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EXISTENCE AND UNIQUENESS IN NONCLASSICAL DIFFUSION*

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Abstract. We consider a class of diffusion models that arise in certain nonclassical physical situations and discuss existence and uniqueness of the resulting evolution equations.

1. Introduction. A quite general program to consider diffusion processes in solids was proposed recently by Aifantis [1]. It was based on the principles of continuum mechanics, that is, the differential statements of conservation of mass and momentum together with appropriate constitutive equations. It avoided unclear thermodynamic questions related to the existence of chemical potentials and produced new classes of diffusion behavior within a purely mechanical framework. Several of these classes are discussed in [1] where effects of viscosity, memory, and nonlocality associated with the diffusing species were considered.

The program can briefly be described as follows. Instead of adopting the usual practice of expressing the diffusion flux as the gradient of a chemical potential, we determine it as a solution of a more general statement of momentum balance for the diffusing species. This statement is a differential equation containing the divergence of a symmetric second-order tensor $\mathbf{T}$ representing the stress supported by the diffusing species, and a drag vector $\mathbf{f}$ modeling the resistance felt by the diffusing substance as it travels through the matrix. The type of constitutive equations made for $\mathbf{T}$ and $\mathbf{f}$ define corresponding classes of diffusion behavior.

Among them, Fick's law is derived as a special case when $\mathbf{T}$ is a function of the solute density ρ and $\mathbf{f}$ is proportional to the flux $\mathbf{j}$. If T also depends on a viscosity-like term $\nabla j\,(j_{i,j})$, it turns out that the usual parabolic diffusion equation is replaced by a pseudoparabolic one of the form discussed by Ting [2] and Showalter [3]. If instead, T is assumed to depend on the second density gradient $\nabla^2\rho(\rho_{ij})$, a fourth-order equation is

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obtained with a leading term $\Delta^2\rho$ in addition to the usual ones entering the classical diffusion equation. This is exactly the equation of spinodal decomposition introduced by Cahn [4] to model uphill diffusion and phase separation in metallic alloys. Similar terms of the form $\partial_t(\Delta\rho)$ and $\Delta^2\rho$ also arise by uncoupling the equations of double diffusivity as originally proposed by Aifantis [5] and later discussed further by Aifantis and Hill [6], [7].

It should be emphasized that analogous higher-order derivatives have also appeared previously in the literature of heat transfer and flow through porous media and the resulting class of differential equations are known as "metaparabolic" or "Sobolev" type [3]. It is pointed out, however, that for diffusion problems such equations were first derived in [1], thus providing a new interpretation on the physical origin of this class of implicit evolution equations and additional justification for the study of their mathematical properties. In this connection, we remark that while certain explicit solutions of these newly derived diffusion equations were obtained (e.g., [4]–[8]) and classical uniqueness results have partially been studied (e.g., [6], [8]), their weak formulation [3] has not been explored in the context of diffusion theory and time and space dependency of the phenomenological coefficients occurring in the corresponding implicit evolution equation.

Section 2 outlines the derivation of a new nonclassical diffusion equation whose straight-forward generalization yields the corresponding inhomogeneous implicit evolution equation. Existence, uniqueness, and continuous dependence results pertaining to this equation are established in Sec. 3 by utilizing results of Showalter [9] and Kuttler [10]. Finally, in Sec. 4 specific boundary value problems are considered.

2. The nonclassical diffusion equation. For a solute diffusing through a solid matrix the following equations of mass and momentum balance hold [1]:

$$\left.\begin{aligned} \rho_t + \operatorname{div}\mathbf{j} &= 0, \\ \operatorname{div}\mathbf{T} + \mathbf{f} &= \mathbf{j}_t, \end{aligned}\right\} \tag{2.1}$$

where ρ is the density and $\mathbf{j}$ the flux of the diffusing species, while $\mathbf{T}$ and $\mathbf{f}$ denote, respectively, the symmetric stress tensor the diffusing substance exerts on itself and the resistance force vector exerted upon it by the solid matrix. The index t denotes partial differentiation with respect to time and the symbol div is the divergence operator

$$\left(\operatorname{div}\mathbf{j} = \frac{\partial j_i}{\partial x_i}, \operatorname{div}\mathbf{T} = \frac{\partial T_{ij}}{\partial x_j}\right).$$

Equations 2.1 hold for all diffusion processes and it is only the nature of the constitutive equations for $\mathbf{T}$ and $\mathbf{f}$ that determines how diffusion processes differ from one to another.

The constitutive equations to be assumed here are of the form

$$\left\{\begin{matrix}\mathbf{T}\\ \mathbf{f}\end{matrix}\right\} \text{ functions of } \{\rho, \mathbf{j}, \nabla\rho, \nabla\mathbf{j}, \nabla^2\rho\}, \tag{2.2}$$

where $\nabla\rho$ is the first gradient of density ($\nabla\rho = \partial\rho/\partial x_i$), $\nabla\mathbf{j}$ is a viscosity-like tensor ($\nabla\mathbf{j} = \partial j_i/\partial x_k$), and $\nabla^2\rho$ is the second gradient of $\rho(\nabla^2\rho = \partial^2\rho/\partial x_i\partial x_j)$. Thus, viscosity is considered through $\nabla\mathbf{j}$ and short-range nonuniformities (pseudo-nonlocality) through

$\nabla^2\rho$. Equations (2.2) are restricted by the invariance considerations of frame indifference and material symmetry. For isotropic diffusion, the quantities $\mathbf{T}$ and $\mathbf{f}$ in (2.2) are isotropic functions of their arguments. If, in addition, they are linear and also independent of the position vector $\mathbf{x}$, the following representations can be deduced by using standard results of modern continuum mechanics [1]; see also [11].

$$\left.\begin{aligned} \mathbf{T} &= -\pi\rho\mathbf{l} + \lambda(\operatorname{tr}\mathbf{D})1 + 2\mu\mathbf{D} + a\Delta\rho\mathbf{l} + b\nabla^2\rho, \\ \mathbf{f} &= -\alpha\mathbf{j} - \beta\nabla\rho, \end{aligned}\right\} \tag{2.3}$$

where the various phenomenological coefficients are constants and $\mathbf{D}$ is the symmetric part of $\nabla\mathbf{j}$ [$\mathbf{D} = \frac{1}{2}(\nabla\mathbf{j} + (\nabla\mathbf{j})^T)$]. By substituting (2.3) into (2.1) we have

$$\alpha\mathbf{j} + \mathbf{j}_t = -F_1\nabla\rho + \lambda\nabla(\operatorname{tr}\mathbf{D}) + 2\mu\operatorname{div}(\mathbf{D}) + F_2\nabla(\Delta\rho), \tag{2.4}$$

with $F_1 = \pi + \beta$ and $F_2 = a + b$. On using the easily verified identities

$$\left.\begin{aligned} &\operatorname{div}(\mathbf{D}) = \frac{1}{2}[\Delta\mathbf{j} + \nabla(\operatorname{div}\mathbf{j})], \qquad \Delta\mathbf{j} = -\operatorname{curl}\operatorname{curl}\mathbf{j} + \nabla(\operatorname{div}\mathbf{j}), \\ &\nabla(\operatorname{tr}\mathbf{D}) = \nabla(\operatorname{div}\mathbf{j}), \qquad \Delta(\operatorname{curl}\mathbf{j}) = \operatorname{curl}(\Delta\mathbf{j}), \end{aligned}\right\} \tag{2.5}$$

together with (2.4) we obtain

$$\mathbf{j} + D^*\mathbf{j}_t = -D\nabla\rho + \tilde{D}_1\nabla(\operatorname{div}\mathbf{j}) + \tilde{D}_2\Delta\mathbf{j} + E\nabla(\Delta\rho), \tag{2.6}$$

where $D^* = \alpha^{-1}$, $D = \alpha^{-1}F_1$, $\tilde{D}_1 = \alpha^{-1}(\lambda + \mu)$, $\tilde{D}_2 = \alpha^{-1}\mu$, and $E = \alpha^{-1}F_2$.

By taking the curl of both sides of (2.6) and noting that the curl of a gradient vanishes, we derive the following equation for $\operatorname{curl}\mathbf{j} \equiv \mathbf{v}$:

$$\mathbf{v}_t + \alpha\mathbf{y} = \mu\Delta\mathbf{v}. \tag{2.7}$$

Proceeding formally and assuming that $\mathbf{v}$ and its derivatives are in $L^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$ we apply the Fourier transform in both sides of (2.7) to obtain

$$\hat{\mathbf{v}}_t + \alpha\hat{\mathbf{v}} = -\mu|q|^2\hat{\mathbf{v}}, \tag{2.8}$$

with $\hat{\mathbf{v}} \equiv \iiint e^{-i\mathbf{q}\cdot\mathbf{x}}\mathbf{v}(x)\,dv$ denoting the Fourier transform of $\operatorname{curl}\mathbf{j}$. Since (2.8) is linear, it follows that $\hat{\mathbf{v}} = 0$ (and therefore $\mathbf{v} = 0$) if $\mathbf{v} = 0$ at $t = 0$ or if initially $\mathbf{j}$ is given by the gradient of a scalar field. Of course, the vanishing of $\mathbf{v}$ is guaranteed if $\mathbf{j}$ is delivered by a gradient for all times, as assumed in the classical theory. We also note that a similar argument implies $\mathbf{v} = 0$, regardless of the value of $\mathbf{v}$ at $t = 0$, if the inertia term $\mathbf{j}_t$ of (2.1) is neglected entirely. Thus, with the result $\operatorname{curl}\mathbf{j} = 0$, a combination of $(2.1)_1$, (2.5), and (2.6) yields

$$\mathbf{j} + D^*\mathbf{j}_t = \nabla(-D\rho - \tilde{D}\rho_t + E\Delta\rho), \tag{2.9}$$

with $\tilde{D} = \tilde{D}_1 + \tilde{D}_2$.

It is this last equation that we shall regard as the appropriate generalization of the first Fick's law of diffusion. It models physical situations where viscosity and short-range inhomogeneities associated with the diffusing species are important. On substituting (2.9) into the mass balance equation $(2.1)_1$, we obtain the following equation:

$$\rho_t + D^*\rho_{tt} = D\Delta\rho + \tilde{D}\Delta\rho_t - E\Delta^2\rho. \tag{2.10}$$

An extension of this derivation to consider situations where the diffusion properties are varying in time and space can easily be obtained by adopting the following set of simplified constitutive equations:

$$\left.\begin{aligned} \mathbf{T} &= c_1\rho\mathbf{I} + c_2(\operatorname{tr}\mathbf{D})\mathbf{I} + c_3\operatorname{tr}(\nabla^2\rho)\mathbf{I}, \\ \mathbf{f} &= -\alpha\mathbf{j}, \end{aligned}\right\} \tag{2.11}$$

where α is a nonsingular symmetric matrix depending on $\mathbf{x}$ and t and the $c_i (i = 1, 2, 3)$ are constants. The reason for neglecting the nonhydrostatic terms in $(2.11)_1$ and the $\nabla\rho$ term in $(2.11)_2$ is that these terms did not contribute to the final form of the generalized Fick's law (2.9) and (2.10) for the case of constant coefficients. We will also neglect the inertia term $\mathbf{j}_t$ since this quantity is small in usual diffusion situations. Under these conditions, it follows that $(2.1)_2$ yields, with $\boldsymbol{\beta} = \boldsymbol{\alpha}^{-1}$, the flux expression

$$\mathbf{j} = \boldsymbol{\beta}\nabla(c_1\rho - c_2\rho_t + c_3\Delta\rho), \tag{2.12}$$

which with $(2.1)_1$ gives the following diffusion equation:

$$\frac{\partial}{\partial t}(\rho - c_2\operatorname{div}(\boldsymbol{\beta}\nabla\rho)) = -c_1\operatorname{div}(\boldsymbol{\beta}\nabla\rho) - c_3\operatorname{div}(\boldsymbol{\beta}\nabla(\Delta\rho)). \tag{2.13}$$

We note that more general equations could have been obtained by letting the coefficients c_i be tensors also depending on $\mathbf{x}$ and t. For brevity, we do not present the details of such derivations here, but we remark that they all lead to differential equations that are special cases of the following general fourth-order equation:

$$\frac{\partial}{\partial t}\left(\rho - \sum_{i,j}\partial_i\left(\tilde{D}_{ij}(\mathbf{x},t)\partial_j\rho\right)\right) - \sum_{i,j}\partial_i\left(D_{ij}(\mathbf{x},t)\partial_j\rho\right) + \sum_{|\alpha|,|\beta|\le 2}(-1)^{|\beta|}D^\beta\left(E_{\alpha\beta}(\mathbf{x},t)D^\alpha\rho\right) = g(\mathbf{x},t), \tag{2.14}$$

where α, β are multiindices and the various function coefficients will be specified later.

3. Existence and uniqueness. Motivated by the developments in the previous section, we consider the following implicit evolution equation with time-dependent coefficients,

$$\frac{\partial}{\partial t}\left(\rho - \sum_{i,j}\partial_i\left(\tilde{D}_{ij}(\mathbf{x},t)\partial_j\rho\right)\right) - \sum_{i,j}\partial_i\left(D_{ij}(\mathbf{x},t)\partial_j\rho\right) + \sum_{|\alpha|,|\beta|\le 2}(-1)^{|\beta|}D^\beta\left(E_{\alpha\beta}(\mathbf{x},t)D^\alpha\rho\right) = g(\mathbf{x},t), \tag{3.1}$$

and an initial condition of the type[3]

$$\rho(0,x) = \rho_0(x), \tag{3.2}$$

[3]In actuality, the initial condition (3.2) will involve an operator $B(t)$ to be defined later and reads: $\lim_{t\to 0+}\langle B(t)\rho(t,\cdot), v\rangle = \langle B(0)\rho_0(\cdot), v\rangle$ for all v in an appropriate Banach space of functions. This is because the abstract theory to be used also applies to degenerate evolution equations in which the part of the equation that is differentiated with respect to t could vanish entirely, thus making the initial condition (3.2) [but not its proposed generalization] inappropriate.

where the prescribed function $\rho_0(x)$ and the source function $g(\mathbf{x}, t)$ will be specified later. Appropriate boundary conditions, of a form also to be discussed later, are attached to (3.1) and the following assumptions are made on the various coefficients which are all real-valued functions:

$$\left.\begin{aligned} &\tilde{D}_{ij} = \tilde{D}_{ji},\\ &\tilde{D}_{ij} \text{ is bounded, measurable, and } C^1 \text{ in } t,\\ &\tilde{D}_{ij}\xi_i\xi_j \geqslant 0 \text{ for } \xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3,\\ &D_{ij} \in L^\infty(\Omega \times [0, T]),\\ &E_{\alpha\beta} \in L^\infty(\Omega \times [0, T]). \end{aligned}\right\} \tag{3.3}$$

Next we state two basic definitions and a main hypothesis as follows:

DEFINITION 1. Let V, W, and H be such that the following hold true:

$$C_0^\infty(\Omega) \subseteq V \subseteq H^2(\Omega), \qquad W = H^1(\Omega), \qquad H = L^2(\Omega). \tag{3.4}$$

DEFINITION 2. We introduce the following time-dependent operators:

$$\langle B(t)u, v\rangle = \int_\Omega \left[u(\mathbf{x})\overline{v(\mathbf{x})} + \tilde{D}_{ij}(\mathbf{x}, t)\partial_i u(\mathbf{x})\,\overline{\partial_j v(\mathbf{x})}\right] dx, \quad u, v \in W,$$

$$\langle A(t)u, v\rangle = \int_\Omega \left[D_{ij}(\mathbf{x}, t)\partial_i u(\mathbf{x})\partial_j \overline{v(x)} + \sum_{|\alpha|, |\beta| \leqslant 2} E_{\alpha\beta}(\mathbf{x}, t) D^\alpha u(\mathbf{x}) D^\beta \overline{v(\mathbf{x})}\right] dx, \tag{3.5}$$

$u \in H^2(\Omega)$, and $v \in V$.

We adopt the notation that if $u \in L^2(0, T; H^2(\Omega))$, Au will denote the element of $L^2(0, T; V')$ with $Au(t) = A(t)u(t)$ and a similar convention will apply to the term Bu.

Main Hypothesis. We assume that there exists at least one $\lambda \in \mathbf{R}$ for which the following coercivity inequality holds:

$$2\langle A(t)u, u\rangle + \lambda\langle B(t)u, u\rangle + \langle B'(t)u, u\rangle \geqslant C_1\|u\|_V^2, \tag{3.6}$$

where $C_1 > 0$.

Before stating the main results on existence and uniqueness of solutions for the abstract equation containing initial-boundary value problems for (3.1) as special cases, we need to define what is meant by a solution. This is done in the following.

DEFINITION 3. Let $f \in L^2(0, T; V')$. We define f' in the sense of V' valued distributions by

$$f'(\phi) = -\int_0^T f(t)\phi'(t)\,dt \quad \text{for all } \phi \in C_0^\infty(0, T). \tag{3.7}$$

For $f \in L^2(0, T; V')$, we say that $f' \in L^2(0, T; V')$ if there exists $g \in L^2(0, T; V')$, such that

$$f'(\phi) = \int_0^T g(t)\phi(t)\,dt \quad \text{for all } \phi \in C_0^\infty(0, T). \tag{3.8}$$

It can be shown that the function g is unique; thus we may set $f' = g$. We also introduce

$$X = \left\{ u \in L^2(0,T;V) \colon (i^*Bu)' \in L^2(0,T;V') \right\}, \tag{3.9}$$

$$\|u\|_X = \|u\|_{L^2(0,T;V)} + \left\|(i^*Bu)'\right\|_{L^2(0,T;V')}. \tag{3.10}$$

For convenience, we let $Lu = (i^*Bu)'$ for $u \in X$, and we also make use of the following proposition whose proof is in [10].

PROPOSITION 1. For $u, v \in X$, the following hold.

(a) $t \to \langle B(t)u(t), v(t)\rangle_{W',W}$ equals an absolutely continuous function a.e. [denoted by $\langle Bu, v\rangle(\cdot)$],

(b) $\langle Lu(t), u(t)\rangle = \frac{1}{2}[\langle Bu, u\rangle'(t) + \langle B'(t)u(t), u(t)\rangle]$ a.e.,

(c) $t \to B(t)u(t)$ equals a function of $C(0,T;W')$ a.e. [denoted by $Bu(\cdot)$],

(d) $\sup\{\|Bu(t)\|_{W'}, t \in [0,T]\} \leqslant C\|u\|_X$ for some $C > 0$,

(e) $|\langle Bu, v\rangle(t)| \leqslant C\|u\|_X\|v\|_X$ for some $C > 0$ and all $t \in [0,T]$.

(f) If $K_t\colon X \to X'$ is given by $\langle K_t u, v\rangle_{X',X} = \int_0^t \langle Lu(s), v(s)\rangle\, ds + \langle Bu, v\rangle(0)$, then K_t is linear, continuous, and weakly continuous.

(g) $\langle K_t u, u\rangle \geqslant \frac{1}{2}[\langle Bu, u\rangle(t) + \langle Bu, u\rangle(0)] + \frac{1}{2}\int_0^t \langle B'(s)u(s), u(s)\rangle\, ds.$

With the above, we can state the following existence and uniqueness theorem, which is a special case of the results obtained in [10]. The existence part can also be obtained as a specialization of the existence theorem of [9].

THEOREM 1. Let[4] $f \in L^2(0,T;V')$ and let $u_0 \in W$. Then there exists a unique solution $u \in X$ to the problem

$$\left.\begin{aligned} &Lu + Au = f \text{ in } L^2(0,T;V'),\\ &i^*Bu(0) = i^*B(0)u_0 \text{ in } V'. \end{aligned}\right\} \tag{3.11}$$

THEOREM 2. The solution of Theorem 1 satisfies the estimates

$$\langle Bu, u\rangle(t) + \frac{3}{4}C_1\int_0^t \|u\|_V^2\, ds \leqslant \frac{4}{C_1}\int_0^t \|f(s)\|_{V'}^2 + \lambda\int_0^t \langle Bu, u\rangle(s)\, ds + \langle Bu, u\rangle(0), \tag{3.12}$$

$$\langle Bu, u\rangle(t) \leqslant \left[\frac{4}{C_1}\int_0^T \|f\|_{V'}^2\, ds + \langle B(0)u_0, u_0\rangle\right]e^{\lambda t}, \tag{3.13}$$

where λ is given in (3.6).

Proof. Estimate (3.12) follows from multiplying $(3.11)_1$ by u and using Proposition 1 to integrate by parts. It is shown in Theorem 3 of [10] that $\langle Bu, u\rangle(0) = \langle Bu(0), u_0\rangle$. Therefore, using (c) of Proposition 1, $\langle Bu, u\rangle(0) = \langle Bu(0), u_0\rangle \leqslant (\langle Bu, u\rangle(0))^{1/2}(\langle B(0)u_0, u_0\rangle)^{1/2}$. Replacing $\langle Bu, u\rangle(0)$ in (3.12) by $\langle Bu_0, u_0\rangle$, an application of Gronwall's inequaltiy [12] yields (3.13).

[4] Note that $L^2(0,T;H) \subseteq L^2(0,T;V')$. By requiring f to just be in $L^2(0,T;V')$, the consideration of general boundary conditions for (3.1) is simplified. This will be illustrated in the example of boundary value problems considered in Sec. 4.

COROLLARY 1. Let $u_n \in X$ be the solution to $(3.11)_1$ and the initial condition $i^*Bu_n(0) = i^*B(0)u_{0n}$. Then if u is a solution of (3.11) and $\lim_{n\to\infty}\langle B(0)(u_0 - u_{0n}), u_0 - u_{0n}\rangle = 0$, it follows that $\lim_{n\to\infty}\|u_n - u\|_X = 0$.

Proof. Let $w_n = u_n - u$. Then w_n satisfies

$$\left.\begin{aligned} &Lw_n + Aw_n = 0 \text{ in } L^2(0,T;V'),\\ &i^*Bw_n(0) = i^*B(0)(u_{0n} - u_0) \text{ in } V'.\end{aligned}\right\} \tag{3.14}$$

Estimate (3.13) implies that $\langle Bw_n, w_n\rangle(t)$ converges uniformly to zero on $[0,T]$. This along with estimate (3.12) implies that $\|w_n\|_{L^2(0,T;V)}$ converges to zero. Therefore, Aw_n and consequently Lw_n converge to zero in $L^2(0,T;V')$, and this completes the proof. Next we consider the question of when $(3.11)_2$ can be replaced with the stronger condition

$$Bu(0) = B(0)u_0 \text{ in } W', \tag{3.15}$$

and prove

THEOREM 3. Let $u_0 \in W$ and assume that there exists a sequence $\{u_{0n}\}_{n=1}^{\infty} \subseteq V$ such that $\lim_{n\to\infty}\langle B(0)(u_0 - u_{0n}), u_0 - u_{0n}\rangle = 0$. Then if u is the solution of (3.11), condition $(3.11)_2$ can be replaced with (3.15).

Proof. Suppose first that $u_0 \in V$. Then $w(t) = u(t) - u_0$ implies $w \in X$ and $i^*Bw(0) = 0$. It is shown in [10] that there exists a sequence $\{w_n\} \subseteq X$ such that $w_n(t) = 0$ near $t = 0$ and $\lim_{n\to\infty}\|w_n - w\|_X = 0$; therefore, from (d) of Proposition 1, $Bw(0) = \lim_{n\to\infty} Bw_n(0) = 0$, and thus $Bu(0) = B(0)u_0$. Next, let u_n be the solution of (3.11) with u_{0n} in place of u_0 and $w_n = u - u_n$. Then w_n is the solution of

$$\left.\begin{aligned} &Lw_n + Aw_n = 0 \text{ in } L^2(0,T;V'),\\ &i^*Bw_n(0) = i^*B(0)(u_0 - u_{0n}),\end{aligned}\right\} \tag{3.16}$$

and (3.13) establishes that $\lim_{n\to\infty}\langle Bw_n, w_n\rangle(t) = 0$ uniformly for $t \in [0,T]$. Inequality (3.12) now implies that $\lim_{n\to\infty}\|w_n\|_{L^2(0,T;V)} = 0$ and $(3.16)_1$ that $\lim_{n\to\infty} Lw_n = 0$ because of the continuity of A. Therefore,

$$Bu(0) = \lim_{n\to\infty} Bu_n(0) = \lim_{n\to\infty} B(0)u_{0n} = B(0)u_0, \tag{3.17}$$

which completes the proof.

In concluding this section we note that the condition of Theorem 3 that allows the replacement of $(3.11)_2$ by (3.15) amounts to the imposition of a compatibility constraint between boundary and initial conditions. This will be seen in the applications to specific initial-boundary value problems discussed in the next section.

4. Boundary value problems. In this section we discuss specific initial-boundary value problems obtained as special cases of the theorems given in Sec. 3. Let $f \in L^2(0,T;V')$, such that

$$\langle f, v\rangle = \int_0^T \Big[(g(t), v(t))_H - \big\langle (i^*Bw)'(t), v(t)\big\rangle - \big\langle Aw(t), v(t)\big\rangle\Big]\,dt$$
$$+ \int_0^T \left[(l(t), v(t))_{L^2(\partial\Omega)} + \left(k(t), \frac{\partial v(t)}{\partial n}\right)_{L^2(\partial\Omega)}\right] dt, \tag{4.1}$$

where $l, k \in L^2(0,T;L^2(\partial\Omega))$, $g \in L^2(0,T;H)$, $w \in H^1(0,T;H^2(\Omega))$ are all given; $\partial\Omega$ is a smooth two-dimensional manifold with $\mathbf{n}$ its outward normal, and the terms involving $L^2(\partial\Omega)$ make sense because $v(t) \in H^2(\Omega)$.

Suppose that (3.6) holds so that there exists a unique solution, $u \in X$, to

$$\left.\begin{aligned} &Lu + Au = f \text{ in } L^2(0,T;V'),\\ &i^*Bu(0) = i^*B(0)(u_0 - w(0)) \text{ in } V',\end{aligned}\right\} \tag{4.2}$$

and let $\rho = u + w$, $v \in C_0^\infty(0,T;V)$. On multiplying $(4.2)_1$ by v and integrating by parts we have

$$\begin{aligned} -\int_0^T \langle B\rho(t), v'(t)\rangle\,dt + \int_0^T \langle A\rho(t), v(t)\rangle\,dt &= \int_0^T (g(t), v(t))_H\,dt \\ &+ \int_0^T \left[(l(t), v(t))_{L^2(\partial\Omega)} + \left(k(t), \frac{\partial v(t)}{\partial n}\right)_{L^2(\partial\Omega)}\right] dt. \end{aligned} \tag{4.3}$$

On rearranging (4.3) we obtain

$$\begin{aligned} &\int_0^T \left[-\langle (Bv)', \rho\rangle_W + \langle B'v, \rho\rangle_W\right] dt + \int_0^T \langle A\rho, v\rangle\,dt \\ &\quad= \int_0^T (g(t), v(t))_H\,dt + \int_0^T \left[(l(t), v(t))_{L^2(\partial\Omega)} + \left(k(t), \frac{\partial v(t)}{\partial n}\right)_{L^2(\partial\Omega)}\right] dt \end{aligned} \tag{4.4}$$

for all $v \in C_0^\infty(0,T;V)$. Let $\tilde{X} = \{v \in L^2(0,T;V)\colon (Bv)' \in L^2(0,T;W')\}$, $\|u\|_{\tilde{X}} = \|u\|_{L^2(0,T;V)} + \|(Bu)'\|_{L^2(0,T;W')}$, and $\hat{X}$ the closure of $C_0^\infty(0,T;V)$ in $\tilde{X}$. Then (4.4) holds for all $v \in \hat{X}$.

Next we define $\gamma\colon V \to L^2(\partial\Omega) \times L^2(\partial\Omega)$, such that

$$\gamma u(x) = \left(u(x), \frac{\partial u}{\partial n}(x)\right); \tag{4.5}$$

thus $\ker\gamma = H_0^2(\Omega)$ [15]. On letting $\hat{X}_0 = \{v \in \hat{X}\colon \gamma v(t) = 0 \text{ a.e.}\}$, $\mathscr{B} = \gamma(\hat{X})$, and $\hat{\gamma}\colon \hat{X}/\hat{X}_0 \to \mathscr{B}$ be given by

$$\hat{\gamma}([u]) = \gamma u, \tag{4.6}$$

we define $\|b\|_{\mathscr{B}} = \|\hat{\gamma}^{-1}b\|_{\hat{X}/\hat{X}_0}$. Also for $v \in \hat{X}$, $\mathscr{S}\rho \in \hat{X}'$, and $S\rho \in \hat{X}_0'$ are defined by

$$\left.\begin{aligned} \langle \mathscr{S}\rho, v\rangle &= -\int_0^T \langle (Bv)', \rho\rangle_W\,dt + \int_0^T \langle B'v, \rho\rangle_W\,dt + \int_0^T \langle A\rho, v\rangle\,dt,\\ S\rho &= i^*\mathscr{S}\rho,\end{aligned}\right\} \tag{4.7}$$

where $i\colon \hat{X}_0 \to \hat{X}$ is the inclusion map.

Now by letting $v \in \hat{X}_0$, it follows that

$$\langle S\rho, v\rangle = \langle \mathscr{S}\rho, v\rangle = \int_0^T (g(t), v(t))_H\,dt. \tag{4.8}$$

Since both $\hat{X}_0$ and $\hat{X}$ are dense in $L^2(0,T;H)$, (4.8) implies that $S\rho \in L^2(0,T;H')$ and that for all $v \in \hat{X}_0$,

$$\langle \mathscr{S}\rho, v\rangle - \langle S\rho, v\rangle_{L^2(0,T;H)} = 0. \tag{4.9}$$

Also if $\pi\colon \hat{X} \to \hat{X}/\hat{X}_0$ is given by $\pi u = [u]$, it follows that there exists a unique element of $\mathscr{B}'$, denoted by $\partial\rho$, satisfying the condition $\langle \mathscr{S}\rho, v\rangle - \langle S\rho, v\rangle_{L^2(0,T;H)} = \langle \pi^*\gamma^*\partial\rho, v\rangle$ for $v \in \hat{X}$. Thus for all $v \in \hat{X}$,

$$\langle \mathscr{S}\rho, v\rangle - \langle S\rho, v\rangle_{L^2(0,T;H)} = \langle \partial\rho, \gamma v\rangle_{\mathscr{B}'\mathscr{B}}. \tag{4.10}$$

The discussion leading to (4.10) is similar to the treatment of boundary value problems given in [15] where ∂ is called the abstract Green's operator. Summarizing (4.10), (4.8), and (4.4), we have

$$\left.\begin{aligned} \langle \mathscr{S}\rho, v\rangle &= \int_0^T (g(t), v(t))_H\, dt \text{ for all } v \in \hat{X}_0,\\ \rho &= u + w \text{ where } u(t) \in V \text{ a.e.},\\ \langle \partial\rho, \gamma v\rangle_{\mathscr{B}'\mathscr{B}} &= \int_0^T (l(t), v(t))_{L^2(\partial\Omega)}\, dt + \int_0^T \left(k(t), \frac{\partial v(t)}{\partial n}\right)_{L^2(\partial\Omega)} dt, \end{aligned}\right\} \tag{4.11}$$

where $(4.11)_1$ implies that ρ is a weak solution of the appropriate differential equation, $(4.11)_2$ gives the forced boundary conditions, and $(4.11)_3$, which holds for all $v \in \hat{X}$, yields the variational boundary conditions.

4.1. *Dirichlet boundary conditions.* Here we let $V = H_0^2(\Omega)$ and assume that each $E_{\alpha\beta}$ is continuous with $\sum_{|\alpha|,|\beta|=2} E_{\alpha\beta}\xi^\alpha\xi^\beta \geqslant C|\xi|^4$ for some $C > 0$ and all $\xi \in \mathbf{R}^3$. Thus the conditions of Gardings inequality [13] are satisfied, and (3.6) holds for some λ and $C_1 > 0$. In this case $(4.11)_3$ is trivial while $(4.11)_2$ yields the following weak form of time-dependent boundary conditions:

$$\rho(t) - w(t) \in H_0^2(\Omega) \text{ a.e.} \tag{4.12}$$

To see that $(4.11)_1$ is the weak form of the relevant differential equation, let $v \in C_0^\infty((0,T)\times\Omega)$ and use the definitions of $A(t)$ and $B(t)$ in (4.4). If a measurable representative for ρ is chosen, $(4.11)_1$ and $(4.7)_1$ imply

$$\begin{aligned} &-\int_0^T \int_\Omega \rho(\mathbf{x},t)v_t(\mathbf{x},t) + \tilde{D}_{ij}(\mathbf{x},t)\partial_i\rho(\mathbf{x},t)\partial_j v_t(\mathbf{x},t)\, dx\, dt\\ &+\int_0^T \int_\Omega D_{ij}(\mathbf{x},t)\partial_i\rho\partial_j v\, dx\, dt + \int_0^T \int_\Omega \sum_{|\alpha|,|\beta|\leqslant 2} E_{\alpha\beta}(\mathbf{x},t) D^\alpha\rho D^\beta v\, dx\, dt\\ &\qquad = \int_0^T \int_\Omega g(\mathbf{x},t)v(\mathbf{x},t)\, dx\, dt, \end{aligned} \tag{4.13}$$

which shows that ρ is indeed a weak solution of (3.1).

On returning to the initial condition, we note that $(4.2)_2$ implies that

$$i^*B\rho(0) = i^*B(0)u_0. \tag{4.14}$$

If $u_0 - w(0) \in \overline{V}$ where the closure is taken in W, then Theorem 3 implies $B\rho(0) = B(0)u_0$. In particular, $u_0 - w(0) \in V$ is the requirement that the initial conditions on $\partial\Omega$ are compatible with the boundary conditions at $t = 0$. Moreover, if $D_{ij}(\mathbf{x},t)$ do not depend on t, the condition $B\rho(0) = B(0)u_0$ implies $\rho(0) = u_0$.

In closing, we note that (4.12) is equivalent to the more familiar form of boundary conditions:

$$\left.\begin{aligned} \rho(\mathbf{x},t) &= w(\mathbf{x},t), \quad x \in \partial\Omega, \\ \partial_i\rho(\mathbf{x},t) &= \partial_i w(\mathbf{x},t), \quad x \in \partial\Omega \end{aligned}\right\}. \tag{4.15}$$

4.2. *Variational boundary conditions.* In discussing boundary conditions of the variational type, we consider a special case of (3.1) where

$$\langle A(t)u, v\rangle = \int_\Omega \left(D_{ij}(\mathbf{x},t)\partial_i u\partial_j v + \Delta u\Delta v\right) dx, \tag{4.16}$$

and $V = \{u \in H^2(\Omega): \partial u/\partial n = 0 \text{ on } \partial\Omega\}$. With $\|\ \|_0$ and $\|\ \|_i$ denoting the $L^2(\Omega)$ and $H^i(\Omega)$ norms respectively, we have

LEMMA 1. There exists a constant $C_0 > 0$ such that for all $u \in V$,

$$\|u\|_0 + \|\Delta u\|_0 \geqslant C_0\|u\|_2. \tag{4.17}$$

Proof. For any $f \in L^2(\Omega)$, there exists a unique $u \in V$ such that $u - \Delta u = f$, as implied by elliptic regularity [15]. Thus, if $L: V \to L^2(\Omega)$ is given by $Lu = u - \Delta u$, L is one to one, onto, and continuous. Because of the open mapping theorem [16], L^{-1} is continuous, $\|u\|_V = \|L^{-1}Lu\|_V \leqslant \|L^{-1}\|(\|u\|_0 + \|\Delta u\|_0)$, and this completes the proof.

LEMMA 2. For given $\varepsilon > 0$ there exists $K > 0$ depending only on ε and Ω such that $\|u\|_1 \leqslant \varepsilon\|u\|_2 + \|u\|_0 K$ for all $u \in H^2(\Omega)$.

Proof. See reference [14].

Lemmas 1 and 2 may be used to verify (3.6) for the case where V is specified as above, and then $(4.11)_1$ implies ρ is a weak solution of

$$\frac{\partial}{\partial t}\Big(\rho - \sum_{i,j}\partial_i\big(\tilde{D}(\mathbf{x},t)\partial_j\rho\big)\Big) - \sum_{i,j}\partial_i\big(D_{ij}(\mathbf{x},t)\partial_j\rho\big) + \Delta^2\rho = g(\mathbf{x},t). \tag{4.18}$$

The condition $(4.11)_2$ yields a forced boundary condition of the form

$$\frac{\partial\rho}{\partial n}(\mathbf{x},t) = \frac{\partial w}{\partial n}(\mathbf{x},t), \quad \mathbf{x} \in \partial\Omega. \tag{4.19}$$

Finally, $(4.11)_3$ is a weak formulation of a corresponding variational boundary condition which can be determined by assuming sufficient smoothness and then applying the Divergence Theorem. This yields the boundary condition

$$\frac{\partial}{\partial t}\left(\tilde{D}_{ij}\partial_j\rho n_i\right) + \left(D_{ij}\partial_j\rho\right)n_i - \frac{\partial(\Delta\rho)}{\partial n} = l \text{ on } \partial\Omega, \tag{4.20}$$

while, as before, the initial condition is given by

$$i^*B\rho(0) = i^*B(0)u_0. \tag{4.21}$$

For our last example, let $V = \{u \in H^2(\Omega)$ such that $\int_{\partial\Omega}(\partial u/\partial n)\,ds = 0$, and $u =$ an unspecified constant on $\partial\Omega\}$. If $L: V \to L^2(\Omega)$ is given by $Lu = u - \Delta u$, it can be shown (by employing the techniques used in [13], [15] to obtain elliptic regularity) that L is injective, onto, and continuous; thus by the open mapping theorem [16], L^{-1} is also continuous. It follows that for $u \in V$,

$$\|u\|_2 = \|L^{-1}(Lu)\|_2 \leqslant \|L^{-1}\| \, \|Lu\|_0 \leqslant \|L^{-1}\|(\|u\|_0 + \|\Delta u\|_0),$$

and, as in the preceding example, the basic hypothesis (3.6) holds. Therefore, we have obtained existence and uniqueness of a weak solution for the equation (4.18) subject to the initial condition (4.21) and the boundary conditons $(4.11)_{2,3}$. For this example, $(4.11)_2$ yields

$$\left.\begin{aligned} &\rho(\mathbf{x}, t) = C(t) + w(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega, \\ &\int_{\partial\Omega} \frac{\partial \rho}{\partial n}(\mathbf{x}, t)\, ds = \int_{\partial\Omega} \frac{\partial w(\mathbf{x}, t)}{\partial n}\, ds, \end{aligned}\right\} \tag{4.22}$$

where $C(\cdot)$ is an unknown function of t only, and $(4.11)_3$ is a weak formulation of the variational boundary conditions of the form

$$\left.\begin{aligned} &\int_{\partial\Omega} \left[\frac{\partial}{\partial t}(\tilde{D}_{ij}\partial_j \rho) n_i + D_{ij}\partial_j \rho n_i - \frac{\partial(\Delta\rho)}{\partial n}\right] ds = \int_{\partial\Omega} l(\mathbf{x}, t)\, ds, \\ &\int_{\partial\Omega} \Delta\rho(\mathbf{x}, t) \frac{\partial v}{\partial n}(\mathbf{x}, t)\, ds = \int_{\partial\Omega} k(\mathbf{x}, t) \frac{\partial v}{\partial n}(\mathbf{x}, t)\, ds, \end{aligned}\right\} \tag{4.23}$$

with $(4.23)_2$ holding for all $v \in V$. In this connection, we note that $(4.23)_2$ has a local description of the form

$$\Delta\rho(\mathbf{x}, t) - k(\mathbf{x}, t) = r(t), \tag{4.24}$$

where $r(\cdot)$ is an unknown function of t. To see this, let $H_0 = \{f \in L^2(\partial\Omega) \colon f = \text{constant}\}$ so that $\partial v/\partial n \in H_0^{\perp}$ for each $v \in V$ and thus $\Delta\rho - k$ is in $H_0^{\perp\perp} = H_0$. Since this holds for each t, the result (4.24) is established.

4.3. *Other boundary conditions*. Obviously, other types of boundary conditions may be discussed by these methods. Time dependent, nonhomogeneous ones are obtained by choosing $f \in L^2(0, T; V')$ and $V \subseteq H^2(\Omega)$. Then, the abstract weak formulation of the auxiliary conditions is given by $(4.11)_{2,3}$. In every case, the crucial hypothesis to be verified is (3.6). We point out that if V is chosen to be all of $H^2(\Omega)$ in Sec. (4.2), the relevant boundary conditions are of purely variational type, that is,

$$\left.\begin{aligned} &\frac{\partial}{\partial t}(\tilde{D}_{ij}\partial_j \rho n_i) + D_{ij}\partial_j \rho n_j - \frac{\partial(\Delta\rho)}{\partial n} = l(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega, \\ &\Delta\rho = k(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega, \end{aligned}\right\} \tag{4.25}$$

which generalize a corresponding situation discussed in [8].

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