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A RIGOROUS JUSTIFICATION OF THE REYNOLDS EQUATION*

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Abstract. A small parameter technique is used to derive Reynolds' lubrication equation from the Stokes equation. The error associated with the approximation is estimated in suitable norms.

1. Introduction. In a classical paper [9], O. Reynolds formulated a differential equation which gives an asymptotic approximation of the Navier-Stokes equation. In the one-dimensional case this equation reads as follows:

$$(H^3P')' = 6\mu\overline{V}H'. \tag{1.1}$$

P is the pressure in a thin film of a viscous fluid of constant density, thickness H and viscosity μ. $\overline{V}$ is the X_1-component of the moving surface. Equation (1.1) applies when the height of the fluid is very small compared to the span and the length. The justifications given in the literature for the transition from the full Navier-Stokes equations to (1.1) are mainly heuristic; we quote in particular the papers of Elrod [3] and Wannier [10].

A possible approach to the Reynolds equation, and thus to the theory of lubrication, is to suppose true the expressions of the Couette-Poiseuille flow,

$$V_1 = \frac{1}{2\mu}\frac{\partial P}{\partial X_1}X_2(X_2 - H) + \overline{V}(H - X_2)/H,$$

even when H is not a constant. In the first part of this paper (Secs. 2 and 3) we examine in detail the asymptotic validity of this assumption of the theory.

In Sec. 4 we develop a formal expansion which gives in the first term the Reynolds equation. Various rigorous estimates of the remainder are proved. In this paper we limit ourselves to the two-dimensional linear case (i.e., we consider as "exact" the solution given by the linear Stokes equations).

The transition from the Stokes equations to the Reynolds equation in the three-dimensional case (but only to the first order) is studied in a paper by G. Bayada and M.

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Chambat [11], which recently appeared and which we saw after completing the present work. No use is made in [11] of the technique of the stream function on which this paper is largely based.

2. Generalized Couette-Poiseuille flow. Let $H(X_1) \in C^\infty(R^1)$ be a periodic function of period L such that $H_M \geqslant H \geqslant H_m > 0$. Define

$$S = \{(X_1, X_2); -\infty < X_1 < \infty, 0 < X_2 < H(X_2)\},$$
$$\Omega = \{(X_1, X_2); 0 < X_1 < L, 0 < X_2 < H(X_1)\}.$$

First of all we study the following.

Problem 1. To find $\mathbf{V} = (V_1, V_2)$ such that

$$\mu \Delta \mathbf{V} = \nabla P \text{ in } S, \tag{2.1}$$
$$\nabla \cdot \mathbf{V} = 0, \tag{2.2}$$
$$V_1 = \bar{V}, V_2 = 0 \text{ on } X_2 = 0, \qquad \mathbf{V} = 0 \text{ on } X_2 = H(X_1), \tag{2.3}$$
$$\mathbf{V}(X_1 + L, X_2) = \mathbf{V}(X_1, X_2), \quad L > 0 \text{ a given constant.} \tag{2.4}$$

$\mathbf{V}$ is the velocity of the plane flow we discuss, P the pressure, μ the (constant) viscosity and $\bar{V}$ a given constant. If $H = H_m$ the solution of Problem 1 is of course the well-known Couette-Poiseuille formula

$$\mathbf{V} = \mathbf{V}^{(C)} + \mathbf{V}^{(P)} \tag{2.5}$$

where

$$V_1^{(C)} = \frac{K}{2\mu L} X_2(X_2 - H_m), \qquad V_1^{(P)} = \bar{V}(1 - X_2/H_m), \qquad V_2^{(C)} = 0, \qquad V_2^{(P)} = 0. \tag{2.6}$$

$$P = \frac{K}{L} X_1, \tag{2.7}$$

with K an arbitrary constant. Hence in this elementary case (a) the solution of Problem 1 is unique when K is assigned or if

$$C = \int_0^{H_m} V_1(X_2)\, dX_2$$

is prescribed. The one-to-one correspondence between C and K is easily computed:

$$C = \frac{-KH_m^3}{12 L\mu} + \frac{H_m \bar{V}}{2}.$$

Moreover (b) the pressure is periodic only if $K = 0$ and in this case $\mathbf{V} = \mathbf{V}^{(P)}$. We are going to show that properties (a) and (b) remain true when H is an arbitrary periodic function. If $\mathbf{V}$ is a regular solution of Problem 1 then P_{X_1} is periodic with respect to X_1; thus

$$P(X_1, X_2) = G(X_1, X_2) + A(X_2) X_1 + B(X_2)$$

where $G(X_1, X_2)$ is L-periodic in X_1. Hence

$$P(X_1 + L, X_2) - P(X_1, X_2) = A(X_2)L.$$

But P_{X_2} is also periodic in X_1; thus A is constant. Defining $K = AL$ we have

$$P(X_1 + L, X_2) - P(X_1, X_2) = K. \tag{2.8}$$

We give now a weak formulation of Problem 1 in which the constant K appears explicitly following the standard functional approach (see e.g. [6]).

Let $\dot{J}(S)$ be the set of all sufficiently smooth solenoidal two-dimensional vectors vanishing near ∂S which are L-periodic in X_1. In $\dot{J}(S)$ we introduce the scalar product[1]

$$[\mathbf{U}, \mathbf{V}] = \int_\Omega \mathbf{U}_{X_k} \cdot \mathbf{V}_{X_k}\, dX, \qquad dX = dX_1\, dX_2$$

and the norm

$$\|\mathbf{U}\| = [\mathbf{U}, \mathbf{U}]^{1/2}. \tag{2.9}$$

Denote by $H(S)$ the completion of $\dot{J}(S)$ with respect to (2.9). Multiplying (2.1) by $\mathbf{\Phi} \in H$ and integrating by parts over Ω we have, recalling (2.8),

$$-\mu[\mathbf{V}, \mathbf{\Phi}] = K\int_0^{H(0)} \Phi_1(0, X_2)\, dX_2, \tag{2.10}$$

for all $\mathbf{\Phi} \in H$. The integral on the right-hand side of (2.10) makes sense since $\Phi_1 \in L^2(\partial\Omega)$. Let $\mathbf{A}$ be any smooth solenoidal vector field which satisfies (2.3) and (2.4). If we define $\mathbf{U} = \mathbf{V} - \mathbf{A}$ we can write (2.10) in the form

$$\mathbf{U} \in H, \qquad -\mu[\mathbf{U}, \mathbf{\Phi}] = K\int_0^{H(0)} \Phi_1(0, X_2)\, dX_2 - [\mathbf{A}, \mathbf{\Phi}]\mu. \tag{2.11}$$

The right-hand side in (2.11) defines a linear functional in $H(S)$. Thus by the Riesz representation theorem we have

THEOREM 2.1. For every $K \in \mathbf{R}^1$ there exists one and only one solution of Problem (2.10).

The regularity of the solution together with other properties, is better obtained using a stream function $\Psi(X_1, X_2)$ defined by

$$\Psi_{X_2} = V_1, \qquad \Psi_{X_1} = -V_2. \tag{2.12}$$

In terms of Ψ, Problem 1 becomes

Problem 2. To find Ψ such that

$$\Delta^2\Psi = 0 \quad \text{in} \quad S, \tag{2.13}$$

$$\Psi = 0, \quad \Psi_{X_2} = \overline{V} \quad \text{on} \quad X_2 = 0, \tag{2.14}$$

$$\Psi = C, \quad \frac{d\Psi}{dn} = 0 \quad \text{on} \quad X_2 = H(X_1), \tag{2.15}$$

$$\Psi(X_1 + L, X_2) = \Psi(X_1, X_2). \tag{2.16}$$

d/dn is the normal derivative along ∂S. We note that the stream function is constant on ∂S by the boundary conditions imposed on $\mathbf{V}$. Thus we can assume $\Psi = 0$ on $X_2 = 0$; however, since ∂S consists of two disjunct arcs, we have $\Psi = C$ on $X_2 = H(X_1)$ and in

[1] Use is made of the summation convention.

general $C \neq 0$. The physical meaning of C is readily seen. Indeed

$$C = \int_0^{H(0)} \Psi_{X_2}(0, X_2)\, dX_2 = \int_0^{H(0)} V_1(0, X_2)\, dX_2.$$

With C prescribed, the theory of linear boundary value problems applies to (2.13)–(2.16) with the minor changes required by the condition of periodicity. Hence the solution of Problem 2 exists, is unique and of class $C^\infty(\bar{S})$ since we suppose $H \in C^\infty(\bar{S})$ and $\partial S \in C^\infty$. In the following theorem we show that the formulations of Problems 1 and 2 are perfectly equivalent. This, in particular, proves the C^∞ regularity of the solution of Problem 1.

Theorem 2.2. The correspondence $C = F(K)$ is one-to-one from $\mathbf{R}^1$ on $\mathbf{R}^1$.

Proof. Let $K \in \mathbf{R}^1$. By Theorem 2.1 there exists one and only one solution of Problem 1; thus

$$C = F(K) = \int_0^{H(0)} V_1(0, X_2)\, dX_2$$

is well-defined. We say that F is one-to-one. Let $\mathbf{V}^1$ and $\mathbf{V}^2$ be the solutions of Problem 1 corresponding to K_1 and K_2 with $K_1 \neq K_2$. Suppose by contradiction $C_1 = C_2$ and define $\mathbf{W} = \mathbf{V}^1 - \mathbf{V}^2$. From (2.10) we have

$$-\mu[\mathbf{W}, \mathbf{\Phi}] = (K_1 - K_2) \int_0^{H(0)} \Phi_1(0, X_2)\, dX_2.$$

With $\mathbf{\Phi} = \mathbf{W}$ we get

$$-\mu \|\mathbf{W}\|_H^2 = (K_1 - K_2)(C_1 - C_2) = 0$$

and this cannot be since $\|\mathbf{W}\|_H \neq 0$.

To prove that $F(\mathbf{R}^1) = \mathbf{R}^1$ we pick $\bar{C} \in \mathbf{R}^1$ and let Ψ be the corresponding unique solution of Problem 2. Now $\mathbf{V} = (\Psi_{X_2}, -\Psi_{X_1})$ is a solution to Problem 1. Let P be related pressure. Since ∇P is periodic in X_1 there exists a constant $\bar{K}$ such that

$$P(X_1 + L, X_2) - P(X_1, X_2) = \bar{K}$$

and clearly $\bar{C} = F(\bar{K})$. $\square$

In analogy with the case $H = H_m$ we call $\mathbf{V}^{(P)}$ the solution to Problem 1 when $K = 0$. If K is arbitrary and $\mathbf{V}$ the corresponding solution we put

$$\mathbf{V}^{(C)} = \mathbf{V} - \mathbf{V}^{(P)}.$$

Theorem 2.3. The pressure is periodic with respect to X_1 (i.e., $K = 0$) if and only if

$$\int_0^{H(0)} V_1(0, X_2)\, dX_2 = \int_0^{H(0)} V_1^{(P)}(0, X_2)\, dX_2. \tag{2.17}$$

Proof. If $K = 0$ then (2.17) follows by definition. Vice versa suppose (2.17) holds. We have

$$\int_0^{H(0)} V_1^{(C)}(0, X_2)\, dX_2 = 0. \tag{2.18}$$

Now $\mathbf{V}^{(C)}$ satisfies

$$\Delta \mathbf{V}^{(C)} = \nabla P^{(C)}, \qquad \nabla \cdot \mathbf{V}^{(C)} = 0 \quad \text{in } S, \tag{2.19}$$

$$\mathbf{V}^{(C)} = 0 \quad \text{on} \quad \partial S, \qquad \mathbf{V}^{(C)}(X_1 + L, X_2) = \mathbf{V}^{(C)}(X_1, X_2) \quad \text{in} \quad S. \tag{2.20}$$

Multiplying $(2.19)_1$ by $\mathbf{V}^{(C)}$ we obtain from $(2.19)_2$ and (2.18)

$$\|\mathbf{V}^{(C)}\|_H = 0.$$

Hence $\mathbf{V}^{(C)} = 0$. This implies $K = 0$. □

3. Asymptotic properties of the generalized Couette-Poiseuille flow. In this section we reformulate Problems 1 and 2 in the variable domain:

$$S_\varepsilon = \{(X_1, X_2) \in \mathbf{R}^2, |X_1| < \infty, 0 < X_2 < \varepsilon H(X_1)\}$$

and then let $\varepsilon \to 0$. Since ε is a small parameter we prefer to work with nondimensional variables. Thus we define

$$x = (x_1, x_2),\ x_1 = X_1/L,\ x_2 = X_2/H_m,\ h(x_1) = H(Lx_1)/H_m,\ \mathbf{v} = \mathbf{V}/\bar{V},\ \mathbf{v} = (v_1, v_2),$$

$$p = \frac{PL}{\mu\bar{V}},\ k = \frac{KH_m^2}{\mu\bar{V}L},\ c = \frac{C}{\bar{V}H_m},\ \psi = \frac{\Psi}{\bar{V}H_m},\ h_m \geqslant h(x_1) \geqslant h_m,$$

$$s_\varepsilon = \{(x_1, x_2);\ |x_1| < \infty, 0 < x_2 < \varepsilon h(x_1)\},$$

$$o_\varepsilon = \{(x_1, x_2);\ 0 < x_1 < 1, 0 < x_2 < \varepsilon h(x_1)\},$$

ε a nondimensional parameter. Problems 1 and 2 become

Problem 1′. To find $\mathbf{v}$ such that

$$\Delta \mathbf{v} = \nabla p, \qquad \nabla \cdot \mathbf{v} = 0 \text{ in } s_\varepsilon, \tag{3.1}$$

$$v_1 = 1,\ v_2 = 0 \quad \text{on} \quad x_2 = 0, \qquad \mathbf{v} = 0 \quad \text{on} \quad x_2 = \varepsilon h(x_1), \tag{3.2}$$

$$\mathbf{v}(x_1 + 1, x_2) = \mathbf{v}(x_1, x_2). \tag{3.3}$$

Problem 2′. To find ψ such that

$$\Delta^2 \psi = 0 \quad \text{in} \quad s_\varepsilon, \tag{3.4}$$

$$\psi = 0,\ \psi_{x_2} = 1 \quad \text{on} \quad x_2 = 0, \qquad \psi = c,\ \frac{d\psi}{dn} = 0 \quad \text{on} \quad x_2 = \varepsilon h(x_1), \tag{3.5}$$

$$\psi(x_1 + 1, x_2) = \psi(x_1, x_2). \tag{3.6}$$

The Couette-Poiseuille solution corresponding to $h(x_1) = 1$ is, in this notation,

$$v_1(x_2) = v_1^{(P)} + v_1^{(C)}, \qquad v_2 = 0, \tag{3.7}$$

where

$$v_1^{(C)} = \frac{k}{2} x_2 (x_2 - \varepsilon), \qquad v_1^{(P)} = (\varepsilon - x_2)/\varepsilon.$$

We are interested in solutions of Problems 1′ for which $v_1^{(C)}$ does not vanish and does not diverge as $\varepsilon \to 0$. Of course this is equivalent to saying that $k = k_0/\varepsilon^2$. This is the case in the theory of lubrication and hereafter we will always make this assumption. We easily verify, when $k = k_0/\varepsilon^2$ in (3.7),

$$\|v_1\|_{L^1(o_\varepsilon)} = C_1\varepsilon, \tag{3.8}$$

$$\|v_1\|_{L^p(o_\varepsilon)} = C_p\varepsilon^{1/p}, \quad 1 < p < \infty, \tag{3.9}$$

$$\|v_1\|_{L^\infty(o_\varepsilon)} = C_\infty, \tag{3.10}$$

$$\|\nabla v_1\|_{L^2(o_\varepsilon)} = \tilde{C}\varepsilon^{-1/2}. \tag{3.11}$$

Moreover, if $k = k_0/\varepsilon^2$ in (3.7), then

$$c = c_0\varepsilon, \qquad c_0 = -k_0/12 + 1/2. \tag{3.12}$$

In terms of c_0, (3.7) becomes

$$v_1(x_2) = 3(1 - 2c_0)x_2^2/\varepsilon^2 + 2(3c_0 - 2)x_2/\varepsilon + 1. \tag{3.13}$$

The basic idea of the theory of lubrication is to consider (3.13) still valid when h depends on x_1 and ε is very small. To make this point precise we prove that the estimates (3.8)–(3.12) remain valid for the generalized Couette-Poiseuille flow and that the exact solution of Problem 1′ converges in suitable norms to the flow of the theory of lubrication.

Let $H^1_{p0}(s_\varepsilon)$ be the completion with respect to the norm

$$\|u\|_{H^1_{p0}} = \left(\int_{o_\varepsilon} |\nabla u|^2\, dx\right)^{1/2}, \quad dx = dx_1\, dx_2$$

of the C^∞-functions vanishing near ∂s_ε and 1-periodic in x_1. For the proof of the following Poincaré-like lemma compare also [6], p. 8.

LEMMA 3.1. Suppose $u \in H^1_{p0}$. We have

$$\|u\|_{L^{2p}(o_\varepsilon)} \leqslant C(p)\varepsilon^{1/p}\|\nabla u\|_{L^2(o_\varepsilon)}$$

where (3.14)

$$C(p) = \left(h_M(p!)^2/2^{p-1}\right)^{1/p}, \quad 1 \leqslant p < \infty.$$

Proof. It suffices to prove (3.14) when $u \in C^\infty(s_\varepsilon)$ vanishes near ∂s_ε and is 1-periodic in x_1. For $p = 1$ (3.14) certainly holds. Now

$$\begin{aligned}\int_{o_\varepsilon} u^{2(p+1)}\, dx &\leqslant (p+1)^2\int_{o_\varepsilon} |u^p u_{x_1}|\, dx \int_{o_\varepsilon} |u^p u_{x_2}|\, dx \\ &\leqslant \frac{1}{2}(p+1)^2\|u\|^2_{L^{2p}}\|\nabla u\|^2_{L^2}.\end{aligned} \tag{3.15}$$

Suppose (3.14). Then by (3.15)

$$\|u\|_{L^{2(p+1)}} \leqslant \left(\varepsilon h_M(p+1)!/2^p\right)^{1/(p+1)}\|\nabla u\|_{L^2}.$$

Hence by induction the result follows. □

In the following three lemmas we recall various "a priori" estimates for the solutions of harmonic and biharmonic problems.

LEMMA 3.2. (C. Miranda [7]) Let Ω be an open, bounded, and connected subset of $\mathbf{R}^2$ with a regular boundary (e.g. $\partial\Omega \in C^2$). Suppose $v \in C^1(\overline{\Omega}) \cap C^4(\Omega)$ and

$$\Delta^2 v = 0 \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{on} \quad \partial\Omega.$$

Then

$$|\nabla v| \leqslant (56)^{1/2} \max_{\partial\Omega} |dv/dn|.$$

Take Ω as in Lemma 3.2. Let $B_x(r_i)$, $(B_x(r_e))$ be the greatest open disk of radius r_i (r_e) internally (externally) tangent to $\partial\Omega$ in x, i.e., $B_x(r_i) \subset \Omega$ $(B_x(r_e) \cap \Omega = \varnothing)$. Define

$$\delta_i = \inf\{ r_i(x),\, x \in \partial\Omega \}, \qquad \delta_e = \inf\{ r_e(x),\, x \in \partial\Omega \}, \qquad \delta = \min\{ \delta_i, \delta_e \}.$$

Since $\partial\Omega \in C^2$ we have $\delta > 0$.

LEMMA 3.3. (G. Adler [1]) If $v \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega})$ is harmonic we have

$$\max_{\Omega} |\nabla v| \leqslant C_1 \frac{1}{\delta} \max_{\partial\Omega} |v(s)| + C_2 \max_{\partial\Omega} |v'(s)| + C_3 \delta \max_{\partial\Omega} |v''(s)|$$

where s is the arc length along $\partial\Omega$ and C_1, C_2, C_3 are absolute constants.

LEMMA 3.4. (D. Gilbarg and N. S. Trudinger [4], p. 37) Let $v \in C^0(\Omega) \cap C^2(\overline{\Omega})$ and f bounded in Ω. If $\Delta v = f$ we have

$$d(x)|\nabla v(x)| \leqslant C\Big(\sup_{\Omega} |v| + \sup_{\Omega} d^2(x) |f(x)| \Big)$$

where $d(x) = \operatorname{dist}\{x, \partial\Omega\}$.

The conclusions of the above lemmas remain true if $\Omega = s_\varepsilon$ and the function v is 1-periodic in x_1. This can be seen quite easily if we modify the proofs, reasoning in o_ε and treating points like $(0, x_2)$, $(1, x_2)$ as coincident interior points. In particular, we can take $\delta = \varepsilon/h_M$ in Lemma 3.3.

Define now in s_ε the function $\psi_0(x_1, x_2)$ with the problem

$$\partial^4\psi_0/\partial x_2^4 = 0 \quad \text{in} \quad s_\varepsilon, \tag{3.16}$$

$$\psi_0 = 0,\ \partial\psi_0/\partial x_2 = 1 \quad \text{on} \quad x_2 = 0, \qquad \psi_0 = \varepsilon c_0,\ \partial\psi_0/\partial x_2 = 0 \quad \text{on} \quad x_2 = \varepsilon h(x_1).$$

We find

$$\psi_0(x_1, x_2) = \frac{(h - 2c_0)x_2^3}{\varepsilon^2 h^3} + \frac{(3c_0 - 2h)x_2^2}{\varepsilon h^2} + x_2. \tag{3.17}$$

We may regard ψ_0 as the stream function of the theory of lubrication. Indeed, let $v_{01} = \partial\psi_0/\partial x_2$; we have

$$v_{01} = \frac{3(h - 2c_0)x_2^2}{\varepsilon^2 h^3} + \frac{2(3c_0 - 2h)x_2}{\varepsilon h^2} + 1 \tag{3.18}$$

and this is precisely the (nondimensional) x_1-component of the velocity given by the theory of lubrication.

THEOREM 3.1. Suppose $c = c_0\varepsilon$ in Problem 2′. If $\mathbf{v}$ is the corresponding velocity and $k(\varepsilon)$ is given by

$$p(x_1 + 1, x_2) - p(x_1, x_2) = k,$$

then we have[2]

$$\|\nabla v_1\|_{L^2(o_\varepsilon)} \leqslant C\varepsilon^{-1/2}, \qquad \|\nabla v_2\|_{L^2(o_\varepsilon)} \leqslant C, \tag{3.19}$$

$$\|v_1\|_{L^p(o_\varepsilon)} \leqslant C\varepsilon^{1/p}, \qquad \|v_2\|_{L^p(o_\varepsilon)} \leqslant C\varepsilon^{2/p}, \quad 2 < p < \infty, \tag{3.20}$$

$$|k(\varepsilon)| \leqslant C\varepsilon^{-2}. \tag{3.21}$$

Proof. Define $\eta = \psi - \psi_0$ where ψ is the solution of Problem 2′ and ψ_0 is given by (3.17). η satisfies

$$\Delta^2\eta = f_1/\varepsilon^2 + f_2/\varepsilon + f_3, \tag{3.22}$$

$$\eta = 0, \qquad d\eta/dn = 0 \quad \text{on} \quad \partial s_\varepsilon, \tag{3.23}$$

with f_1, f_2, and f_3 functions not depending on ε. Multiplying (3.22) by η and integrating by parts over o_ε we get, recalling that η is 1-periodic in x_1,

$$\|\eta_{x_1x_1}\|^2_{L^2(o_\varepsilon)} + 2\|\eta_{x_1x_2}\|^2_{L^2(o_\varepsilon)} + \|\eta_{x_2x_2}\|^2_{L^2(o_\varepsilon)} \leqslant \frac{C}{\varepsilon^2}\|\eta\|_{L^2(o_\varepsilon)}. \tag{3.24}$$

By the Poincaré inequality we have

$$\|\eta\|_{L^2(o_\varepsilon)} \leqslant C\varepsilon\|\eta_{x_2}\|_{L^2(o_\varepsilon)}, \qquad \|\eta_{x_2}\|_{L^2(o_\varepsilon)} \leqslant C\varepsilon\|\eta_{x_2x_2}\|_{L^2(o_\varepsilon)}.$$

Thus

$$\|\eta\|_{L^2(o_\varepsilon)} \leqslant C\varepsilon^2\|\eta_{x_2x_2}\|_{L^2(o_\varepsilon)}.$$

Hence by (3.24),

$$\|\eta_{x_1x_1}\|^2_{L^2(o_\varepsilon)} + 2\|\eta_{x_1x_2}\|^2_{L^2(o_\varepsilon)} + \|\eta_{x_2x_2}\|^2_{L^2(o_\varepsilon)} \leqslant C\|\eta_{x_2x_2}\|_{L^2(o_\varepsilon)}.$$

It follows that

$$\|\nabla\eta_{x_1}\|^2_{L^2} + \|\nabla\eta_{x_2}\|^2_{L^2} \leqslant C, \tag{3.25}$$

$$\|\eta_{x_1}\|_{L^2} \leqslant C\varepsilon, \qquad \|\eta_{x_2}\|_{L^2} \leqslant C\varepsilon, \qquad \|\eta\|_{L^2} \leqslant C\varepsilon^2. \tag{3.26}$$

Moreover, we have

$$\|\nabla v_1\|_{L^2} = \|\nabla\psi_{x_2}\|_{L^2} \leqslant \|\nabla\eta_{x_2}\|_{L^2} + \|\nabla\psi_{0_{x_2}}\|_{L^2},$$

$$\|\nabla v_2\|_{L^2} = \|\nabla\psi_{x_1}\|_{L^2} \leqslant \|\nabla\eta_{x_1}\|_{L^2} + \|\nabla\psi_{0_{x_1}}\|_{L^2},$$

and by direct computation

$$\|\nabla\psi_{0_{x_2}}\|_{L^2} \leqslant C\varepsilon^{-1/2}, \qquad \|\nabla\psi_{0_{x_1}}\|_{L^2} \leqslant C\varepsilon^{1/2}.$$

Hence (3.19) holds true. By Lemma 3.1 and (3.25) we have

$$\|\eta_{x_1}\|_{L^p} \leqslant C\varepsilon^{2/p}, \qquad \|\eta_{x_2}\|_{L^p} \leqslant C\varepsilon^{2/p}. \tag{3.27}$$

[2] The various C's denote constants generally different, not depending on ε.

Thus the estimates (3.20) follow if we note that

$$\|\psi_{0_{x_1}}\|_{L^p} \leqslant C\varepsilon^{1+1/p}, \qquad \|\psi_{0_{x_2}}\|_{L^p} \leqslant C\varepsilon^{1/p}.$$

Finally we prove (3.21). Take $c = \varepsilon c_0$ and $c = 2\varepsilon c_0$ in Problem 2′ and let $\mathbf{v}$ and $\tilde{\mathbf{v}}$ be the corresponding velocities. Letting $\boldsymbol{\phi} = \mathbf{v} - \tilde{\mathbf{v}}$ in the integral identity

$$-[\mathbf{v}, \boldsymbol{\phi}] = k(\varepsilon)\int_0^{\varepsilon h(0)} \phi_1(0, x_2)\, dx_2$$

we obtain

$$\varepsilon c_0 k(\varepsilon) = -[\mathbf{v}, \tilde{\mathbf{v}}] + \|\mathbf{v}\|_H^2.$$

Since, by (3.19),

$$\|\mathbf{v}\|_H \leqslant C_1\varepsilon^{-1/2}, \qquad \|\tilde{\mathbf{v}}\|_H \leqslant C_2\varepsilon^{-1/2},$$

we get as a result

$$c_0|k(\varepsilon)| \leqslant C\varepsilon^{-2}. \quad \square$$

THEOREM 3.2. Suppose again $c = \varepsilon c_0$ in Problem 2′ and let $\mathbf{v}$ be the corresponding velocity. Then

$$\|\mathbf{v}\|_{L^\infty} \leqslant C. \tag{3.28}$$

Proof. Let $\omega(x_1, x_2)$ be harmonic, 1-periodic in x_1 and such that $\omega = \psi$ on ∂s_ε. By Lemma 3.3, which is applicable in the present case, $|\nabla\omega|$ remains bounded as $\varepsilon \to 0$. Defining $\zeta = \psi - \omega$ we have

$$\Delta^2\zeta = 0 \quad \text{in} \quad s_\varepsilon, \qquad \zeta = 0 \quad \text{on} \quad \partial s_\varepsilon.$$

Hence by Lemma 3.2,

$$|\nabla\zeta| \leqslant \sqrt{56}\max_{\partial s_\varepsilon}|\partial\zeta/\partial n| \leqslant \sqrt{56}\left(\max_{\partial s_\varepsilon}|\partial\psi/\partial n| + \max_{\partial s_\varepsilon}|\partial\omega/\partial n|\right).$$

Since $|\partial\psi/\partial n| \leqslant 1$ and $|\nabla\omega|$ is bounded, we get $|\nabla\zeta| \leqslant C$. It follows that

$$\|\nabla\psi\|_{pL^\infty} \leqslant C,$$

which implies (3.28). $\square$

In the following theorem we derive an interior pointwise estimate.

THEOREM 3.3. Let ψ be the solution of Problem 2′ when $c = \varepsilon_0$ and let ψ_0 be given by (3.17). Define $\eta = \psi - \psi_0$. Then

$$d(x)|\nabla\eta(x)| \leqslant C\varepsilon^2 \tag{3.29}$$

where $d(x) = \operatorname{dist}\{x, \partial s_\varepsilon\}$.

Proof. The function η satisfies (3.22) and (3.23). Let α be the solution of the problem

$$\Delta\alpha = f_2/\varepsilon^2 + f_1/\varepsilon + f_0 \quad \text{in} \quad s_\varepsilon, \qquad \alpha = 0 \quad \text{on} \quad \partial s_\varepsilon, \qquad \alpha(x_1 + 1, x_2) = \alpha(x_1, x_2) \tag{3.30}$$

with f_i as in (3.22). Define $\alpha_B = 3Nx_2(\varepsilon h_M - x_2)/(2\varepsilon^2)$, $0 \leqslant x_2 \leqslant \varepsilon h_M$ where the constant N is a pointwise bound for $|f_i|$ $(i = 1, 2, 3)$. By the maximum principle we have

$|\alpha| \leqslant \alpha_B$ in s_ε when $\varepsilon < 1$. Hence

$$|\alpha| \leqslant M, \qquad M = 3Nh_M^2/8. \tag{3.31}$$

Suppose β is defined by

$$\Delta\beta = \alpha \quad \text{in} \quad s_\varepsilon, \qquad \beta = 0 \quad \text{on} \quad \partial s_\varepsilon, \qquad \beta(x_1 + 1, x_2) = \beta(x_1, x_2)$$

and let $\beta_B = Mx_2(\varepsilon h_M - x_2)/2$, $0 \leqslant x_2 \leqslant \varepsilon h_M$. Using again a comparison argument we find $|\beta| \leqslant \beta_B$; hence

$$|\beta| \leqslant Mh_M^2\varepsilon^2/8. \tag{3.32}$$

Assume for the moment

$$|\partial\beta/\partial n| \leqslant C\varepsilon. \tag{3.33}$$

Since $d(x) \leqslant \varepsilon h_M$ and $\Delta\beta = \alpha$, we have by Lemma 3.4, recalling (3.31) and (3.32),

$$d|\nabla\beta| \leqslant C_1\Big(\sup_{s_\varepsilon}|\beta| + \sup_{s_\varepsilon} d^2|\alpha|\Big) \leqslant C_2\varepsilon^2. \tag{3.34}$$

Define $\gamma = \eta - \beta$. We have

$$\Delta^2\gamma = 0 \quad \text{in} \quad s_\varepsilon, \qquad \gamma = 0 \quad \text{on} \quad \partial s_\varepsilon.$$

Thus we get, by Lemma 3.2 and taking into account (3.23),

$$|\nabla\gamma| \leqslant \sqrt{56}\max_{\partial s_\varepsilon}|\partial\gamma/\partial n| = \sqrt{56}\max_{\partial s_\varepsilon}|\partial\beta/\partial n|. \tag{3.35}$$

On the other hand, by (3.33), (3.34), and (3.35), we obtain

$$d|\nabla\eta| \leqslant d|\nabla\gamma| + d|\nabla\beta| \leqslant C\varepsilon^2.$$

It remains to verify (3.33). Let δ be given by

$$\Delta\delta = M \quad \text{in} \quad s_\varepsilon, \qquad \delta = 0 \quad \text{on} \quad \partial s_\varepsilon, \qquad \delta(x_1 + 1, x_2) = \delta(x_1, x_2). \tag{3.36}$$

By (3.31) and the maximum principle we get $|\beta| \leqslant \delta$. On the other hand $\beta = \delta$ on ∂s_ε; consequently

$$|\partial\beta/\partial n| \leqslant |\partial\delta/\partial n| \quad \text{on} \quad \partial s_\varepsilon. \tag{3.37}$$

To estimate $|\partial\delta/\partial n|$ we transform (3.36) in a harmonic problem by letting $\xi = \delta - \beta_B$. We have

$$\Delta\xi = 0 \quad \text{in} \quad s_\varepsilon,$$

$$\xi(x_1, 0) = 0, \qquad \xi(x_1, \varepsilon h(x_2)) = \varepsilon^2 Mh(x_2)(h(x_2) - h_M)/2,$$

and by Lemma 3.3, $|\nabla\xi| \leqslant C_3\varepsilon$. Taking into account that $|\nabla\beta_B| \leqslant C_4\varepsilon$ it follows that $|\nabla\delta| \leqslant C_5\varepsilon$. Thus $|\partial\delta/\partial n| \leqslant C_5\varepsilon$ on ∂s_ε. This by (3.37) implies (3.33). □

From (3.29) we deduce the estimates

$$d(x)|v_1 - v_{01}| \leqslant C\varepsilon^2. \tag{3.38}$$

4. A perturbation technique. In this section we develop a perturbation scheme in ε for Problem 2′. Since we prefer to work in a fixed domain we introduce the new variables

$$y_1 = x_1, \qquad y_2 = x_2/\varepsilon. \tag{4.1}$$

In this way s_ε and o_ε become, respectively,

$$s = \{(y_1, y_2); -\infty < y_1 < \infty, 0 < y_2 < h(y_1)\},$$
$$o = \{(y_1, y_2); 0 < y_1 < 1, 0 < y_2 < h(y_1)\}.$$

Hereafter we use for partial differentiation the notation

$$D^q_{q_1,q_2} f = \partial^q f / \partial^{q_1} y_1 \partial^{q_2} y_2, \quad q = q_1 + q_2.$$

Moreover, we put

$$\nabla_\varepsilon = \left(\varepsilon D^1_{1,0}, D^1_{0,1}\right), \qquad \Delta_\varepsilon = \varepsilon^2 D^2_{2,0} + D^2_{0,2}.$$

Problem 2′ can be rewritten as

Problem 3. To find ϕ such that

$$\Delta^2_\varepsilon \phi = 0 \quad \text{in} \quad s, \tag{4.2}$$

$$\phi = 0, \qquad D_{0,1}\phi = \varepsilon \quad \text{on} \quad y_2 = 0, \tag{4.3}$$

$$\phi = \varepsilon c_0, \qquad D_{0,1}\phi = 0 \quad \text{on} \quad y_2 = h(y_1), \tag{4.4}$$

$$\phi \text{ is 1-periodic in } y_1, \tag{4.5}$$

where of course $\phi(y_1, y_2) = \psi(y_1, \varepsilon y_2)$. Let $u_i(y_1, y_2) = v_i(y_1, \varepsilon y_2)$. Recalling the definition of stream function we have

$$u_1 = \frac{1}{\varepsilon} D_{0,1}\phi, \qquad u_2 = -D_{1,0}\phi. \tag{4.6}$$

Furthermore, for the pressure gradient, we get from (3.1) and (4.6)

$$D_{1,0}p = \frac{1}{\varepsilon} D^3_{2,1}\phi + \frac{1}{\varepsilon^3} D^3_{0,3}\phi, \tag{4.7}$$

$$\frac{1}{\varepsilon} D_{0,1}p = -D^3_{3,0}\phi - \frac{1}{\varepsilon^2} D^3_{1,2}\phi. \tag{4.8}$$

LEMMA 4.1. The solution ϕ_ε of Problem 3 is odd with respect to ε.

Proof. Because $\Delta_\varepsilon = \Delta_{-\varepsilon}$ and $\Delta^2_\varepsilon = \Delta^2_{-\varepsilon}$, $-\phi_{-\varepsilon}$ satisfies Problem 3. Thus by the uniqueness of the solution we infer $\phi_\varepsilon = -\phi_{-\varepsilon}$. □

By (4.6) u_1 is even with respect to ε and u_2 is odd. Moreover, from (4.7) and (4.8), we see that p is an even function of ε.

Set formally

$$\phi = \phi^{(0)} + \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \varepsilon^3\phi^{(3)} + \cdots. \tag{4.9}$$

Substituting (4.9) into Problem 3 and equating like powers of ε we obtain, in agreement with Lemma 4.1,

$$\phi^{(2i)} = 0, \quad i = 0, 1, \ldots. \tag{4.10}$$

The coefficients of the odd powers of ε satisfy the following hierarchy of boundary value problems:

$$D_{0,4}^4\phi^{(1)} = 0 \quad \text{in} \quad s, \tag{4.11}$$

$$\phi^{(1)} = 0,\ D_{0,1}\phi^{(1)} = 1 \quad \text{on} \quad y_2 = 0, \qquad \phi^{(1)} = c_0,\ D_{0,1}\phi^{(1)} = 0 \quad \text{on} \quad y_2 = h(h_1), \tag{4.12}$$

$$D_{0,4}^4\phi^{(3)} = -2D_{2,2}^4\phi^{(1)} \quad \text{in} \quad s, \tag{4.13}$$

$$\phi^{(3)} = 0,\ D_{0,1}\phi^{(3)} = 0 \quad \text{on} \quad y_2 = 0 \text{ and } y_2 = h(y_1), \tag{4.14}$$

$$D_{0,4}^4\phi^{(2j+1)} = -D_{4,0}^4\phi^{(2j-3)} - 2D_{2,2}^4\phi^{(2j-1)}, \quad j = 2,3,\dots, \tag{4.15}$$

$$\phi^{(2j+1)} = 0,\ D_{0,1}\phi^{(2j+1)} = 0 \quad \text{on} \quad y_2 = 0 \text{ and } y_2 = h(y_1). \tag{4.16}$$

Problem (4.11)–(4.12) can be solved easily. We get (cf. (3.17))

$$\phi^{(1)} = \left(h(y_1) - 2c_0\right)y_2^3/h^3(y_1) + \left(3c_0 - 2h(y_1)\right)y_2^2/h^2(y_1) + y_2. \tag{4.17}$$

Recalling (4.10) we solve problems (4.13)–(4.14) and (4.15)–(4.16) recursively. We are led to the following formal expansion of ϕ in odd powers of ε:

$$\phi = \sum_{j=0}^{\infty} \varepsilon^{2j+1}\phi^{(2j+1)}. \tag{4.18}$$

Define

$$u_1^{(2j)} = D_{0,1}^1\phi^{(2j+1)}, \qquad u_2^{(2j+1)} = -D_{1,0}^1\phi^{(2j+1)}. \tag{4.19}$$

By (4.6) we have

$$u_1 = \sum_{j=0}^{\infty} \varepsilon^{2j}u_1^{(2j)}, \qquad u_2 = \sum_{j=0}^{\infty} \varepsilon^{2j+1}u_2^{(2j+1)}. \tag{4.20}$$

Next we consider for p an expansion of the form

$$p = \sum_{k=-2}^{\infty} \varepsilon^k p^{(k)}. \tag{4.21}$$

Putting (4.21) into (4.7) and (4.8) we have:

$$D_{1,0}p^{(2j+1)} = 0, \qquad D_{0,1}p^{(2j+1)} = 0, \quad j = -1,0,1,\dots, \tag{4.22}$$

$$D_{1,0}^1p^{(-2)} = D_{0,3}^3\phi^{(1)}, \tag{4.23}$$

$$D_{1,0}^1p^{(2j)} = D_{2,1}^3\phi^{(2j+1)} + D_{0,3}^3\phi^{(2j+3)}, \quad j = 0,1,2,\dots, \tag{4.24}$$

$$D_{0,1}^1p^{(-2)} = 0, \qquad D_{0,1}^1p^{(0)} = -D_{1,2}^3\phi^{(1)}, \tag{4.25}$$

$$D_{0,1}^1p^{(2j)} = -D_{3,0}^3\phi^{(2j-1)} - D_{1,2}^3\phi^{(2j+1)}. \tag{4.26}$$

By (4.22), $\nabla p^{(2j+1)} = 0$; therefore, apart from an inessential constant, we arrive at the formal expansion

$$p = \sum_{j=-1}^{\infty} \varepsilon^{2j}p^{(2j)}. \tag{4.27}$$

Let us examine in greater detail the first term of $(4.20)_1$. From (4.17), we obtain (cf. (3.18)):

$$u_1^{(0)} = 3(h(y_1) - 2c_0)y_2^2/h^3(y_1) + 2(3c_0 - 2h(y_1))y_2/h^2(y_1) + 1. \tag{4.28}$$

Furthermore, by $(4.25)_1$, $p^{(-2)}$ does not depend on y_2 and from (4.23) we have:

$$dp^{(-2)}/dy_1 = 6(h(y_1) - 2c_0)/h^3(y_1). \tag{4.29}$$

We arrive, in conclusion, at the classical one-dimensional Reynolds equation:

$$\frac{d}{dy_1}\left(h^3\frac{dp^{(-2)}}{dy_1}\right) = 6\frac{dh}{dy_1}. \tag{4.30}$$

Moreover, from (4.28) and (4.29) we deduce:

$$u_1^{(0)} = \frac{1}{2}\frac{dp^{(-2)}}{dy_1}y_2(y_2 - h(y_1)) + (h(y_1) - y_2)/h(y_1). \tag{4.31}$$

Our goal is now to give a rigorous justification of the formal expansions (4.18), (4.20), and (4.21).

LEMMA 4.2. Define

$$r(n) = \phi - \sum_{j=0}^{n} \varepsilon^{2j+1}\phi^{(2j+1)}. \tag{4.32}$$

We have

$$\| D_{1,0}r(n) \|_{L^2(o)} \leqslant C\varepsilon^{2n+3}, \qquad \| D_{0,1}r(n) \|_{L^2(o)} \leqslant C\varepsilon^{2n+3}, \tag{4.33}$$

$$\| D_{2-i,i}^2 r(n) \|_{L^2(o)} \leqslant C\varepsilon^{2n+3}, \quad i = 0,1,2. \tag{4.34}$$

Proof. By (4.2)

$$\Delta_\varepsilon^2 r(n) = -\sum_{j=0}^{n} \varepsilon^{2j+1}\Delta_\varepsilon^2\phi^{(2j+1)}.$$

We claim that

$$\Delta_\varepsilon^2 r(n) = \varepsilon^{2n+3}F(y_1, y_2; \varepsilon, n), \tag{4.35}$$

where

$$F(y_1, y_2; \varepsilon, n) = -\left[D_{4,0}^4\phi^{(2n-1)} + 2D_{2,2}^4\phi^{(2n+1)} + \varepsilon^2 D_{4,0}^4\phi^{(2n+1)}\right].$$

The proof is by induction. For $n = 1$, (4.35) is true. Suppose (4.35) with n arbitrary; then

$$\begin{aligned}\Delta_\varepsilon^2 r(n+1) &= \Delta_\varepsilon^2 r(n) - \varepsilon^{2n+3}\Delta_\varepsilon^2\phi^{(2n+3)}\\ &= -\varepsilon^{2n+3}\left[D_{4,0}^4\phi^{(2n-1)} + 2D_{2,2}^4\phi^{(2n+1)} + D_{0,4}^4\phi^{(2n+3)}\right]\\ &\quad -\varepsilon^{2n+5}\left[D_{4,0}^4\phi^{(2n+1)} + 2D_{2,2}^4\phi^{(2n+3)} + \varepsilon^2 D_{4,0}^4\phi^{(2n+3)}\right].\end{aligned}$$

The first term in square brackets in the left-hand side vanishes by (4.15); thus (4.35) follows. We conclude that $F(y_1, y_2; \varepsilon, n)$ is bounded pointwise by a constant not depending on ε. Because $r(n)$, r_{y_1}, and r_{y_2} vanish on ∂s, we have

$$\| \Delta_\varepsilon r(n) \|_{L^2} \leqslant C\varepsilon^{2n+3}. \tag{4.36}$$

On the other hand,

$$\| \Delta_\varepsilon r \|_{L^2}^2 = \varepsilon^4\| D_{2,0}^2 r \|_{L^2}^2 + 2\varepsilon^2\| D_{1,1}^2 r \|_{L^2}^2 + \| D_{0,2}^2 r \|_{L^2}^2.$$

Hence

$$\| D^2_{2-i,i} r(n) \|_{L^2} \leqslant C\varepsilon^{2n+3-(2-i)}, \quad i = 0,1,2. \tag{4.37}$$

Now (4.37) holds for every n and

$$r(n) = r(n+1) + \varepsilon^{2n+3}\phi^{(2n+3)}.$$

Inequality (4.34) follows because

$$\| D^2_{2-i,i} r(n) \|_{L^2} \leqslant \| D^2_{2-i,i} r(n+1) \|_{L^2} + C_1\varepsilon^{2n+3} \leqslant C\varepsilon^{2n+3}.$$

On the other hand,

$$\| D^1_{1,0} r(n) \|_{L^2} \leqslant C \| D^2_{1,1} r(n) \|_{L^2}, \qquad \| D^1_{0,1} r(n) \|_{L^2} \leqslant C \| D^2_{0,2} r(n) \|_{L^2};$$

thus (4.33) holds. □

Let

$$\rho(u_1; n) = u_1 - \sum_{j=0}^{n} \varepsilon^{2j} u_1^{(2j)}, \qquad \rho(u_2; n) = u_2 - \sum_{j=0}^{n} \varepsilon^{2j+1} u_2^{(2j+1)}.$$

Since

$$\rho(u_1; n) = \frac{1}{\varepsilon} D^1_{0,1} r(n), \qquad \rho(u_2; n) = -D^1_{1,0} r(n),$$

we have the following as an immediate consequence of Lemma 4.1.

THEOREM 4.1. The following estimates for $\rho(u_1; n)$ and $\rho(u_2; n)$ hold true:

$$\| \rho(u_1; n) \|_{L^2} \leqslant C\varepsilon^{2n+2}, \qquad \| \rho(u_2; n) \|_{L^2} \leqslant C\varepsilon^{2n+3}, \tag{4.38}$$

$$\| D^1_{1-i,i} \rho(u_1; n) \|_{L^2} \leqslant C\varepsilon^{2n+2}, \qquad \| D^1_{1-i,i} \rho(u_2; n) \|_{L^2} \leqslant C\varepsilon^{2n+3}, \quad i = 0,1. \tag{4.39}$$

Remark 4.1. The classical theory of lubrication deals with $u_1^{(0)}$ and $u_2^{(1)}$. For these terms we have, in particular,

$$\| u_1 - u_1^{(0)} \|_{L^2} \leqslant C\varepsilon^2, \qquad \| u_2 - \varepsilon u_2^{(1)} \|_{L^2} \leqslant C\varepsilon^3.$$

An interior estimate for the L^2-norms of higher-order derivatives of the stream function ϕ is given in the following.

LEMMA 4.3. Let ζ be a C^∞ function vanishing near ∂s and 1-periodic in y_1. Define

$$g = \Delta_\varepsilon^N r(n), \quad N = 1,2,\ldots.$$

Then we have

$$\| \nabla_\varepsilon(\zeta g) \|_{L^2} \leqslant C\varepsilon^{2n+3}, \tag{4.40}$$

$$\| \Delta_\varepsilon(\zeta g) \|_{L^2} \leqslant C\varepsilon^{2n+3}. \tag{4.41}$$

Proof. Let $N = 1$. We have

$$\begin{aligned} \zeta g \Delta_\varepsilon(\zeta g) &= \zeta g(\zeta \Delta_\varepsilon g + 2\nabla_\varepsilon \zeta \cdot \nabla_\varepsilon g + g\Delta_\varepsilon \zeta) \\ &= \zeta^2 g \Delta_\varepsilon g + \zeta g^2 \Delta_\varepsilon \zeta + 2g\nabla_\varepsilon \zeta \cdot \nabla_\varepsilon(\zeta g) - 2g^2 |\nabla_\varepsilon \zeta|^2. \end{aligned} \tag{4.42}$$

By (4.35) and (4.36) we get

$$\|g\|_{L^2} \leqslant C\varepsilon^{2n+3}, \qquad \|\Delta_\varepsilon g\|_{L^2} \leqslant C\varepsilon^{2n+3}. \tag{4.43}$$

Integrating by parts in the left-hand side of (4.42) we get, taking into account (4.43),

$$\|\nabla_\varepsilon(\zeta g)\|_{L^2}^2 \leqslant C\varepsilon^{2n+3}\big(\varepsilon^{2n+3} + \|\nabla_\varepsilon(\zeta g)\|_{L^2}\big).$$

Hence (4.40) holds when $N = 1$. In particular, we have from (4.40),

$$\|\zeta D^1_{1,0} g\|_{L^2} \leqslant C\varepsilon^{2n+2}, \qquad \|\zeta D^1_{0,1} g\|_{L^2} \leqslant C\varepsilon^{2n+3}. \tag{4.44}$$

To prove (4.41) we note that:

$$\begin{aligned}\zeta g \Delta_\varepsilon^2(\zeta g) = \zeta g\big\{&\zeta\Delta_\varepsilon^2 g + 4\nabla_\varepsilon\zeta\cdot\nabla_\varepsilon(\Delta_\varepsilon g) + 2\Delta_\varepsilon\zeta\Delta_\varepsilon g\\ &+ \big[4\big(\varepsilon^2\nabla_\varepsilon D_{1,0}\zeta\cdot\nabla_\varepsilon D_{1,0} g + \nabla_\varepsilon D_{0,1}\zeta\cdot\nabla_\varepsilon D_{0,1} g\big) + 4\nabla_\varepsilon g\cdot\nabla_\varepsilon(\Delta_\varepsilon\zeta)\big] + g\Delta_\varepsilon^2\zeta\big\}.\end{aligned} \tag{4.45}$$

Recalling (4.35) we have

$$\|\nabla_\varepsilon(\Delta_\varepsilon g)\|_{L^2} \leqslant C\varepsilon^{2n+3}. \tag{4.46}$$

Hence by (4.43) and (4.46) the only terms in (4.45) which cannot be immediately estimated are those in square brackets. On the other hand, we can write

$$\zeta g\nabla_\varepsilon g\cdot\nabla_\varepsilon(\Delta_\varepsilon\zeta) = \big(g\nabla_\varepsilon(\zeta g) - g^2\nabla_\varepsilon\zeta\big)\cdot\nabla_\varepsilon(\Delta_\varepsilon\zeta);$$

thus we estimate this term using (4.40) and $(4.43)_1$. Since the other terms in square brackets in (4.45) can be similarly treated, we obtain (4.41) after integrating by parts in the left-hand side of (4.45). A suitable induction argument yields (4.40) and (4.41) when N is arbitrary. □

LEMMA 4.4. Let ζ be as in Lemma 4.3. We have, for $q \geqslant 2$ and $i = 0, 1, \dots, q$,

$$\|\zeta D^q_{q-i,i} r(n)\|_{L^2} \leqslant C\varepsilon^{2n+3}. \tag{4.47}$$

Proof. We first prove by induction

$$\|\zeta D^q_{q-i,i} r(n)\|_{L^2} \leqslant C\varepsilon^{2n+3-(q-i)} \tag{4.48}$$

when $2 \leqslant q \leqslant 2n+3$ and $i = 0, 1, \dots, q$. Inequality (4.48) holds when $q = 2$ by (4.37). Suppose (4.48) true for $q = 3, \dots, 2N$. We claim, for $i = 0, 1, \dots, 2N+1$,

$$\|\zeta D^{2N+1}_{2N+1-i,i} r(n)\|_{L^2} \leqslant C\varepsilon^{2n+3-(2N+1-i)}. \tag{4.49}$$

Let

$$g = \Delta_\varepsilon^N r(n) = \sum_{k=0}^{N} \binom{N}{k} \varepsilon^{2(N-k)} D^{2N}_{2(N-k),2k} r(n). \tag{4.50}$$

We want to rewrite (4.40) after collecting in the left-hand side all terms like

$$\zeta D^{2N+1}_{2N+1-i,i} r(n). \tag{4.51}$$

We have

$$\|D^1_{1,0}(\zeta g)\|^2_{L^2} = \int_O \sum_{k,h=0}^{N} \left[\binom{N}{k}\varepsilon^{2(N-k)} D^1_{1,0}\left(\zeta D^{2N}_{2(N-k),2k} r(n)\right)\right]$$
$$\times \left[\binom{N}{h}\varepsilon^{2(N-h)} D^1_{1,0}\left(\zeta D^{2N}_{2(N-h),2h} r(n)\right)\right] dy_1\, dy_2. \tag{4.52}$$

Let

$$D^1_{1,0}\left(\zeta D^{2N}_{2(N-k),2k} r(n)\right) = \zeta D^{2N+1}_{2(N-k)+1,2k} r(n) + \alpha_k$$

where by the induction hypothesis

$$\|\alpha_k\|_{L^2} \leqslant C\varepsilon^{2n+3-2(N-k)}. \tag{4.53}$$

The diagonal terms in (4.52) can be estimated as follows:

$$\left\|D^1_{1,0}\left(\zeta D^{2N}_{2(N-k),2k} r(n)\right)\right\|^2_{L^2} \geqslant \left\|\zeta D^{2N+1}_{2(N-k)+1,2k} r(n)\right\|^2_{L^2} - 2\|\alpha_k\|_{L^2}\left\|\zeta D^{2N+1}_{2(N-k),2k} r(n)\right\|_{L^2}. \tag{4.54}$$

For the off-diagonal terms in (4.52), we put

$$D^1_{1,0}\left(\zeta D^{2N}_{2(N-k),2k} r(n)\right) = D^{2N+1}_{2(N-k)+1,2k}(\zeta r(n)) + \beta_k \tag{4.55}$$

where

$$\|\beta_k\|_{L^2} \leqslant C\varepsilon^{2n+3-2(N-k)}. \tag{4.56}$$

Using (4.55) we have, with successive integrations by parts,

$$\int_O D^1_{1,0}\left(\zeta D^{2N}_{2(N-k),2k} r(n)\right) \times D^1_{1,0}\left(\zeta D^{2N}_{2(N-h),2h} r(n)\right) dy_1\, dy_2$$
$$= \int_O \left(D^{2N+1}_{2N+1-h-k,h+k}(\zeta r)\right)^2 dy_1\, dy_2 + \int_O \beta_h D^{2N+1}_{2(N-k)+1,2k}(\zeta r)\, dy_1\, dy_2$$
$$+ \int_O \beta_k D^{2N+1}_{2(N-h)+1,2h}(\zeta r)\, dy_1\, dy_2 + \int_O \beta_h \beta_k\, dy_1\, dy_2. \tag{4.57}$$

To transform further the first term in the right-hand side of (4.57), we let

$$D^{2N+1}_{2N+1-h-k,h+k}(\zeta r) = \zeta D^{2N+1}_{2N+1-h-k,h+k} r + \gamma$$

where

$$\|\gamma\|_{L^2} \leqslant C\varepsilon^{2n+3-(2N+1-h-k)}. \tag{4.58}$$

Thus

$$\left\|\zeta D^{2N+1}_{2N-h-k,h+k} r(n)\right\|^2_{L^2} \leqslant \left\|D^{2N+1}_{2N+1-h-k,h+k}(\zeta r)\right\|^2_{L^2} + 2\|\gamma\|_{L^2}\left\|\zeta D^{2N+1}_{2N+1-h-k,h+k} r(n)\right\|_{L^2}. \tag{4.59}$$

Let us repeat the same calculations from (4.52) to (4.59) but starting with

$$\|D^1_{0,1}(\zeta g)\|_{L^2}.$$

We finally obtain

$$\sum_{i=0}^{2N+1} \varepsilon^{2(2N+1-i)}\left\|\zeta D^{2N+1}_{2N+1-i,i} r(n)\right\|^2_{L^2} \leqslant C\varepsilon^{2n+3}\left(\varepsilon^{2n+3} + \sum_{i=0}^{2N+1} \varepsilon^{2N+1-i}\left\|\zeta D^{2N+1}_{2N+1-i,i} r(n)\right\|_{L^2}\right).$$

Hence (4.49) follows. Let now (4.48) be true for $q = 3, \dots, 2N+1$. We claim, for $i = 0, 1, \dots, 2N+2$,

$$\| \zeta D^{2N+2}_{2N+2-i,i} r(n) \|_{L^2} \leqslant C\varepsilon^{2n+3-(2N+2-i)}. \tag{4.60}$$

Proceed now in a way similar to the first part of the proof, starting from (4.41). This inequality can be written as

$$\varepsilon^4 \| D^2_{2,0}(\zeta g) \|^2_{L^2} + 2\varepsilon^2 \| D^2_{1,1}(\zeta g) \|^2_{L^2} + \| D^2_{0,2}(\zeta g) \|^2_{L^2} \leqslant C\varepsilon^{4n+6} \tag{4.61}$$

where g is given again by (4.50). Our goal is to put in evidence and collect in the left-hand side of inequality (4.61) the L^2-norms of all terms like

$$\zeta D^{2N+2}_{2N+2-i,i} r(n).$$

This can be done with estimates similar to those of the first part of the induction argument. As final result we obtain

$$\sum_{i=0}^{2N+2} \varepsilon^{2(2N+2-i)} \| \zeta D^{2N+2}_{2N+2-i,i} r(n) \|^2_{L^2} \leqslant C\varepsilon^{2n+3} \left(\varepsilon^{2n+3} + \sum_{i=0}^{2N+2} \varepsilon^{2N+2-i} \| \zeta D^{2N+2}_{2N+2-i,i} r(n) \|_{L^2} \right).$$

Thus (4.60) follows. This completes the proof by induction of (4.48).

Take now m such that $2m > q$ and note that

$$r(n) = r(n+m) + \sum_{j=n+1}^{n+m} \varepsilon^{2j+1} \phi^{(2j+1)}. \tag{4.62}$$

Since (4.48) holds for every n we have, for $i = 0, 1, \dots, q$,

$$\begin{aligned} \| \zeta D^q_{q-i,i} r(n) \|_{L^2} &\leqslant \| \zeta D^q_{q-i,i} r(n+m) \|_{L^2} + \varepsilon^{2n+3} \| \zeta \phi^{(2n+3)} \|_{L^2} \\ &\leqslant C\left(\varepsilon^{2(n+m)+3-(q-i)} + \varepsilon^{2n+3} \right) \\ &\leqslant C\varepsilon^{2n+3}. \end{aligned}$$

This completes the proof of (4.47). $\square$

By the definitions of $\rho(u_1, n)$ and $\rho(u_2, n)$ we have, for $i = 0, 1, \dots, m$,

$$\zeta D^m_{m-i,i} \rho(u_1, n) = \frac{1}{\varepsilon} \zeta D^{m+1}_{m-i,i+1} r(n), \qquad \zeta D^m_{m-i,i} \rho(u_2, n) = -\zeta D^{m+1}_{m-i+1,i} r(n).$$

Therefore, as an immediate consequence of Lemma 4.3, we obtain

THEOREM 4.2. For $m \geqslant 2$ and $i = 0, \dots, m$,

$$\| \zeta D^m_{m-i,i} \rho(u_1, n) \|_{L^2} \leqslant C\varepsilon^{2n+2}, \tag{4.63}$$

$$\| \zeta D^m_{m-i,i} \rho(u_2, n) \|_{L^2} \leqslant C\varepsilon^{2n+3}. \tag{4.64}$$

Now we have, for $k = 1, 2$,

$$\| \zeta D^m_{m-i,i} \rho(u_k, n) \|_{C^0(\bar{\omega})} \leqslant \| \zeta D^{m+2}_{m-i+1,i+1} \rho(u_k, n) \|_{L^2}.$$

Furthermore, the right-hand side in (4.63) and (4.64) does not depend on m. Thus we obtain, for $i = 0, 1, \dots, m$,

$$\| \zeta D^m_{m-i,i} \rho(u_1, n) \|_{C^0(\bar{\omega})} \leqslant C\varepsilon^{2n+2}, \qquad \| \zeta D^m_{m-i,i} \rho(u_2, n) \|_{C^0(\bar{\omega})} \leqslant C\varepsilon^{2n+3}.$$

To estimate the error associated with the pressure we define

$$\tau(n) = p - \sum_{k=-1}^{n} \varepsilon^{2k} p^{(2k)}.$$

By (4.7), (4.8), and (4.23)–(4.26), we have

$$D^1_{1,0}\tau(n) = \frac{1}{\varepsilon^3} D^3_{0,3} r(n+1) + \frac{1}{\varepsilon} D^3_{2,1} r(n)$$

and

$$D^1_{0,1}\tau(n) = -\varepsilon D^3_{3,0} r(n-1) - \frac{1}{\varepsilon} D^3_{1,2} r(n).$$

Therefore we obtain, by (4.47) for $m \geqslant 1$ and $i = 0, 1, \ldots, m$,

$$\| \zeta D^m_{m-i,i} \tau(n) \|_{L^2} \leqslant C \varepsilon^{2n+2}.$$

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