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NONLINEAR ACOUSTIC WAVE PROPAGATION IN ATMOSPHERE*

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Abstract. In this paper we consider a model problem that simulates an atmospheric acoustic wave propagation situation that is nonlinear. The model is derived from the basic Euler equations for the atmospheric flow and from the regular perturbations for the acoustic part. The nonlinear effects are studied by obtaining two successive linear problems in which the second one involves the solution of the first problem. Well-posedness of these problems is discussed and approximations of the radiation boundary conditions that can be used in numerical simulations are presented.

1. Introduction. In this paper we are interested in a two-dimensional model of acoustic wave propagation in the atmosphere. The propagation originates from a point source with a high intensity of sound. It is well known that acoustic wave propagation in the atmosphere is rather a complex phenomenon. It is influenced by atmospheric conditions such as pressure, density, temperature, and wind variations. To analyze the complete problem is a difficult task. However, numerical methods have proven capabilities of handling such problems, but it has not yet been carried out for this class of problems. During the 1960s approximate analytical methods have been attempted for simplified models of the atmosphere. Axisymmetric three-dimensional time-dependent models were done by Cole and Greifinger [1] and [11], and discussions on the time harmonic case were done by Pierce [2]. These models essentially handle only linear wave propagation but allow inhomogeneities in the atmospheric conditions on pressure and density. Our ultimate goal is to treat the full nonlinear model which can incorporate all variations of atmospheric conditions. However, at the present time we shall be concerned with the simplified situations of the above model but retaining nonlinearity.

The goal of the paper is twofold. First, one is to define the problem that governs the nonlinear behavior. It turns out that the problem can be decomposed into two linear ones. We examine the well-posedness of these problems, i.e., devise the mechanism that will lead

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to the existence and uniqueness of the solution of these problems. The second part will contain a brief discussion on radiation or absorbing conditions that are suitable for numerical calculations. It turns out that these are derived from the appropriate dispersion relations for the linear problem. The question of well-posedness of the problem also plays a crucial role in the numerical simulation. In this particular study, modelling the acoustic source is a crucial part. The source should be of the nature that does not violate the well-posedness of the problem. Here we are interested in including sources that are rather nonsmooth. An example is a blast wave sound such as a space shuttle takeoff situation. The sound sources may be pulses, i.e., "delta functions" in both space and time. However, other standard sinusoidal types of sources can be included without difficulty. It turns out that the well-posedness of the problem yields the regularity of the source and thus gives a guideline to approximate a source such as the delta function type in a proper manner. The analysis of this class of problems can be treated according to the theory of Kreiss [3] for hyperbolic systems. Unfortunately, what turns out is a characteristic problem. This does not conform with Kreiss's analysis entirely. We take a slight deviation from his approach. As an outcome of this analysis, one can also derive a family of boundary conditions that can be used for numerical computations. The numerical results will be reported elsewhere.

As we mentioned earlier, the governing equations are derived from the Euler equations for the atmosphere. As in Cole and Greifinger [11], we consider an isothermal atmosphere above a ground plane with sound produced by instantaneous energy release at a point on the ground. We will also consider cases other than that of instantaneous release rate such as sources of smooth sinusoidal type.

We shall begin with the statement of the fluid flow problem which governs the acoustic phenomena. The model assumes strictly a two-dimensional field of atmosphere with a source that produces a release of sound energy at a point z_0 above the ground. The case we treat is of an isothermal atmosphere with the standard model of exponentially varying pressure and density fields. We shall not be concerned with wind speed so that the atmosphere is in equilibrium. Then the equilibrium atmosphere is characterized by exponential distributions for pressure and density with a scale height h,

$$\frac{P_0(z)}{P^*} = e^{-z} = \frac{\rho_0(z)}{\rho^*}, \tag{1.1}$$

where P^*, ρ^*, T^* are sea level pressure, density, and temperature respectively. Also, the scale height h is given by

$$h = \frac{RT^*}{g}. \tag{1.2}$$

To nondimensionalize the problem we need

$$c^* = \sqrt{\gamma R T^*}, \tag{1.3}$$

which is the isentropic sound speed and

$$c_g = \sqrt{gh}, \tag{1.4}$$

the gravity sound speed. Then the nondimensional form of the Euler equations (the equations of continuity, balance of momentum, and energy) is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{q}) = 0, \tag{1.5}$$

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q} = \frac{-1}{\gamma\rho}\nabla p - \frac{1}{\gamma}\hat{\mathbf{k}}, \tag{1.6}$$

$$\left[\frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla\right]\frac{\bar{p}}{\rho^{\gamma}} = \frac{1}{\rho^{\gamma-1}}\varepsilon f(x, z, t). \tag{1.7}$$

Note that in equation (1.6) the forcing term, $-(1/\gamma)\hat{\mathbf{k}}$ ($\hat{\mathbf{k}}$ is the unit vector in the z direction), arises due to the forcing term per unit mass $-g\hat{\mathbf{k}}$ in the original variables which is due to gravity. In equation (1.7), $f(x, z, t)$ dictates the space time dependency of the source (see Fig. 1) and ε measures the energy release per unit volume. For the case of an instantaneous energy release, ε is given by

$$\varepsilon = \frac{(\gamma - 1)Q_0}{h^3\rho^*} \tag{1.8}$$

where Q_0 is the total energy released at time $t = 0$. The initial conditions are

$$\bar{p} = \rho = e^{-z}, \qquad \mathbf{q} = \mathbf{0} \quad \text{at } t = 0. \tag{1.9}$$

The boundary conditions at $z = 0$ are

$$\mathbf{q}_z = \mathbf{0}, \tag{1.10}$$

which states that the vertical component of the flow is zero at $z = 0$..

2. Formulation of the acoustics problem. The acoustic expansion is based on $\varepsilon \ll 1$ and represents the flow as small changes superimposed on the flow of the ambient state. We note that the ambient velocity is zero, but pressure and density have the form e^{-z}. Thus

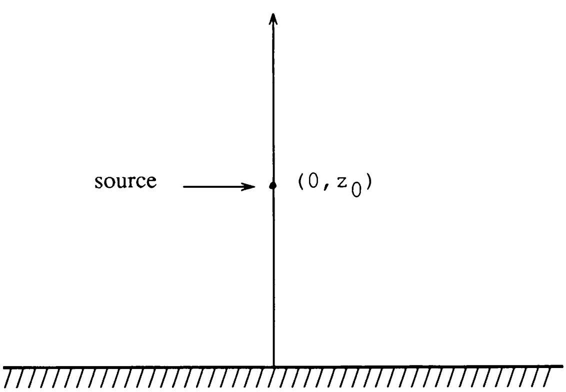

FIG. 1.

the expansions are:

$$\mathbf{q} = \varepsilon\mathbf{u} + \varepsilon^2\mathbf{u}_1 + \cdots, \tag{2.1}$$

$$\bar{p} = e^{-z}\{1 + \varepsilon p + \varepsilon^2 p_1 + \cdots\}, \tag{2.2}$$

$$\rho = e^{-z}\{1 + \varepsilon\sigma + \varepsilon^2\sigma_1 + \cdots\}, \tag{2.3}$$

where $\mathbf{u} = (u, w)$, and $\mathbf{u}_1 = (u_1, w_1)$. Quantities u, u_1 and w, w_1 are the x and the z components of the acoustic velocities, respectively. We substitute expansions (2.1)–(2.3) into equations (1.5)–(1.7), initial conditions (1.9), and boundary conditions (1.10) to obtain the field equations.

The problem that results from order ε is linear and is similar to the one reported by Cole and Greifinger [1]. This is as follows:

$$\sigma_t + u_x + w_z - w = 0, \tag{2.4}$$

$$u_t + \frac{1}{\gamma}p_x = 0, \tag{2.5}$$

$$w_t + \frac{1}{\gamma}p_z - \frac{p - \sigma}{\gamma} = 0, \tag{2.6}$$

$$p_t - \gamma\sigma_t + (\gamma - 1)w = f(x, z, t). \tag{2.7}$$

We rewrite (2.7) using (2.4) as

$$p_t + \gamma u_x + \gamma w_z - w = f(x, z, t). \tag{2.8}$$

Initial and boundary conditions for these perturbations are:

$$p = \sigma = u = w = 0, \tag{2.9}$$

$$w = 0 \quad \text{at } z = 0,\ t > 0. \tag{2.10}$$

For convenience we will call (2.4)–(2.10) problem P_{I}.

Similarly the problem that results from order ε^2 has the form:

$$\sigma_{1,t} + u_{1,x} + w_{1,z} - w_1 = f_1, \tag{2.11}$$

$$u_{1,t} + (1/\gamma)p_{1,x} = f_2, \tag{2.12}$$

$$w_{1,t} + (1/\gamma)p_{1,z} - (p_1 - \sigma_1)/\gamma = f_3, \tag{2.13}$$

$$p_{1,t} - \gamma\sigma_{1,t} + (\gamma - 1)w_1 = f_4, \tag{2.14}$$

with *initial conditions*

$$p_1 = \sigma_1 = u_1 = w_1 = 0 \quad \text{at } t = 0 \tag{2.15}$$

and *boundary conditions*

$$w_1 = 0 \quad \text{at } z = 0,\ t > 0. \tag{2.16}$$

Here,

$$
\begin{aligned}
f_1 &= -[(\sigma u)_x + (\sigma w)_z - \sigma w], \\
f_2 &= \sigma p_x/\gamma - (u u_x + w u_z), \\
f_3 &= \sigma(p_z - p)/\gamma - (u w_x + w w_z), \\
f_4 &= \gamma\{p\sigma - (\gamma + 1)\sigma^2/2\}_t - u(p - \gamma\sigma)_x \\
&\quad + w\{(p - \gamma\sigma)_z + (\gamma - 1)(p - \gamma\sigma)\} - (\gamma - 1)\sigma f(x, z, t).
\end{aligned}
\tag{2.17}
$$

Again, for convenience, we shall refer to equations (2.11)–(2.16) as problem P_{II}. We allow sufficient smoothness on the right-hand side which contains terms given by (2.17) so that P_{II} is well-posed. In fact this gives us the regularity of solution of P_{II}. Once a numerical procedure is constructed for the solution of problem P_{I}, the same procedure can be used to compute the solutions of P_{II} since the differential operator on both problems is the same with the bonus of identical initial and boundary conditions. Also, once the components of the solution (p, p_1), (σ, σ_1), and $(\mathbf{u}, \mathbf{u}_1)$ are known then the solutions of the nonlinear field are given by:

$$
\begin{aligned}
p &= \varepsilon p + \varepsilon^2 p_1, \\
\sigma &= \varepsilon\sigma + \varepsilon^2\sigma_1, \\
\mathbf{q} &= \varepsilon\mathbf{u} + \varepsilon^2\mathbf{u}_1.
\end{aligned}
\tag{2.18}
$$

This procedure can be continued to obtain higher-order approximations for the nonlinear problem. However, a sequence of study reported by Hariharan and Lester [4, 5] for one-dimensional problems and by Hariharan [6] for two-dimensional problems of nonlinear acoustic calculations shows that only two terms are needed to investigate the nonlinearity, even for the case of shock waves. A natural question one may ask is why not solve the nonlinear problem directly, as in the above references, including discontinuities in the solutions such as shock waves. The solutions may form shock discontinuities in the vicinity of the source, in which case considering two linear problems P_{I} and P_{II} separately will not be uniformly valid. However, we are interested in the sound field far away from the source, and the region of possible shock discontinuities is still considered as a source region. A full mathematical justification may be a difficult task.

3. Formal solutions and estimates. Here, we discuss the existence and uniqueness of the initial boundary value problems P_{I} and P_{II}. We shall accomplish this by obtaining proper energy estimates. The first step is to write the governing equations in the following form:

$$
\mathbf{u}_t = \mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_z + \mathbf{C}\mathbf{u} = \mathbf{f} \tag{3.1}
$$

where

$$
\begin{aligned}
&\mathbf{A} = a_{ij},\ a_{12} = 1,\ a_{24} = 1/\gamma,\ a_{42} = \gamma \quad \text{and all other } a_{ij} = 0, \\
&\mathbf{B} = b_{ij},\ b_{13} = 1,\ b_{34} = 1/\gamma,\ b_{43} = \gamma \quad \text{and all other } b_{ij} = 0, \\
&\mathbf{C} = c_{ij},\ c_{13} = -1,\ c_{31} = -c_{34} = 1/\gamma,\ c_{43} = -1, \quad \text{the rest } 0.
\end{aligned}
$$

Also, $\mathbf{u} = (\sigma, u, w, p)^{\mathrm{T}}$ for P_{I} and $\mathbf{u} = (\sigma_1, u_1, w_1, p_1)^{\mathrm{T}}$ for P_{II}. Similarly, the right-hand side $\mathbf{f}$ has the following definition:

$$\mathbf{f} = (0,0,0,f)^{\mathrm{T}} \text{ for } P_{\mathrm{I}} \text{ and } \mathbf{f}_1 = (f_1, f_2, f_3, f_4 + \gamma f_1)^{\mathrm{T}} \text{ for } P_{\mathrm{II}}.$$

The boundary conditions are:

$$w = 0 \quad \text{on } z = 0 \text{ for } P_{\mathrm{I}},$$
$$w_1 = 0 \quad \text{on } z = 0 \text{ for } P_{\mathrm{II}}.$$

Initial conditions are: $\mathbf{u} = \mathbf{0}$ at $t = 0$ for both P_{I} and P_{II}. We want to treat the problems in the context of hyperbolic equations. Let us collect needed relevant information from the theory of hyperbolic equations. First consider the definition of hyperbolicity. Let $\mathbf{A}_j(\mathbf{u})$ and $\mathbf{C}(\mathbf{u})$ be such that

$$\mathbf{u}_t + \sum \mathbf{A}_j(\mathbf{u})\mathbf{u}_{x_j} + \mathbf{C}(\mathbf{u})\mathbf{u} = \mathbf{f}(\mathbf{x}, t). \tag{3.2}$$

DEFINITION 3.1.

If the eigenvalues of

$$\mathbf{A}(\mathbf{u}, \mathbf{w}) = \sum \mathbf{A}_j(\mathbf{u}) w_j$$

are real for real vectors $\mathbf{u}$ and $\mathbf{w}$ then the system (3.2) is said to be *hyperbolic*. If the eigenvalues are real and distinct, then the system is said to be *strongly hyperbolic*.

According to this definition, it is easy to verify from (3.1) that the eigenvalues of $\mathbf{A}w_1 + \mathbf{B}w_2$ are

$$0, 0, \left(w_1^2 + w_2^2\right)^{1/2} \text{ and } -\left(w_1^2 + w_2^2\right)^{1/2}$$

satisfying hyperbolicity, but not strong hyperbolicity. The next notion we require is the symmetry property. Again we consider the system (3.2) for this purpose. In general, $\mathbf{A}_j(\mathbf{u})$ need not be symmetric. There are varieties of procedures which are equivalent to saying the system (3.2) can be written in symmetric form. One of these is in the Freidrichs sense; i.e., there exists a matrix-valued function $\mathbf{E}(\mathbf{u})$ which depends on $\mathbf{u}$ such that the matrices

$$\mathbf{B}_j(\mathbf{u}) = \mathbf{E}(\mathbf{u})^{-1}\mathbf{A}_j(\mathbf{u})\mathbf{E}(\mathbf{u})$$

are symmetric and then system (3.2) can be written in the symmetric form:

$$\mathbf{v}_t + \sum \mathbf{B}_j(\mathbf{v})\mathbf{v}_{x_j} + \mathbf{R}\mathbf{v} = \mathbf{E}(\mathbf{u})\mathbf{f}(\mathbf{x}, t). \tag{3.3}$$

In our considerations $\mathbf{A}$ and $\mathbf{B}$ do not depend on $\mathbf{u}$, implying $\mathbf{E}$ will not depend on $\mathbf{u}$ either. Our first goal is to obtain this matrix $\mathbf{E}$ which we call the symmetrizer. The construction follows from:

LEMMA 3.1. There exists a matrix $\mathbf{E}$ such that the system (3.1) can be written in a symmetric form:

$$\mathbf{v}_t + \mathbf{P}\mathbf{v}_x + \mathbf{Q}\mathbf{v}_z - \mathbf{R}\mathbf{v} = \mathbf{E}\mathbf{f}, \tag{3.4}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are symmetric and $\mathbf{v} = \mathbf{E}\mathbf{u}$.

Proof. The procedure consists of finding a matrix which will simultaneously symmetrize both $\mathbf{A}$ and $\mathbf{B}$. The first step is to find a matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is diagonal. This is easily accomplished by forming the matrix $\mathbf{T}$ using eigenvectors of $\mathbf{A}$. In this case, $\mathbf{T}$ is given by

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \gamma \end{bmatrix}$$

so that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \operatorname{diag}(0,0,1,-1). \tag{3.5}$$

The diagonal elements are simply the eigenvalues of $\mathbf{A}$. This in turn yields:

$$\mathbf{T}^{-1}\mathbf{B}\mathbf{T} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1 \end{bmatrix}. \tag{3.6}$$

This matrix is not symmetric. So we further investigate possibilities of symmetrizing this matrix which preserve the symmetry of $\mathbf{A}$. Consider a diagonal matrix $\mathbf{D} = \operatorname{diag}(\alpha, \beta, \gamma, \delta)$. This gives the above property for properly chosen constant diagonal elements. We observe that

$$\mathbf{D}^{-1}\mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{D} = \operatorname{diag}(0,0,1,-1), \tag{3.7}$$

which simply shows the diagonal form of $\mathbf{A}$ is preserved while

$$\mathbf{D}^{-1}\mathbf{T}^{-1}\mathbf{B}\mathbf{T}\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma/\beta & \delta/\beta \\ 0 & \beta/(2\gamma) & 0 & 0 \\ 0 & \beta/(2\delta) & 0 & 0 \end{bmatrix}. \tag{3.8}$$

We choose α, β, γ, and δ in (3.8) so that the right-hand side will be symmetric. This restriction gives us the following relations:

$$\gamma/\beta = \beta/(2\gamma), \ \delta/\beta = \beta/(2\delta), \text{ and } \alpha \text{ arbitrary}.$$

Upon solving these equations, we find that the one solution is $\beta = \sqrt{2}$, $\gamma = \delta = 1$. Since α is arbitrary we choose it to be 1. Then $\mathbf{D} = \operatorname{diag}(1, \sqrt{2}, 1, 1)$. This the matrix $\mathbf{G} = (\mathbf{T}D)^{-1}$ gives both $\mathbf{G}^{-1}\mathbf{A}\mathbf{G}$ and $\mathbf{G}^{-1}\mathbf{B}\mathbf{G}$ as symmetric. Hence, we have the following symmetric hyperbolic system:

$$\mathbf{v}_t + \mathbf{P}_x\mathbf{v} + \mathbf{Q}_z\mathbf{v} - \mathbf{R}\mathbf{v} = \mathbf{F}, \tag{3.9}$$

where $\mathbf{v} = \mathbf{G}\mathbf{u}$, $\mathbf{P} = \mathbf{G}\mathbf{A}\mathbf{G}^{-1}$, $\mathbf{Q} = \mathbf{G}\mathbf{B}\mathbf{G}^{-1}$, $\mathbf{R} = -\mathbf{G}\mathbf{C}\mathbf{G}^{-1}$, and $\mathbf{F} = \mathbf{G}\mathbf{f}$.

The next step is to consider the well-posedness of the problems P_{I} and P_{II}. Any definition of well-posedness of an initial boundary value problem consists of several steps. Namely, they are:

a) specification of spaces $H_{\mathbf{F}}$ to which $\mathbf{F}$ belongs,

b) the space in which the solution $\mathbf{v}$ is sought,

c) existence and uniqueness of the solution $\mathbf{v} \in H_{\mathbf{v}}$, for any $\mathbf{F} \in H_{\mathbf{F}}$,

d) continuous dependence of the solution on the function $\mathbf{F}$.

Composition of all these steps leads to a detailed analysis of the problem. The machinery to establish such steps follows from Friedrichs [7], provided a suitable energy estimate is derived. Therefore, we shall be concerned only with deriving an estimate for these problems. It turns out that the energy estimate indicates the regularity of **F**, which gives a guide to modelling the source in our acoustics problem.

To derive energy estimates for both problems, we define the following quantities.

Denote the inner product of two vectors by

$$(\mathbf{u}, \mathbf{v}) = \mathbf{u}^{\mathrm{T}}\mathbf{v}.$$

Let $\Omega \subset \mathbf{R}^2$ be the half space $z \geqslant 0$, $-\infty < x < \infty$. Define the L_2 norm of a vector in Ω by

$$\|\mathbf{u}\|_0^2 = \int_\Omega (\mathbf{u}, \mathbf{u})\, dx\, dz = \int_0^\infty \int_{-\infty}^\infty (\mathbf{u}, \mathbf{u})\, dx\, dz. \tag{3.10}$$

We introduce functions $\mathbf{w} = e^{-\eta t}\mathbf{v}$, $\mathbf{H} = e^{-\eta t}\mathbf{F}$, for some positive constant η. Then (3.9) becomes

$$\mathbf{w}_t + \mathbf{P}\mathbf{w}_x + \mathbf{Q}\mathbf{w}_z - \mathbf{R}\mathbf{w} + \eta\mathbf{I}\mathbf{w} = \mathbf{H}. \tag{3.11}$$

Now consider the derivative of the inner product,

$$(\mathbf{w}, \mathbf{w})_t = (\mathbf{w}, \mathbf{w}_t) + (\mathbf{w}_t, \mathbf{w}).$$

Using equation (3.11) we have

$$(\mathbf{w}, \mathbf{w})_t = -(\mathbf{P}\mathbf{w}, \mathbf{w}_x) - (\mathbf{w}, \mathbf{P}\mathbf{w}_x) - (\mathbf{w}, \mathbf{Q}\mathbf{w}_z) - (\mathbf{Q}\mathbf{w}_{x_z}, \mathbf{w})$$
$$+ \left(\mathbf{w}, (\mathbf{R} + \mathbf{R}^{\mathrm{T}} - 2\eta\mathbf{I})\mathbf{w}\right) + 2(\mathbf{w}, \mathbf{H}).$$

Now using the symmetry properties of **P** and **Q** derived in Lemma 3.1, we obtain:

$$(\mathbf{w}, \mathbf{w})_t = -(\mathbf{w}, \mathbf{P}\mathbf{w})_x - (\mathbf{w}, \mathbf{Q}\mathbf{w})_z + \left(\mathbf{w}, (\mathbf{R} + \mathbf{R}^T - 2\eta\mathbf{I})\mathbf{w}\right) + 2(\mathbf{w}, \mathbf{H}). \tag{3.12}$$

Integrating (3.12) over Ω, we obtain the following energy integral:

$$\frac{d}{dt}\|\mathbf{w}\|_0^2 = \int_{-\infty}^\infty (\mathbf{w}, \mathbf{Q}\mathbf{w})\|_{z=0}\, dx + \int_\Omega \left\{\left(\mathbf{w}, (\mathbf{R} + \mathbf{R}^{\mathrm{T}} - 2\eta\mathbf{I})\mathbf{w}\right\}\, dx\, dz + 2\int_\Omega (\mathbf{w}, \mathbf{H})\, dx\, dz. \tag{3.13}$$

Recall our aim is to obtain an energy integral inequality. At this point we need the notion called "maximal dissipativity." A discussion of this concept can be found in Kreiss [8]. Suppose a boundary condition of the form $\mathbf{B}_1\mathbf{w} = \mathbf{0}$ at $z = 0$ is posed where $\mathbf{B}_1$ is a rectangular matrix. Then we have the following:

DEFINITION 3.2. The boundary condition $\mathbf{B}_1\mathbf{w} = \mathbf{0}$ at $z = 0$ is maximally dissipative provided

$$(\mathbf{y}, \mathbf{Q}\mathbf{y})|_{z=0} \leqslant 0 \text{ for all } \mathbf{y} \text{ satisfying } \mathbf{B}_1\mathbf{y} = \mathbf{0}.$$

For the moment we shall assume there is a boundary operator $\mathbf{B}_1$ which satisfies the definition 3.2. This means that we need to prove the following:

LEMMA 3.2. There is a boundary operator $\mathbf{B}_1\mathbf{w} = \mathbf{0}$ satisfying maximal dissipativeness with $\mathbf{w}(x, 0, t) = 0$.

REMARK 3.1. The resulting boundary operator $\mathbf{B}_1$ is exactly the rectangular matrix:

$$\mathbf{B}_1 = [0 \quad 0 \quad 1 \quad 0].$$

Returning to the equation (3.13), the energy integral, we use Lemma 3.2 and obtain the following inequality:

$$\frac{d}{dt}\|\mathbf{w}\|_0^2 \leqslant \int_\Omega \{(\mathbf{w}, (\mathbf{R} + \mathbf{R}^{\mathrm{T}} - 2\eta\mathbf{I})\mathbf{w})\}\, dx\, dz + 2\int_\Omega (\mathbf{w}, \mathbf{H})\, dx\, dz. \tag{3.14}$$

If η_0 is big enough, $\eta_0\mathbf{I} > \mathbf{R} + \mathbf{R}^{\mathrm{T}}$, for instance, set

$$\eta_0 = 2\|\mathbf{R}\|/\delta \quad (\delta < 1).$$

Then for $\eta > 2\eta_0$,

$$-(\mathbf{w}, (\mathbf{R} + \mathbf{R}^{\mathrm{T}})\mathbf{w}) + 2\eta(\mathbf{w}, \mathbf{w}) \geqslant 2(\eta - \eta_0)(\mathbf{w}, \mathbf{w}) \geqslant \eta(\mathbf{w}, \mathbf{w}). \tag{3.15}$$

On the other hand we have the inequality,

$$(\mathbf{w}, \mathbf{H}) \leqslant \|\mathbf{w}\|\,\|\mathbf{H}\| \leqslant (\varepsilon/2)\|\mathbf{w}\|^2 + (1/(2\varepsilon))\|\mathbf{H}\|^2.$$

Take $\varepsilon = \eta\delta/2$, so that

$$(\varepsilon/2)\|\mathbf{w}\| = (\eta\delta/4)\|\mathbf{w}\|^2 \leqslant (\eta/4)\|\mathbf{w}\|^2.$$

Thus the inequality (3.14) becomes

$$\frac{d}{dt}\|\mathbf{w}\|_0^2 \leqslant -\eta\|\mathbf{w}\|_0^2 + (\eta/2)\|\mathbf{w}\|_0^2 + (2\eta^{-1}/\delta)\|\mathbf{H}\|_0^2. \tag{3.16}$$

Integrating (3.16) from time $t = 0$ to $t = T$ and using the zero initial conditions, we obtain the following inequality:

$$\|e^{-\eta T}\mathbf{v}(\mathbf{x}, T)\|_0^2 + (\eta/2)\int_0^T \|e^{-\eta t}\mathbf{v}(\mathbf{x}, t)\|_0^2\, dt \leqslant C\eta^{-1}\int_0^T \|e^{-\eta t}\mathbf{F}(\mathbf{x}, t)\|_0^2\, dt. \tag{3.17}$$

Here, $\mathbf{x} = (x, z)$, $\eta > 2\eta_0$, and C is a constant independent of $\mathbf{F}(\mathbf{x}, t)$. Inequality (3.17) holds for both problems P_{I} and P_{II} with an appropriate forcing function $\mathbf{F}$. In summary, the above procedure yields the desired result.

THEOREM 3.1. Problem P_{I} is well-posed; i.e., for any $\mathbf{F} \in L_2(\Omega)$, there exists a unique solution $\mathbf{v}$ in $L_2(\Omega)$, satisfying the estimate

$$\|\mathbf{v}(\mathbf{x}, T)e^{-\eta T}\|_0^2 + (1/2)\eta\int_0^T \|\mathbf{v}(\mathbf{x}, t)e^{-\eta t}\|_0^2 \leqslant K\eta^{-1}\int_0^T \|\mathbf{F}(\mathbf{x}, t)e^{-\eta t}\|_0^2\, dt \tag{3.18}$$

for any $\eta \geqslant 2\eta_0 > 0$ and for some constant $\mathbf{K}$ independent of $\mathbf{F}$.

REMARK 3.3. Observe that $\mathbf{u} = \mathbf{G}^{-1}\mathbf{v}$, $\mathbf{f} = \mathbf{G}^{-1}\mathbf{F}$, and it is readily verified that $\mathbf{u}$ satisfies the same estimate as that of (3.18) with $\mathbf{F}$ replaced by $\mathbf{f}$.

REMARK 3.4. Theorem 3.1 suggests that the forcing function $\mathbf{f}$ should be at least in $L_2(\Omega)$. Thus for practical considerations, even if the acoustics are generated by pulse sources (e.g., of the "delta function" type), they should be modelled by a function which is in $L_2(\Omega)$.

Proof of the well-posedness of the problem P_{II} is similar as mentioned earlier and we merely state it.

THEOREM 3.2. For any $\mathbf{f}_1 \in L_2(\Omega)$ there exists a unique solution $\mathbf{u}_1 \in L_2(\Omega)$ such that

$$\|\mathbf{u}_1(\mathbf{x}, t)e^{-\eta T}\|_0^2 + \tfrac{1}{2}\eta\int_0^T \|\mathbf{u}_1(\mathbf{x}, t)e^{-\eta t}\|_0^2\, dt \leqslant K\eta^{-1}\int_0^T \|\mathbf{f}_1(\mathbf{x}, t)e^{-\eta t}\|_0^2\, dt \tag{3.19}$$

and the solution of P_{I}, i.e., $\mathbf{u}$ and the forcing term $\mathbf{f}$ are both in $H^1(\Omega)$.

REMARK 3.5. The requirement $\mathbf{u} \in H^1(\Omega)$ (which is the Sobolev space of order 1) arises because the forcing term for P_{II} contains derivatives of the solution of P_{I}.

Recall that the nonlinear solution is sought in the form

$$\mathbf{u}_n = \varepsilon\mathbf{u} + \varepsilon^2\mathbf{u}_1. \tag{3.20}$$

This is as stated in equation (2.18) the linear combination of problems P_{I} and P_{II}. Combining Theorems 3.1 and 3.2, we have the following:

THEOREM 3.3. There exists a unique nonlinear solultion $\mathbf{u}_n \in L_2(\Omega)$ for the two term linear solutions of P_{I} and P_{II} for the nonlinear acoustic problem provided that the forcing function $\mathbf{f}$ for P_{I} is in $L_2(\Omega)$.

This theorem tells us that the smoothness of the source of acoustics should be more than a square integrable function. Its first derivative must also be a square integrable function. In such a situation, it is sufficient to assume enough smoothness on it. This becomes crucial in the numerical computations. If one uses a second-order finite difference scheme, all the spatial derivatives need to be at least in $C^3(\Omega)$. Thus, rather than considering step by step the regularity of the source, it is easier to approximate it by a C^∞ function. For example, if we consider a source term of the form $f(x, z, t) = \delta(x)g(z, t)$, where g is a smooth function, then one may approximate f by f_m where

$$f = (m/\pi)\exp(-m^2x^2)g(z, t). \tag{3.21}$$

Similar modifications are easily made when the source is a pulse in the other independent variables z and t.

To end this section, let us conclude with the proof of Lemma 3.2.

Proof (*Lemma* 3.2). Same proof holds for both P_{I} and P_{II}. The given boundary condition in both problems is the normal velocity component zero; i.e., $\mathbf{w} \equiv \mathbf{0}$, $w_1 = 0$ on $z = 0$. Therefore, let $\mathbf{y} = (\rho, u, 0, p)^{\mathrm{T}}$. For any vector in this form, we compute $(\mathbf{w}, \mathbf{Q}\mathbf{w})$, that is,

$$(\mathbf{w}, \mathbf{Q}\mathbf{w}) = \mathbf{y}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{G}\mathbf{B}\mathbf{G}^{-1}\mathbf{G}\mathbf{y}. \tag{3.22}$$

Noting that $\mathbf{G} = (\mathbf{T}\mathbf{D})^{-1}$, we have

$$\mathbf{G}^{-1} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \gamma & \gamma \end{bmatrix} \text{ and } \mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & -1/\gamma \\ 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/2 & 0 & 1/(2\gamma) \\ 0 & -1/2 & 0 & 1/(2\gamma) \end{bmatrix}.$$

Then simple matrix manipulations yield

$$\mathbf{G}^{\mathrm{T}}\mathbf{G}\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\gamma} \\ 0 & 0 & \frac{1}{2\gamma} & 0 \end{bmatrix}$$

and $(\mathbf{w}, \mathbf{Q}\mathbf{w}) = 0$ and the corresponding boundary conditions have the form

$$[0 \quad 0 \quad 1 \quad 0]\begin{bmatrix} \rho \\ u \\ w \\ p \end{bmatrix} = 0$$

and the rectangular matrix $\mathbf{B}_1$ is easily identified.

4. Radiation boundary conditions. For computational purposes, it is essential to truncate half space Ω into a finite region Ω'. For example, if one uses a finite difference scheme, then it makes computations easier if Ω' is a rectangle as indicated in Fig. 2.

Then the pieces of the boundaries Γ_1, Γ_2, and Γ_3 need to be nonreflecting or radiating boundaries since they must correspond to wave behavior at far distances. For the simple wave equation such discussions are extensively known. A summary of these may be found in Hariharan [9]. In this reference, particularly the work of Engquist and Majda [10] is noted. What follows is an attempt to extend the idea in [10] to obtain boundary conditions for the linear problems under consideration. For this purpose, we shall be concerned with only the problem P_{I}. The same radiation conditions are applicable to problem P_{II}. Recall that problem P_{I} is prescribed by equations (2.5)–(2.10). Suppose we are interested in the radiation boundary condition on the boundary Γ_3. It is sufficient to treat this boundary alone for obtaining radiation conditions. That is to say, treat the problem as a half space problem, with Γ_3 playing the role of the x-axis ($-\infty < x < \infty$).

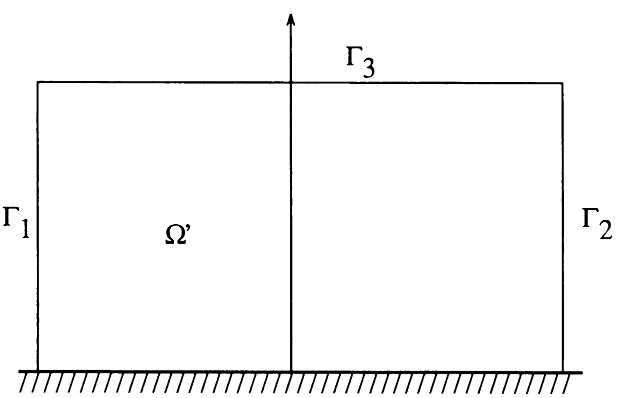

FIG. 2.

To follow the idea of [10], we take the Laplace transform with respect to time t (since the initial values are specified) and the Fourier transform with respect to x of equations (2.5)–(2.8). Using initial conditions (2.9) we obtain:

$$\begin{aligned} s\hat{\sigma} + i\xi\hat{u} + \hat{w}_z - \hat{w} &= 0, \\ s\hat{u} + \left(\frac{i\xi}{\gamma}\right)\hat{p} &= 0, \\ s\hat{w} + (1/\gamma)\hat{p}_z - \frac{\hat{p} - \hat{\sigma}}{\gamma} &= 0, \\ s\hat{p} + i\gamma\xi\hat{u} + \gamma\hat{w}_z - \hat{w} &= \hat{f}. \end{aligned} \tag{4.1}$$

At far distances the effect of the forcing term f vanishes. Thus we seek homogeneous solutions of (4.1) with dependence $\exp(\lambda z)$. This leads to the characteristic equation

$$\det\begin{bmatrix} s & i\xi & \lambda - 1 & 0 \\ 0 & s & 0 & i\xi/\gamma \\ 1/\gamma & 0 & s & (\lambda - 1)/\gamma \\ 0 & i\gamma\xi & \gamma\lambda - 1 & s \end{bmatrix} = 0. \tag{4.2}$$

This has roots

$$\lambda_{1,2} = 1/2 \pm \mu(s, \xi), \tag{4.3}$$

where

$$\mu(s, \xi) = \left(1/4 + s^2 + \xi^2 + \beta^2\xi^2/s^2\right)^{1/2} \tag{4.4}$$

with

$$\beta^2 = (\gamma^2 - 1)/\gamma \quad (> 0). \tag{4.5}$$

For a decaying wave we choose the negative root of (4.3). Moreover, $\mu(s, \xi)$ can be written in the form

$$\mu(s, \xi) = \left(s^2 + w_1^2(\xi)\right)^{1/2}\left(s^2 + \omega_2^2(\xi)\right)^{1/2}/s. \tag{4.6}$$

Indeed one can obtain a theoretical solution of (4.1) by variation of parameters using the homogeneous solution dictated by $\exp(\lambda z)$ with the values of λ given by (4.3). But the difficulty will be to invert the transforms using the boundary conditions. A similar difficulty arises at far distances even without the source term. It is easy to see that the integrand will contain terms of the form $\exp((1/2)z - \mu(s, \xi)z)$. From equation (4.6) we obtain ω_1 and ω_2 explicitly as follows:

$$\begin{aligned} \omega_1(\xi) &= \frac{1}{2}\left[\left(\frac{1}{4} + \xi^2 + 2\beta\xi\right)^{1/2} + \left(\frac{1}{4} + \xi^2 - 2\beta\xi\right)^{1/2}\right], \\ \omega_2(\xi) &= \frac{1}{2}\left[\left(\frac{1}{4} + \xi^2 + 2\beta\xi\right)^{1/2} - \left(\frac{1}{4} + \xi^2 - 2\beta\xi\right)^{1/2}\right]. \end{aligned} \tag{4.7}$$

Let $s = i\tau$. Then equation (4.6) becomes

$$\mu(i\tau, \xi) = 1/(i\tau)\left[\left(\tau^2 - \omega_1^2(\xi)\right)\left(\tau^2 - \omega_2^2(\xi)\right)\right]^{1/2}. \tag{4.8}$$

For propagating waves we require μ to be imaginary. This is ensured by $\tau > \omega_1(\xi)$ or $\tau < \omega_2(\xi)$. From (4.7), one finds such a requirement is satisfied provided $|\xi/\tau| < 1$, or equivalently $|\xi/s| < 1$. Thus we shall be concerned with approximating $\mu(s, \xi)$ for large values of s. To see this let us emphasize that the solution of λ we seek is of the form

$$\lambda = \frac{1}{2} - \mu(s, \xi). \tag{4.9}$$

Multiplying equation (4.9) by $\exp(\lambda z)$, we obtain an associated differential operator

$$\frac{d\hat{p}}{dz} = \frac{1}{2}\hat{p} - \mu(s, \xi)\hat{p}. \tag{4.10}$$

In the pseudo-differential operator terminology, $\mu(s, \xi)$ is the symbol of an associated pseudo-differential operator. To obtain radiating solutions, equation (4.10) needs to be inverted for both the Laplace transform and the Fourier transform. Indeed a perfectly absorbing boundary condition arising from (4.10) is given by the inversion,

$$\frac{\partial}{\partial z}\int_{-i\infty}^{i\infty}\int_{-\infty}^{\infty} \hat{p}(z; \xi, s)e^{st - i\xi x}\, d\xi\, ds = \int_{-i\infty}^{i\infty}\int_{-\infty}^{\infty} \hat{p}(z; \xi, s)\left\{\frac{1}{2} - \mu(s, \xi)\right\}e^{st - i\xi x}\, d\xi\, ds.$$

The above expression simplifies to

$$\frac{\partial p}{\partial z} = \frac{1}{2}p - \int_{-i\infty}^{i\infty}\int_{-\infty}^{\infty} \mu(s, \xi)\hat{p}(z; \xi, s)e^{st - i\xi x}\, d\xi\, ds. \tag{4.11}$$

Equation (4.11) provides a boundary condition that is nonlocal in time and space. A similar procedure to that discussed above will hold for boundary conditions on the boundaries Γ_1 and Γ_2 provided we take the Fourier transform with respect to z and construct differential operators in the direction of x. Boundary condition (4.11) is not easy to implement. However, if we approximate the symbol $\mu(s, \xi)$ for large values of s then it is possible to obtain approximate local boundary conditions from (4.11). To do this we consider $\mu(s, \xi)$ again and investigate its nature when $|\xi/s| \ll 1$. We rewrite μ as

$$\mu(s, \xi) = s\left[1 + \left(\frac{1}{4} + \xi^2\right)/s^2 + \beta^2\xi^2/s^4\right]^{1/2}. \tag{4.12}$$

A crude approximation is $\mu \simeq s$. Substitution of this approximation in equation (4.11) gives the boundary operator

$$\frac{\partial p}{\partial z} = \frac{1}{2}p - \frac{\partial p}{\partial t}. \tag{4.13}$$

This is a possible boundary condition. Using the Taylor approximation of $(1 + x)^{1/2}$ for small x, we see from (4.12) that the next level of approximation is

$$\mu(s, \xi) \simeq s\left[1 + 1/(8s^2) + \frac{1}{2}\xi^2/s^2\right]. \tag{4.14}$$

We differentiate (4.11) with respect to t and substitute the approximation (4.14) to obtain the next order boundary condition

$$p_{zt} = \frac{1}{2}p - \left(p_{tt} + p/8 - \frac{1}{2}p_{xx}\right). \tag{4.15}$$

Similarly, higher-order accurate boundary conditions can be derived. This process seems elegant. However, not all such boundary conditions yield stable results. That is to say, well-posedness of the problem is not guaranteed with all such boundary conditions. At this point it remains to be shown that we can derive energy estimates of the form (3.18) with boundary conditions of the above type on the boundaries Γ_1, Γ_2, and Γ_3. Discussion of these results, including corresponding discrete versions of our problems, will be reported elsewhere.

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