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# A HYPERBOLIC STEFAN PROBLEM*

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**Abstract.** A free-boundary problem of Stefan type is presented under constitutive assumptions on flux and energy which contain an effective time delay. This contains the hyperbolic telegraphers equation and, hence, has the feature that propagation speed of disturbances is bounded. With the appropriate physically consistent condition on the interface this is shown to lead to a well-posed weak formulation of the problem.

**1. Introduction.** The classical theory of heat conduction is based on Fourier's law,

$$\mathbf{q}(t) = -k\nabla u(t), \tag{1.1}$$

which relates the heat flux $\mathbf{q}$ to the temperature $u$ at each time $t > 0$. With the conservation of energy equation,

$$cu_t + \nabla \cdot \mathbf{q} = 0,$$

this gives the usual parabolic heat equation, which has the feature that a thermal disturbance at any point is felt instantly at every other point. As an alternative one could consider a model in which the flux responds to a temperature gradient after a delay period of $\tau > 0$, i.e., [1]

$$\mathbf{q}(t + \tau) = -k\nabla u(t). \tag{1.2}$$

More generally, the theory of heat conduction with memory leads to constitutive equations of the form [8]

$$\mathbf{q}(t) = -\int_0^\infty a(s)\nabla u(t - s)\, ds.$$

We shall consider the special case of $a(s) = (k/\tau)\exp(-s/\tau)$ for which the above specializes to

$$\tau\mathbf{q}'(t) + \mathbf{q}(t) = -k\nabla u(t). \tag{1.3}$$

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This is certainly an approximation of (1.2) and, with the equation of conservation of energy above, it yields the hyperbolic telegrapher's equation,

$$\tau c u_{tt} + c u_t - k \Delta u = 0, \tag{1.4}$$

in place of the classical heat equation ($\tau = 0$).

Our interest is hereafter focused on the constitutive relation (1.3) which has a long history, dating back (at least) to Maxwell [9]. A number of writers have proposed (1.3) as a substitute for (1.1) as a means of placing an upper bound on the speed of propagation of thermal disturbances. This appears to be particularly important in certain models with large variations in temperature or large gradients of temperature; for various studies on the conduction of heat based on (1.3) see [2–6, 10, 14]. Specifically we propose to formulate a well-posed free-boundary problem of Stefan type which is consistent with (1.3) and, hence, contains the telegrapher's equation (1.4). Stefan problems describe the conduction of heat in a medium involving a phase change, that is, the absorption of latent heat and corresponding displacement of the interface between solid and liquid phases of the same material. For perspectives on Stefan problems which contain (1.4) we refer to [7, 11–13]. We shall digress substantially from these works: here the exchange of latent heat energy takes place after an effective delay of the same duration as the flux response to temperature gradients. This seems necessary in order to permit the matching of the flux and latent heat exchange along a single free surface. Independently the specific heat may be delayed or advanced with respect to total energy by an increment depending on the phase. This seems appropriate in order to match wave speeds of the hyperbolic equations on either side of the free surface.

Our plan is as follows. In Sec. 2 we discuss heat energy or enthalpy functions which correspond to (1.1), (1.2), and (1.3), respectively, and then give for each of the three cases an intuitive description of the appropriate conditions on the moving interface or free boundary. These conditions are different from those of either [7, 11] or of [12, 13]. Section 3 contains a derivation of the partial differential equations and free-boundary conditions directly from the relation (1.3), the corresponding constitutive assumption on the energy function, and the principle of conservation of energy. From this local formulation of the problem we then obtain an appropriate weak formulation. In Sec. 4 we show this generalized problem is well posed. This generalized problem is formulated so as to admit a region of "mush", a mixture of solid and liquid phases in equilibrium at the melting temperature. In the classical Stefan problem it follows from the maximum principle that such regions do not occur unless they are either present initially or are created by internal distributed sources. We shall show this is not the case for these hyperbolic Stefan problems when we discuss the one-phase problem in Sec. 5. There we highlight some of the anomalies that arise with hyperbolic models of heat transfer.

**2. Energy and phase-change models.** We have presented above three constitutive assumptions on the flux: the classical Fourier law (1.1) in which flux responds instantly to a temperature gradient, the pure delay assumption (1.2) for this response, and the relaxation assumption (1.3) in which the delay is distributed over a time interval. A corresponding assumption on the energy stored in the medium and its relation to temperature will be presented for each case, and then we discuss the corresponding conditions on an interface where a change-of-phase occurs.

Consider first a simple experiment in which a uniform heat source of intensity $F > 0$ is applied to a unit volume of ice at temperature $u < 0$. The temperature increases at the rate $F/c_1$ until it reaches $u = 0$ where melting begins. The temperature remains at zero until $L$ units of heat have been supplied to convert the ice to water; $L > 0$ is the latent heat. During this period there is a fraction $\xi$ of water co-existing in thermal equilibrium with the ice and $\xi$ increases at the rate $F/L$. When all of the ice has melted, $\xi = 1$ and the temperature begins to increase at the rate $F/c_2$. The constants $c_1$, $c_2$ are specific heats of ice and water, respectively. In summary, we can say that the rate of increase of heat energy or enthalpy,

$$e(t) \equiv C(u(t)) + L\xi(t), \tag{2.1}$$

is given by $F$, where $C(u) = c_1 u$ for $u \leqslant 0$, $C(u) = c_2 u$ for $u \geqslant 0$ and $\xi \in H(u)$, the Heaviside graph given by $H(u) = 1$ for $u > 0$, $H(u) = 0$ for $u < 0$, and $H(0) = [0, 1]$. Thus, from any energy level $e$ we obtain the temperature as a Lipschitz function, $u = (C + LH)^{-1}(e)$ and the fraction of water is $\xi = (1/L)(e - C(u))$. [See Fig. 1.] The energy function (2.1) is the classical assumption that the storage or release of energy as latent heat is instantaneous; this is consistent with the flux assumption (1.1).

Suppose we had assumed (1.2): flux is induced only after a time delay $\tau > 0$. To be consistent, energy balance considerations imply that the storage of latent heat energy should be likewise delayed, hence,

$$e(t) = C(u(t)) + L\xi(t - \tau), \quad \xi \in H(u). \tag{2.2}$$

(A delay of specific heat energy storage would not be consistent with the telegrapher's equation.) However, when we repeat the preceding experiment [Fig. 2] the energy is required to jump instantly, at the time $\tau$ units after the temperature reaches zero, by a magnitude of $L$ units upward. Thus the "superheated ice" sucks in sufficient energy to change the phase instantly to water, a very unstable and unrealistic situation.

Likewise, to be consistent with our intended assumption (1.3) we consider the constitutive energy relation

$$\left(1 + \tau \frac{d}{dt}\right) e(t) = \left(1 + T(u)\frac{d}{dt}\right) C(u(t)) + L\xi(t), \quad \xi \in H(u), \tag{2.3}$$

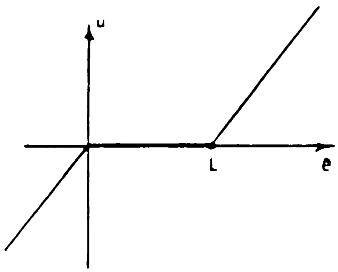


FIG. 1

where $T(u) = \tau_1$ if $u < 0$ and $T(u) = \tau_2$ if $u > 0$. This relation contains three relaxation times which we prescribe below. Note that in the special case of $\tau = \tau_1 = \tau_2$ this corresponds to

$$e(T) = C(u(t)) + (L/\tau)\int_0^\infty \exp(-s/\tau)\xi(t-s)\,ds, \quad \xi \in H(u),$$

which is a smooth approximation of (2.2). The more general relation (2.3) will be useful in the conduction problems with two phases. There the two delay times $\tau_1$, $\tau_2$ will be used to match the propagation speeds in the two phases. Although no formal requirements occur on $\tau \geqslant 0$ in the following, one envisions $\tau$ to be of the same order of magnitude as $\tau_1$ and $\tau_2$. Note that the form of our model equation is essentially independent of $\tau$. The essential assumptions in (2.3) are that latent heat affects the energy after a delay of $\tau$ and that specific heat affects the energy after a delay of $\tau - \tau_j$ $(j = 1, 2)$ in the corresponding phase. Let us repeat our experiment above but with the assumption (2.3). As before, temperature increases at the rate $F/c_1$ until it reaches zero. [See Fig. 3]. Then $\xi$ jumps from 0 to $\tau_1 F/L$; thereafter it increases steadily to 1 where $e = L - \tau F$. Afterward temperature increases again, given by

$$u = (1/c_2)\{e - (L - \tau F) - F\tau_2(1 - \exp[(-e + L - \tau F)/(\tau_2 F)])L\}.$$

The nonclassical feature observed here is that a fraction of water melts instantly when temperature reaches zero. Note that $\tau_1$ must be small, $\tau_1 F \ll L$, for the preceding, and there is a smoothing of the classical temperature response near the end of the melting interval at the expense of the discontinuity in latent heat stored at the beginning of this interval.

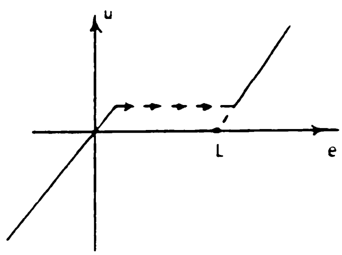


FIG. 2

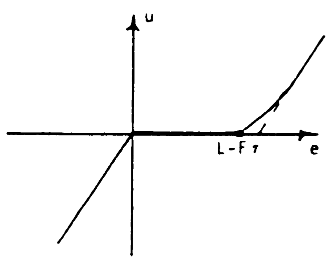


FIG. 3

We next give an heuristic description of the one-phase Stefan problem; a model is the melting of ice at temperature zero in a porous medium $G$ in $\mathbf{R}^m$. Let $\Omega = G \times (0, \infty)$ and note that this space-time region is separated into a region of ice $\Omega_0 = \{(x, t) \in \Omega: u(x, t) = 0\}$ and a region of water $\Omega_+ \equiv \{(x, t) \in \Omega: u(x, t) > 0\}$ by a surface $S$. If $\mathbf{N} = (N_x, N_t)$ is the unit normal to $S$ in $\mathbf{R}^{m+1}$ then at each time $t > 0$ the surface has unit spatial normal $\mathbf{n} = N_x/\|N_x\|$ and moves along $\mathbf{n}$ with speed $V(t) = -N_t/\|N_x\|$. The classical Stefan condition is

$$-\mathbf{q}(t) \cdot \mathbf{n}(t) + LV(t) = 0 \quad \text{on } S \tag{2.4}$$

and it states quantitatively that the change in flux (1.1) across $S$ equals the product of change in energy-per-volume (2.1) across $S$ and the velocity of the interface. Our interest here is on models where flux and energy respond to the history of temperature over an interval of effective duration $\tau > 0$; so we state the energy balance as follows: at a point $(x, t) \in S$ and during the time-interval to $(x, t + \tau)$, the change in flux out of $\Omega_+$ equals the product of change in energy-per-volume and the velocity of the interface. With the assumptions (1.2), (2.2) of pure delay, this takes the form

$$-\mathbf{q}(t + \tau) \cdot \mathbf{n}(t) + \big(L + c_2 u(t + \tau_2)\big)V(t) = 0 \tag{2.5}$$

wherein energy at $t + \tau$ corresponds to specific heat at $t + \tau_2$ and latent heat exchange at $t$. However, if we take instead the relaxation assumptions (1.3) and (2.3) we obtain

$$-\big(\mathbf{q}(t) + \tau\mathbf{q}'(t)\big) \cdot \mathbf{n}(t) + \big(L + \tau_2 c_2 u'(t)\big)V(t) = 0 \tag{2.6}$$

as the interface condition for energy balance. This last constraint will be obtained in the next section.

Finally, we note two features of the pure delay assumptions in the Stefan problem. First, from (1.2) it follows that in a neighborhood of any point on $S$ where $V(t) > 0$ we have flux identically zero. Thus there is no mechanism for energy transfer. Second, the condition (2.5) does not determine the position of $S$: there are two parameters, $\mathbf{q}(t + \tau)$ and $u(t + \tau)$, available in (2.5), either of which could be specified arbitrarily and then the other obtained from the "boundary condition" (2.5) on $S$ for a solution in $\Omega_+$. Thus the pure delay assumptions lead to inconsistencies. We shall show that the relaxation assumptions yield a well-posed problem.

**3. The Stefan problem.** First we shall derive the classical formulation of the Stefan problem from the constitutive assumptions with relaxation and the local conservation of heat energy. This problem will contain the telegrapher's equation as planned. Then we develop a weak formulation of this problem in an appropriate Hilbert space of distributions. Thus let $G$ be a domain in Euclidean space $\mathbf{R}^m$ and set $\Omega = G \times (0, \infty)$. The temperature at point $x \in G$ and time $t > 0$ is $u(x, t)$; from the conductivities of ice and water, $k_1$ and $k_2$, respectively, we define the function $K$ by $K(u) = k_1 u$ for $u \leqslant 0$ and $K(u) = k_2 u$ for $u \geqslant 0$. Recall that $C(u)$ and $T(u)$ are determined similarly from specific heats and delay times. Denote by $\Omega_-$, $\Omega_+$, and $\Omega_0$ the subdomains of $\Omega$ where $u < 0$, $u > 0$, and $u = 0$, respectively. Let $S_+$ be the boundary of $\Omega_+$ and $S_-$ the boundary of $\Omega_-$ in $\Omega$. $\overline{\mathbf{N}} = (N_t, N_x)$ is the normal on either $S_+$ or $S_-$ oriented out of $\Omega_+$ and into $\Omega_-$, respectively, so that it is assigned consistently on $S \equiv S_+ \cap S_-$, the interface between $\Omega_+$

and $\Omega_-$. Hereafter we assume that the flux and energy are related to temperature by

$$\left(1 + \tau \frac{d}{dt}\right)\mathbf{q}(t) = -\nabla K u(t) \tag{3.1}$$

and

$$\left(1 + \tau \frac{d}{dt}\right)e(t) = \left(1 + T(u)\frac{d}{dt}\right)C(u(t)) + L\xi(t), \quad \xi \in H(u). \tag{3.2}$$

The principle of conservation of energy takes the following form. For any subdomain $G^*$ of $G$ with boundary $\partial G^*$ and outward normal $\mathbf{n}$,

$$\frac{d}{dt}\int_{G^*} e(t)\,dx = -\int_{\partial G^*} \mathbf{q}(t)\cdot\mathbf{n}\,ds + \int_{G^*} F(t)\,dx. \tag{3.3}$$

That is, the local rate of increase of energy in a region is given by the flux across its boundary and internal sources in that region. Since $e(t)$ is differentiable in $\Omega$ by (3.2), it follows that the derivative in (3.3) can be taken inside the integral. Since $q(t)$ is diffentiable in $\Omega$ by (3.1) we obtain

$$\frac{d}{dt}\int_{G^*} e'(t)\,dx = -\int_{\partial G^*} \mathbf{q}'(t)\cdot\mathbf{n}\,ds + \int_{G^*} F'(t)\,dx.$$

Multiply by $\tau$ and add to (3.3) to get

$$\frac{d}{dt}\int_{G^*} (e(t) + \tau e'(t))\,dx = -\int_{\partial G^*} (\mathbf{q}(t) + \tau\mathbf{q}'(t))\cdot\mathbf{n}\,ds + \int_{G^*} (F(t) + \tau F'(t))\,dx.$$

As suggested by our constitutive assumptions, the first two integrands are assumed to be piecewise smooth in $\Omega$. Writing the first integral as a sum over the smooth portions and the second as the sum over their boundaries, and denoting the union of their boundaries by $\tilde{S}$ we compute

$$\int_{G^*} \frac{\partial}{\partial t}(e(t) + \tau e'(t))\,dx + \int_{\tilde{S}\cap G^*} [\![ e + \tau e' ]\!] V(t)\,ds \tag{3.4}$$

$$= -\int_{G^*} \nabla\cdot(\mathbf{q}(t) + \tau\mathbf{q}'(t))\,dx + \int_{\tilde{S}\cap G^*} [\![(\mathbf{q} + \tau\mathbf{q}')\cdot\mathbf{n}]\!]\,ds + \int_{G^*} (F(t) + \tau F'(t))\,dx,$$

where $[\![\ ]\!]$ denotes the jump in values along $\tilde{S}$, $V(t)$ is the velocity of the interface, and $\nabla\cdot$ is the divergence. Since (3.4) holds for every $G^*$ and $t > 0$ as above, there follow the two equalities

$$\frac{\partial}{\partial t}(e(t) + \tau e'(t)) + \nabla\cdot(\mathbf{q}(t) + \tau\mathbf{q}'(t)) = F(t) + \tau F'(t) \quad \text{in } \Omega \sim \tilde{S}, \tag{3.5a}$$

$$-[\![ e(t) + \tau e'(t) ]\!] N_t = [\![(\mathbf{q}(t) + \tau\mathbf{q}'(t))\cdot N_x]\!] \quad \text{on } \tilde{S}, \tag{3.5b}$$

which give the differential characterization of energy conservation (3.3) for processes whose smoothness is consistent with (3.1) and (3.2).

The classical formulation of the Stefan problem follows immediately from (3.1), (3.2), and (3.5). The problem is to find a pair of functions $u$, $\xi$ on $\Omega$ such that $\xi \in H(u)$ and

$$\frac{\partial}{\partial t}\left(\tau_2 c_2 \frac{\partial u}{\partial t} + c_2 u\right) - k_2 \Delta u = F + \tau F' \quad \text{in } \Omega_+ - \tilde{S}, \tag{3.6a}$$

$$L\frac{\partial \xi}{\partial t} = F + \tau F' \quad \text{in } \Omega_0 - \tilde{S}, \tag{3.6b}$$

$$\frac{\partial}{\partial t}\left(\tau_1 c_1 \frac{\partial u}{\partial t} + c_1 u\right) - k_1 \Delta u = F + \tau F' \quad \text{in } \Omega_- - \tilde{S}, \tag{3.6c}$$

$$\left(L(1-\xi) + \tau_2 c_2 \frac{\partial u}{\partial t}\right) N_t = k_2 \nabla u \cdot N_x \quad \text{on } S_+, \tag{3.7a}$$

$$\left(-L\xi + \tau_1 c_1 \frac{\partial u}{\partial t}\right) N_t = k_1 \nabla u \cdot N_x \quad \text{on } S_-, \tag{3.7b}$$

$$\tau_i c_i [\![ \frac{\partial u}{\partial t} ]\!] N_t = k_i [\![ \nabla u ]\!] N_x \quad \text{on } \tilde{S} - S, \tag{3.7c}$$

where $i = 1$ or 2 depending upon $u$ being $< 0$ or $> 0$,

$$u(s,t) = 0, \quad s \in \partial G,\ t > 0, \tag{3.7d}$$

$$u(x,0) = u_0(x), \quad x \in G, \tag{3.8a}$$

$$\left(\tau_2 c_2 \frac{\partial u}{\partial t} + c_2 u + L\right)\Big|_{t=0} = v_0(x) \quad \text{where } u_0(x) > 0, \tag{3.8b}$$

$$\left(\tau_1 c_1 \frac{\partial u}{\partial t} + c_2 u\right)\Big|_{t=0} = v_0(x) \quad \text{where } v_0(x) < 0, \tag{3.8c}$$

$$L\xi(x,0^+) = v_0(x) \quad \text{where } u_0(x) = 0,\ x \in G. \tag{3.8d}$$

The hyperbolic telegrapher's equations (3.6a) and (3.6c) determine the temperatures in the water and ice, and the ordinary differential equation (3.6b) regulates the water fraction in the partially frozen region. The interface conditions (3.7a) and (3.7b) are just as anticipated in (2.6), that is, the heat flux from $\Omega_+$ determines the velocity of the free surface $S_+$ by melting the fraction $1 - \xi$ of ice with latent heat $L$ and raising the temperature behind the interface after an effective delay of $\tau_2 > 0$. The classical jump condition (3.7c) describes the propagation of wave fronts. The Dirichlet boundary condition (3.7d) could be replaced by any of the usual boundary conditions. The initial condition (3.8a) establishes the original temperature distribution and from (3.2) it follows that (3.8b)–(3.8d) specify the initial value $(e + \tau e')(0) = v_0$. Nonlinearities arise not only from the free-boundary conditions (3.7) but also from the fact that specific heats $c_1$, $c_2$ and conductivities $k_1$, $k_2$ change their values with the phase. We have allowed the effective delay to change likewise with phase, taking values $\tau_1$ and $\tau_2$ in $\Omega_-$ and $\Omega_+$, respectively. Note that the speeds of propagation of disturbances along the characteristics are given by $(k_2/(c_2\tau_2))^{1/2}$ and $(k_1/(c_1\tau_1))^{1/2}$ in the respective phases. We shall require that there be a global signal speed independent of phase; to fix the ratio of $\tau_1$ and $\tau_2$ to obtain a single velocity equal to $(1/\tau_0)^{1/2}$ with $\tau_0 > 0$, we choose

$$\tau_1 = \tau_0 k_1/c_1, \qquad \tau_2 = \tau_0 k_2/c_2$$

everywhere above. If $[\![\ ]\!]$ denotes as before the jump in values along $S = S_+ \cap S_-$, we obtain the counterpart of the usual Stefan constraint

$$\left(L + \tau_0 [\![ \frac{\partial}{\partial t} K(u) ]\!]\right) N_t = [\![ \nabla K(u) \cdot N_x ]\!] \quad \text{on } S, \tag{3.9}$$

that is, (3.9) is the special case of (3.7a)–(3.7b) that results when $\Omega_0$ has zero measure.

Finally we state the weak formulation of the Stefan problem. Since classical solutions are expected to exhibit jumps in first-order derivations, we expect a generalized solution to belong at best to the Sobolev space $H^1(\Omega)$. Let $u$, $\xi$ be a solution to (3.6); we need only know that the quantities given by (3.1) and (3.2) are functions smooth except possibly on $\tilde{S}$. Then we compute in the sense of distributions on $\Omega$ the following: for each test function $\varphi \in C_0^\infty(\Omega)$,

$$\left\{ \frac{\partial}{\partial t}\left( \tau_0 \frac{\partial K(u)}{\partial t} + C(u) + L\xi \right) - \Delta K(u) \right\}(\varphi)$$

$$= \int_{\Omega_+} \left( \frac{\partial}{\partial t}\left( \tau_2 c_2 \frac{\partial u}{\partial t} + c_2 u \right) - k_2 \Delta u \right)\varphi + \int_{\Omega_0} L \frac{\partial \xi}{\partial t}\varphi$$

$$+ \int_{\Omega_-} \left( \frac{\partial}{\partial t}\left( \tau_1 c_1 \frac{\partial u}{\partial t} + c_1 u \right) - k_1 \Delta u \right)\varphi$$

$$+ \int_{\tilde{S}} \left[\!\!\left[ \nabla K(u) \cdot N_x - \left( L(1-\xi) + \tau_0 \frac{\partial K(u)}{\partial t} \right) N_t \right]\!\!\right] \varphi.$$

Thus (3.6) and (3.7) are equivalent to

$$\frac{\partial}{\partial t}\left( \tau_0 \frac{\partial K(u)}{\partial t} + C(u) + L\xi \right) - \Delta K(u) = F + \tau F'$$

in $\mathscr{D}'(\Omega)$. A *generalized solution* of the Stefan problem (3.6)–(3.8) is a pair of functions

$$u \in W^{1,\infty}(0,T; L^2(G)) \cap L^\infty(0,T; H_0^1(G)), \qquad \xi \in L^\infty(\Omega)$$

which satisfy

$$\frac{d}{dt}\left( \tau_0 \frac{d}{dt} K(u(t)) + C(u(t)) + L\xi(t) \right) - \Delta K(u(t)) = F + \tau F', \quad \text{a.e. } t \in [0,T], \tag{3.10a}$$

$$u(t) \in H_0^1(G) \text{ and } \xi(t) \in H(u(t)) \quad \text{for all } t \in [0,T], \tag{3.10b}$$

$$u(0) = u_0, \qquad \left( \tau_0 \frac{d}{dt} K(u) + C(u) + L\xi \right)(0) = v_0. \tag{3.10c}$$

It will always be assumed that $F \in W^{1,\infty}(0,T; H^{-1}(G))$ so it is implicit in (3.10a) that the term "$\tau_0 (d/dt) K(u)(t) + C(u) + L\xi(t)$" belongs to $W^{1,\infty}(0,T; H^{-1}(G))$. Hence, the initial conditions are meaningful.

**4. Existence-uniqueness of a generalized solution.** We shall prove that the weak formulation (3.10) of the Stefan problem is well posed. This will be achieved by showing that the problem corresponds to an evolution equation whose solutions are determined by a nonlinear semigroup of contractions on a product space. Consider the following general result.

THEOREM 1. Let $V$ and $H$ be Hilbert spaces with $V$ dense and continuously imbedded in $H$. Denote by $\mathscr{A}\colon V \to V'$ and $\mathscr{C}\colon H \to H'$ the Riesz maps onto their respective dual spaces and let $\mathscr{B}\colon V \to V'$ be a (possibly multi-valued) maximal monotone operator. Assume $f \in W^{1,1}(0,T;V')$, $u_0 \in V$, and $v_0 \in V'$ are given such that there is a $b \in \mathscr{B}(u_0)$ with $b - v_0 \in H'$. Then there exists a unique pair $u \in W^{1,\infty}(0,T;H)$, $w \in L^\infty(0,T;V')$ such that $\mathscr{C}u' + w \in W^{1,\infty}(0,T;V')$, $w(t) \in \mathscr{B}(u(t))$ for all $t \in [0,T]$,

$$(\mathscr{C}u' + w') + \mathscr{A}(u) = f \text{ in } L^\infty(0,T;V'), \tag{4.1}$$

and $u(0) = u_0$, $(\mathscr{C}u' + w)(0) = v_0$.

*Proof*. We can write (4.1) as a first-order system on the product space $E \equiv V \times H$. Thus, we define $D(A) \equiv \{\mathbf{x} = [x_1, x_2] \in E$: there exists $b \in \mathscr{B}(x_2)$ with $-\mathscr{A}x_1 + b \in H'\}$ and $A\mathbf{x} = \{[x_2, \mathscr{C}^{-1}(-\mathscr{A}x_1 + b)]$: $\mathbf{x}$ and $b$ as above$\}$. We denote this set-valued function or relation formally by $A\mathbf{x} = [x_2, \mathscr{C}^{-1}(-\mathscr{A}x_1 + \mathscr{B}(x_2))]$ with the understanding that any such equation is to hold for all choices $b \in \mathscr{B}(x_2)$ as above. Since the scalar product on $E$ is

$$(\mathbf{x},\mathbf{y})_E = \mathscr{A}x_1(y_1) + \mathscr{C}x_2(y_2), \quad \mathbf{x},\mathbf{y} \in E,$$

we have (the set of equations)

$$(A\mathbf{x},\mathbf{y})_E = \mathscr{A}x_2(y_1) + (-\mathscr{A}x_1 + \mathscr{B}(x_2))(y_2), \quad \mathbf{x} \in D(A), \mathbf{y} \in E.$$

Specifically, we obtain

$$(A\mathbf{x} - A\mathbf{y}, \mathbf{x} - \mathbf{y})_E = (\mathscr{B}(x_2) - \mathscr{B}(y_2))(x_2 - y_2), \quad \mathbf{x},\mathbf{y} \in D(A),$$

and $A$ is monotone since $\mathscr{B}$ is monotone. Furthermore, the system in $E$,

$$\mathbf{x} + A(\mathbf{x}) \ni \mathbf{f}, \tag{4.2}$$

is equivalent to the pair of equations

$$\mathscr{C}x_2 + \mathscr{A}x_2 + \mathscr{B}(x_2) \ni f_2 + \mathscr{A}f_1, \qquad \mathscr{A}x_1 = -\mathscr{A}x_2 + \mathscr{A}f_1,$$

so it follows that (4.2) has a solution for every $\mathbf{f} \in E$ if and only if the equation

$$\mathscr{C}x_2 + \mathscr{A}x_2 + \mathscr{B}(x_2) \ni g$$

has a solution for every $g \in V'$. This is certainly true since $\mathscr{C} + \mathscr{A}$ is equivalent to the Riesz map of $V$ onto $V'$ and $\mathscr{B}$ is maximal monotone. It follows that $A$ is maximal monotone on the Hilbert space $E$.

From the preceding we know $-A$ generates a nonlinear semigroup of contractions on $E$. Thus, for each $z_0 \in D(A)$ and $g \in W^{1,1}(0,T;E)$ there is a unique $z \in W^{1,\infty}(0,T;E)$ for which

$$z'(t) + A(z(t)) \ni g(t), \quad \text{a.e. } t \in [0,T],$$

$z(0) = z_0$; it is also the case that $z(t) \in D(A)$ at every $t \in [0,T]$. Setting $z_0 = [\mathscr{A}^{-1}v_0, u_0]$ and $g(t) = [\mathscr{A}^{-1}f(t), 0]$ for $0 \leqslant t \leqslant T$, we obtain a pair $z(t) \equiv [v(t), u(t)]$ with $v \in W^{1,\infty}(0,T;V)$, $u \in W^{1,\infty}(0,T;H)$ and a $w \in L^\infty(0,T;V')$ for which $w(t) \in \mathscr{B}(u(t))$ and $w(t) - \mathscr{A}v(t) \in H'$ for each $t \in [0,T]$ and such that

$$\begin{aligned} v'(t) + u(t) &= \mathscr{A}^{-1}f(t) \quad \text{in } L^\infty(0,T;V), \\ u'(t) + C^{-1}(-\mathscr{A}v(t) + w(t)) &= 0 \quad \text{in } L^\infty(0,T;H), \end{aligned} \tag{4.3}$$

and $u(0) = u_0$, $\mathscr{A}v(0) = v_0$. It is clear that the system (4.3) is equivalent to (4.1).

It remains to recover the weak form of the Stefan problem (3.10) as a special case of (4.1). Make the change of variable $U = K(u)$ and observe that $H(U) = H(u)$. Then choose $H = L^2(G) \cong H'$ and $\mathscr{C}u = \tau c u$ for $u \in H$. Similarly, let $V = H_0^1(G)$ and define $\mathscr{A} \in \mathscr{L}(V, V')$ by the scalar product,

$$\mathscr{A}U(v) = \int_G \nabla U \cdot \nabla v, \quad U, v \in H_0^1(G),$$

so $\mathscr{A} = -\Delta$ is a distribution-valued Laplace operator. Finally, we define $\mathscr{B}(U) = C \circ K^{-1}(U) + LH(U)$, the indicated monotone operator obtained from the $L^2$-realization of the (multi-valued) maximal monotone graph $C \circ K^{-1}(\cdot) + LH(\cdot)$, that is, $w \in \mathscr{B}(U)$ if and only if $w = C \circ K^{-1}(U) + L\xi = \tau_0 u + L\xi$ with $U = K(u)$ and $\xi \in H(u)$ a.e. in $G$.

THEOREM 2. Assume that $F \in W^{2,1}(0, T; H^{-1}(G))$, $u_0 \in H_0^1(G)$, $\xi_0 \in L^\infty(G)$ with $\xi_0(x) \in H(u_0(x))$ for a.e. $x \in G$, and $v_0 \in L^2(G)$ are given. Then there exists a unique generalized solution of the Stefan problem (3.10).

*Proof*. It is clear from the above that Theorem 1 holds in our situation and that if $U$ is the solution of (4.1) then $u = K^{-1}(U)$ satisfies the generalized Stefan problem (3.10).

We remark that the initial conditions (3.7c) take different form in $\Omega_+$ (or $\Omega_-$) and $\Omega_0$. In approaching the initial values through $\Omega_+$ we prescribe values of $u$ and $u_t$ as $t \to 0^+$, the classical conditions for the telegrapher's equation. However, in approaching through $\Omega_0$ in the situation of Theorem 2 we prescribe values of $\xi(x, 0^+) \in [0, 1]$.

**5. The one-phase problem.** The single-phase case of the Stefan problem is the description of the melting of a (partially) frozen portion $\Omega_0$ of the region by heat transferred from the melted portion $\Omega_+$ and the corresponding displacement of the common interface $S$. As before $\Omega_+$ and $\Omega_0$ are the regions where $u > 0$ and $u = 0$, respectively, and we let $\tau$, $c$, and $k$ be the parameters corresponding to the melted region. The classical formulation of the single-phase Stefan problem with no internal sources is to find a pair of functions $u$ and $\xi$ on $\Omega$ such that

$$\frac{\partial}{\partial t}\left(\tau c \frac{\partial u}{\partial t} + cu\right) - k\Delta u = 0 \qquad \text{in } \Omega_+, \tag{5.1a}$$

$$L\frac{\partial \xi}{\partial t} = 0 \qquad \text{in } \Omega_0, \tag{5.1b}$$

$$u \geqslant 0 \quad \text{and} \quad \xi \in H(u) \qquad \text{in } \Omega, \tag{5.1c}$$

$$\left(L(1 - \xi) + \tau c \frac{\partial u}{\partial t}\right) N_t = k\nabla u \cdot N_x \quad \text{on } S, \tag{5.1d}$$

$$u(s, t) = 0, \quad s \in \partial G,\ t > 0, \tag{5.1e}$$

$$u(x, 0) = u_0(x), \qquad x \in G, \tag{5.1f}$$

$$\left.\left(\tau c \frac{\partial u}{\partial t} + cu + L\xi\right)\right|_{t=0} = v_0(x), \qquad x \in G. \tag{5.1g}$$

The hyperbolic telegrapher's equation (5.1a) determines the temperatures in the water and the ordinary differential equation (5.1b) regulates the water fraction in the partially frozen region. The conditions in (5.1c) assert that $u$ and $\xi$ correspond to the one-phase Stefan problem. The interface condition (5.1d) is just the one anticipated as (2.6), that is, the heat flux from $\Omega_+$ determines the velocity of the free surface $S$ by melting the fraction $1 - \xi$ of ice with latent heat $L$ and raising the temperature behind the interface after an effective delay of $\tau > 0$. The Dirichlet boundary condition (5.1e) could be replaced by any of the usual boundary conditions. The initial condition (5.1f) establishes the original temperature distribution and (5.1g) specifies the initial value $(e + \tau e')(0) = v_0$. This quantity is approximately given by $cu(\tau) + L\xi(0)$, so these two initial conditions can be regarded as the approximate specification of the initial water-fraction and of temperature history $u(t)$ on the effective delay interval $[0, \tau]$.

From a computation as in Sec. 3 we find that (a), (b), and (d) of (5.1) are equivalent to

$$\frac{\partial}{\partial t}\left(\tau_0 \frac{\partial K(u)}{\partial t} + C(u) + L\xi\right) - \Delta K(u) = 0$$

in $\mathscr{D}'(\Omega)$. This leads to the following definition. A *generalized solution* of (5.1) is a pair of functions $u$, $\xi$ for which

$$u \in W^{1,\infty}(0,T; L^2(G)) \cap L^\infty(0,T; H_0^1(G)), \quad \xi \in L^\infty(\Omega),$$

$$\frac{d}{dt}\big(\tau c u'(t) + cu(t) + L\xi(t)\big) - k\Delta u(t) = 0 \quad \text{a.e. } t \in [0,T], \tag{5.2a}$$

$$u(t) \geqslant 0 \text{ in } H_0^1(G) \quad \text{and} \quad \xi(t) \in H(u(t)) \quad \text{for all } t \in [0,T], \tag{5.2b}$$

$$u(0) = u_0, \qquad (\tau c u' + cu + L\xi)(0) = v_0. \tag{5.2c}$$

As in the classical case, the one-phase problem (5.2) leads to a *variational inequality*. To see this we let $U(t) = \int_0^t u$, known as the *freezing index*, and set

$$E(U) = c\tau U''(t) + cU'(t) + \mathscr{A}U(t) - v_0 + L.$$

An integration in time of (5.2a) yields $E(U) = L(1 - \xi(t))$. Since $\xi(t) \in H(U'(t))$ we obtain the variational inequality

$$E(U) \geqslant 0, \qquad U'(t) \geqslant 0, \qquad E(U)\big(U'(t)\big) = 0, \quad 0 \leqslant t \leqslant T, \tag{5.3}$$

for a solution of (5.2a) and (5.2b). This is easily resolved by Theorem 1. In fact, if we let $\tilde{H}$ be the maximal monotone graph given by $\tilde{H}(r) = \{1\}$ for $r > 0$ and $\tilde{H}(0) = (-\infty, 1]$, then we can write (5.3) formally as

$$\frac{d}{dt}\big(\tau c u'(t) + cu(t) + L\xi(t)\big) - k\Delta u(t) = 0, \quad \text{a.e. } t \in [0,T], \tag{5.4a}$$

$$u(t) \in H_0^1(G) \text{ and } \xi(t) \in \tilde{H}(u(t)) \quad \text{for all } t \in [0,1]. \tag{5.4b}$$

Certainly (5.4b) implies that the constraint $u(t) \geqslant 0$ is satisfied. If we choose $\mathscr{A}$ and $\mathscr{C}$ as in the proof of Theorem 2 but set $\mathscr{B}(u) = cu + L\tilde{H}(u)$, we obtain from Theorem 1 the following.

THEOREM 3. Assume $u_0 \in H_0^1(G)$, $\xi_0 \in L^\infty(G)$ and $v_0 \in L^2(G)$ with $\xi_0(x) \in H(u_0(x))$ and $u_0(x) \geqslant 0$ for a.e. $x \in G$. Then there exists a unique $u \in W^{1,\infty}(0,T; L^2(G)) \cap L^\infty(0,T; H_0^1(G))$ and $\xi\colon [0,T] \to L^2(G)$ which satisfy (5.4) and (5.2c).

Although Theorem 3 provides a solution of the telegrapher's equation on $\Omega_+$ which is nonnegative on $\Omega$, we emphasize that it is *not* necessarily a generalized solution of (5.1). However, it may be appropriate as a model for heat transfer in the present context.

We close with three remarks on the one-phase problem and an example which indicates certain anomalous behavior of solutions. First, from the interface condition (5.1d) it follows that *if*

$$L(1-\xi)+\tau c u_t > 0 \quad \text{on } S,$$

then $N_t < 0$ and the free surface is outward-moving, i.e., $\Omega_+$ is expanding. However, we shall show this is not necessarily so. Second, in the case of the classical Stefan problem ($\tau = 0$) one can use the maximum principle to show that the constraint $u(t) \geqslant 0$ is automatically satisfied by a generalized solution of (5.2a) which starts from nonnegative initial data. We shall show this is not necessarily so for a generalized solution. Third, if the solution of (5.3) were to satisfy $E(U) \leqslant L$, or, equivalently, that the solution $u$ of (5.4) were to satisfy $\xi \geqslant 0$, then $u$ would be a generalized solution of (5.1) in the sense of (5.2). However, this is not necessarily true.

*Example*. Let $a > 0$. Choose $\lambda = [(1+4\tau^2a^2)c/(4\tau k)]^{1/2}$ and define $u(x,t) = \exp(-t/(2\tau))\cos(at)\sin(\lambda x)$ for $x \in G = (0, \pi/\lambda)$ in $\mathbf{R}^1$ and $0 \leqslant t \leqslant \pi/(2a)$. It is easy to check that $u$ satisfies (5.2) and (5.4) (with $\xi = 1$ since $u \geqslant 0$) up until time $t_0 = \pi/(2a)$. At that time $\Omega_+$ decreases instantly to the empty set. In order to continue as a solution of (5.2a) or (5.4a) the jump condition $[\![\tau c u_t + L\xi]\!] = 0$ must be satisfied, that is,

$$\tau c u_t(t_0^+) + L\xi(t_0^+) = L - \tau c a \exp\left(\frac{-\pi}{4a\tau}\right)\sin(\lambda x). \tag{5.5}$$

Suppose we continue along the solution of (5.4) as given by Theorem 3. Then $u_t(t_0^+) \geqslant 0$, so

$$\xi(x, t_0^+) \leqslant 1 - (\tau c a/L)\exp\left(\frac{-\pi}{4a\tau}\right)\sin(\lambda x).$$

If $a$ is sufficiently large there is an interval around $x = \pi/(2\lambda)$ in which $\xi(x, t_0^+) < 0$. Thus (5.2b) is violated and more than $L$ units of latent heat were absorbed by the material. Suppose instead we continue along a solution of (5.4a) for which $\xi \geqslant 0$. Then (5.5) shows that for $a$ sufficiently large there is an interval around $x = \pi/(2\lambda)$ in which $u_t(x, t_0^+) < 0$; hence, $u$ is strictly negative for some time afterward. Thus (5.2b) is violated and the material has a region $\Omega_-$ which is *supercooled*. In particular, there is no generalized solution of (5.2) with the given data.

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