1}\mathbf{R}^{*\mathrm{T}} - \mathbf{Q}^*. \tag{7.12}$$

It should be pointed out that while $\mathbf{N}_3^*$ has the same simple structure as $\mathbf{N}_3$ shown in (6.10), the structure of $\mathbf{N}_3(\omega)$ does not have such simplicity. Nevertheless, it is clear that $-\mathbf{N}_3^*$ and $-\mathbf{N}_3(\omega)$ are, like $-\mathbf{N}_3$, positive semidefinite.

To prove that **L** of (7.1) is positive definite, let **y** be an arbitrary 3×1 constant matrix and consider

$$\mathbf{y}^{\mathrm{T}}\mathbf{L}\mathbf{y} = \frac{1}{\pi}\int_0^{\pi} -\mathbf{y}^{\mathrm{T}}\mathbf{N}_3(\omega)\mathbf{y}\,d\omega. \tag{7.13}$$

By (7.11) the integrand is

$$-\mathbf{y}^{\mathrm{T}}\mathbf{N}_3(\omega)\mathbf{y} = \mathbf{y}^{*\mathrm{T}}(-\mathbf{N}_3^*)\mathbf{y}^*, \tag{7.14}$$

$$\mathbf{y}^* = \mathbf{\Omega y}. \tag{7.15}$$

According to (6.16) and the discussion following it, the right-hand side of (7.14) is either positive or zero. It is zero only when $y_1^* = y_3^* = 0$, which means, by (7.15), $y_3 = 0$ and $\tan\omega = -y_1/y_2$. We see that $\tan\omega = -y_1/y_2$ provides only one value of ω if $0 \le \omega < \pi$ and two if $0 \le \omega < 2\pi$. Therefore the integrand is either positive for all ω (if $y_3 \ne 0$) or positive except at the specific ω at which $\tan\omega = -y_1/y_2$ (if $y_3 = 0$). Consequently the integral in (7.13) is positive and **L** is positive definite.

The fact that **L** is positive definite has been shown indirectly by Chadwick and Smith [4] by considering the strain energy of a line dislocation. The above derivation is a direct proof that **L** is positive definite.

Note added after the paper was accepted. It has come to our attention that an alternate direct proof that **L** is positive definite has recently been presented by Gundersen, Barnett, and Lothe [17].

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