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DYNAMICAL EQUATIONS FOR THE FINITE ELASTIC BENDING, TORSION, AND STRETCHING OF RODS*

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1. Introduction. In a series of papers [1]–[3] Ericksen has considered semi-inverse solutions for finite elastostatics in which a prismatic body is deformed into helical configurations, in the absence of body forces. Since these solutions describe deformations in which strain is independent of one material coordinate, they give stress distributions which are natural candidates as finite elastic generalizations of the Saint-Venant solutions for extension, torsion, and simple bending. They then would seem to provide a natural model for the response of a rod or bar to finite extension, torsion, and curvature, when those parameters vary only gradually with respect to a coordinate measuring distance along the rod. Indeed, for slender rods subjected to large displacements and rotations but only small strains, it has been shown using a perturbation expansion (Parker [4]) that the leading approximation to the configuration of each cross section is given by a linear combination of Saint-Venant solutions. Consequently, for slender rods, the bending and torsional rigidities of Kirchhoff's theory are correctly given by the Saint-Venant solutions. In [5], Parker extends the treatment for slender rods into the range where a nonlinear constitutive law is required. This yields a theory in which the tension, torque, and bending moments are derived from a stored energy function as in Green and Laws [6]. Moreover, it shows that the appropriate stored energy is a "cross-sectional energy" closely related to the integral arising in Ericksen's treatment of helical deformations.

This paper extends the asymptotic approach in three ways: by considering dynamic effects, by allowing finite strain, and by treating reference configurations which are themselves helical. A system of material coordinates natural for technical theories is chosen. One coordinate measures distance along the "reference curve of centers," which is a helix. The other two are orthogonal Cartesian coordinates in planes intersecting the helix orthogonally. Although generally this coordinate system is nonorthogonal, the deformations in which all material cross sections deform into congruent shapes are readily

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described (as intimated by Ericksen [1, 2]) by a generalization of the procedures for analyzing helical deformations of a prism. For helicoidal reference shapes, the analysis relates the stress-resultant and couple-resultant to a "cross-sectional energy" which takes into account the reference shape of a typical cross section and allows for material anisotropy. Unfortunately, the corresponding states of deformation of the cross sections form only a two-parameter family, so yielding constitutive rules which are too restrictive for describing the dynamics of curved rods, twisted rods, and spiral springs.

The family of deformed cross sections is enlarged by considering the "periodic solutions" discussed by Ericksen [7]. For general helicoidal bodies, these solutions determine a four-parameter family of "canonical deformations" of a typical cross section. This family may be used to give constitutive rules for the bending moments, torque, and tension as functions of two curvatures, the twist, and the extension. The resulting theory is a natural nonlinear generalization of Kirchhoff theory, with the familiar constitutive rules based on the Saint-Venant semi-inverse solutions of linear elasticity replaced by rules derived from a certain "canonical energy" of three-dimensional elasticity.

The procedure which relates the constitutive rules to the periodic, static deformations of the rod is outlined in general. Particular attention is given to the case of rods of circular cross section and having anisotropy which is invariant under rotations about the axis. In this case, the full four-parameter theory may be developed in terms of solutions of variational problems over a single circular cross section. This special case illustrates how the distinction between helical solutions and more general periodic solutions involves the "ambiguous twist of Love" (Alexander and Antman [8]).

2. Kinematic description. We shall treat elastic rods, which in their unstressed reference configuration are helicoidal, with helical axis parallel to the vector $\mathbf{I}_3$ of a fixed orthogonal triad $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$ of unit vectors. One representative helix within the material is chosen as the "curve of centers" and in the reference configuration its position has the form

$$\mathbf{x} = \mathbf{R}(Y) = \mathbf{R}_0 + k^{-1}\cos\alpha(\mathbf{I}_1\cos kY + \mathbf{I}_2\sin kY) + \mathbf{I}_3 Y\sin\alpha. \tag{2.1}$$

Along this helix Y measures distance, $k^{-1}\cos\alpha$ is the helical radius, $2\pi k^{-1}\sin\alpha$ is the pitch, and α is the helical angle (see Fig. 1). The outward principal normal is $\mathbf{I}_1\cos kY + \mathbf{I}_2\sin kY$, the curvature is $k\cos\alpha$, and each turn of the helix corresponds to a length $2\pi k^{-1}$ along the reference curve of centers. Three orthonormal vectors $\mathbf{E}_L(Y)$ $(L = 1, 2, 3)$ are defined at each Y by

$$\begin{aligned}
\mathbf{E}_1(Y) &= (\mathbf{I}_1\cos kY + \mathbf{I}_2\sin kY)\cos\phi + (-\mathbf{I}_1\sin kY + \mathbf{I}_2\cos kY)\sin\alpha\sin\phi - \mathbf{I}_3\cos\alpha\sin\phi,\\
\mathbf{E}_2(Y) &= -(\mathbf{I}_1\cos kY + \mathbf{I}_2\sin kY)\sin\phi + (-\mathbf{I}_1\sin kY + \mathbf{I}_2\cos kY)\sin\alpha\cos\phi - \mathbf{I}_3\cos\alpha\cos\phi,\\
\mathbf{E}_3(Y) &= (-\mathbf{I}_1\sin kY + \mathbf{I}_2\cos kY)\cos\alpha + \mathbf{I}_3\sin\alpha.
\end{aligned} \tag{2.2}$$

These vectors make constant angles with the outward principal normal $\mathbf{E}_1\cos\phi - \mathbf{E}_2\sin\phi$, binormal $\mathbf{E}_1\sin\phi + \mathbf{E}_2\cos\phi$, and unit tangent $\mathbf{E}_3$ to the reference helix. Each rotates uniformly about the *initial twist vector* $\bar{\kappa} = \kappa\mathbf{I}_3$ according to

$$\mathbf{E}_L'(Y) = \frac{d\mathbf{E}_L}{dY} = \bar{\kappa}\wedge\mathbf{E}_L. \tag{2.3}$$

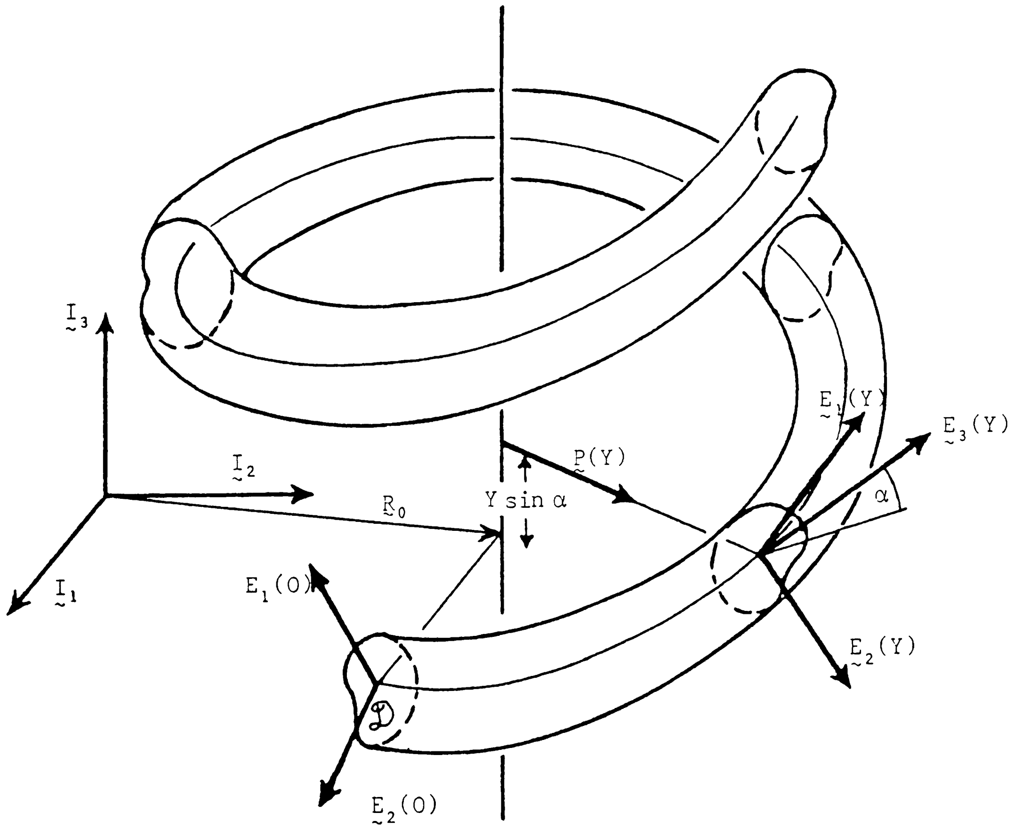

FIG. 1. Helicoidal reference configuration, showing $\mathbf{E}_1(Y)$ and $\mathbf{E}_2(Y)$ in the plane of the outward principal normal $\mathbf{P}(Y) = \mathbf{I}_1 \cos kY + \mathbf{I}_2 \sin kY$.

Material coordinates $Y_J (J = 1, 2, 3)$ are then chosen so that position within the reference configuration is given by

$$\mathbf{x} = \mathbf{R}(Y) + Y_\alpha \mathbf{E}_\alpha(Y) = X_K \mathbf{I}_K, \quad Y \equiv Y_3. \tag{2.4}$$

The equations $Y_1 = Y_2 = 0$ describe the curve of centers, each surface $Y =$ constant describes a plane cutting this helix orthogonally, while Y_1 and Y_2 are orthogonal Cartesian coordinates within each such plane.[1] The region occupied by the body is specified by

$$Y_\alpha \in \mathscr{D}, \quad Y_3 \equiv Y \in (0, L), \quad \operatorname{diam} \mathscr{D} \ll L. \tag{2.5}$$

It may be seen that the choices (2.2) and (2.4) are sufficiently general to permit the conventional choice of $Y_\alpha = 0$ as the locus of the centroids of the cross sections and the choice of the Y_1 and Y_2 axes as principal inertial axes. Identification of the curve of centroids is a nontrivial procedure for general heliform shapes but, anyhow, is not

[1] Roman indices range over the values 1, 2, 3, while Greek indices range over the values 1, 2. The summation convention for repeated indices is used. Primes denote derivatives with respect to Y.

necessarily the most convenient choice in finite elasticity. However, making this choice includes the following reference configurations:

(i) $\alpha = \frac{1}{2}\pi$, a twisted, straight bar of pitch $2\pi k^{-1}$,

(ii) $\alpha = 0$, a planar curved bar of radius k^{-1},

(iii) $\alpha = \frac{1}{2}\pi$, $k = 0$, a straight rod, or beam.

Only in cases (ii) and (iii) do the coordinates Y_J form an orthogonal system in the reference configuration.

Deformed configurations of the rod are described by

$$\mathbf{x} = \mathbf{x}(\mathbf{Y}, t) = \mathbf{r}(Y, t) + u_i(\mathbf{Y}_\alpha, Y, t)\mathbf{e}_i(Y, t) \equiv x_j\mathbf{I}_j, \tag{2.6}$$

where $\mathbf{e}_i(Y, t)$ $(i = 1, 2, 3)$ form a right-handed orthogonal triad of unit vectors having angular velocity $\boldsymbol{\omega}(Y, t)$ and twist vector $\boldsymbol{\kappa}(Y, t)$ such that

$$\dot{\mathbf{e}}_i \equiv \partial\mathbf{e}_i/\partial t = \boldsymbol{\omega} \wedge \mathbf{e}_i, \qquad \mathbf{e}_i' \equiv \partial\mathbf{e}_i/\partial Y = \boldsymbol{\kappa} \wedge \mathbf{e}_i, \quad i = 1, 2, 3. \tag{2.7}$$

The condition

$$u_i(0, Y, t) = 0 \quad \text{all } Y, t \tag{2.8}$$

ensures that $\mathbf{x} = \mathbf{r}(Y, t)$ is the deformed configuration of the *curve of centers*, while the definition

$$\mathbf{r}' \equiv \partial\mathbf{r}/\partial Y = A\mathbf{e}_3 \tag{2.9}$$

selects $\mathbf{e}_3(Y, t)$ as the unit tangent to this curve of centers, with $A = A(Y, t)$ being the *stretch*. The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are related to the local material orientation by

$$u_{2,1}(0, Y, t) = 0, \quad u_{1,1}(0, Y, t) > 0, \tag{2.10}$$

which makes $\mathbf{e}_1(Y, t)$ tangential to the material surface $Y_2 = 0$ at the curve of centers (see Love [9], p. 397). (A more symmetric condition $u_{1,2}(0, Y, t) = u_{2,1}(0, Y, t)$ is frequently applied in linear theories, but the added complexity brings no conceptual benefits in the present theory.) The velocity $\mathbf{V}(Y, t)$ on the curve of centers is defined by

$$\dot{\mathbf{r}} = \mathbf{V}(Y, t). \tag{2.11}$$

3. Dynamics. In the absence of body forces, the momentum equations may be written as

$$T_{Kj:K} = \rho(\mathbf{X})\dot{v}_j, \tag{3.1}$$

where T_{Kj} are the components of Piola–Kirchhoff stress so that the traction on an element $N_K\mathbf{I}_K\,dS$ of the reference surface is $\mathbf{I}_jT_{Kj}N_K\,dS$, where $\mathbf{v} = v_j\mathbf{I}_j = \dot{\mathbf{x}}$ is the material velocity, $\rho = \rho(\mathbf{X})$ is the reference density, and where the colon denotes partial differentiation with respect to the Lagrangian coordinates X_K.

The momentum equations for rod dynamics

$$\frac{\partial\mathbf{F}}{\partial Y} = \dot{\mathbf{q}} = \frac{\partial\mathbf{q}}{\partial t}, \tag{3.2}$$

$$\frac{\partial\mathbf{M}}{\partial Y} + \frac{\partial\mathbf{r}}{\partial Y} \wedge \mathbf{F} = \frac{\partial\boldsymbol{\pi}}{\partial t} + \frac{\partial\mathbf{r}}{\partial t} \wedge \mathbf{q}, \tag{3.3}$$

in the absence of body forces and of tractions on the lateral surface $\partial\mathscr{D} \times (0, L)$ are then obtained by the integration of equations (3.1) over cross sections $\mathscr{D}$ and by defining the stress-resultant $\mathbf{F}$, stress couple $\mathbf{M}$, linear momentum $\mathbf{q}$, and angular momentum $\boldsymbol{\pi}$ (see Green and Laws [6]). In the case of naturally straight rods ($\alpha = \frac{1}{2}\pi$, see (i) and (iii)) with

$Y = X_3$, the appropriate definitions are

$$\mathbf{F} \equiv \iint_{\mathscr{D}} T_{3j}\mathbf{I}_j\, dS = \mathbf{F}(X_3, t), \qquad \mathbf{M} \equiv \iint_{\mathscr{D}} \{\mathbf{x} - \mathbf{r}(X_3, t)\} \wedge T_{3j}\mathbf{I}_j\, dS,$$

$$\mathbf{q} \equiv \iint_{\mathscr{D}} \rho\mathbf{v}\, dS, \qquad \boldsymbol{\pi} \equiv \iint_{\mathscr{D}} \{\mathbf{x} - \mathbf{r}(X_3, t)\} \wedge (\rho\mathbf{v})\, dS. \tag{3.4}$$

The crucial component of any rod theory is the completion of the dynamical system (3.2), (3.3) by constitutive laws relating $\mathbf{F}$, $\mathbf{M}$, $\mathbf{q}$, and $\boldsymbol{\pi}$ to the kinematics of the curve of centers $\mathbf{x} = \mathbf{r}(Y, t)$, and of the associated triad $\{\mathbf{e}_i(Y, t)\}$ (generalizations leading to "director theories" are also possible). Implicitly, any rod theory treats the configuration, stress state, and velocity distribution of each cross section as depending constitutively on only a small number of kinematic parameters. Such assumptions are likely to be valid only when distortions of the cross sections are characterized by parameters which vary significantly with Y only over scales large compared to the cross-sectional diameter $D \equiv \operatorname{diam} \mathscr{D}$. This suggests the use of asymptotic formulations as first introduced by Hay [10] and developed by Rigolot [11], [12] and the present author [4], [5], [13]. Indeed, for the statics of naturally straight, untwisted rods having a linear constitutive law, the distortion of each cross section is shown in [4] to be related to the stretch, curvature, and twist as a superposition of Saint-Venant's solution for stretching, bending, and torsion. That theory, like the generalization [5] to nonlinear, small-strain elasticity, allows large rotations and displacements. Since Saint-Venant's solutions describe deformations with uniform stretch, curvature, or twist, it might appear that a natural generalization to finite elasticity is provided by Ericksen's semi-inverse treatment of heliform configurations [1], [2], [3]. However, since the traction system over each cross section has a resultant $\mathbf{F} = F\mathbf{e}$ which is parallel to the helical axis and has moment about any point $\mathbf{x} = \mathbf{x}_0 + \lambda\mathbf{e}$ of that axis of the form $\hat{\mathbf{M}} = \hat{M}\mathbf{e}$, the helical solutions yield deformations characterized by only two parameters, F and $\hat{M}$. A resulting rod theory is too degenerate since it contains fewer parameters than does elastica theory. Consequently, we base the description on Ericksen's [7] "periodic" solutions in which A and κ_i are independent of t but periodic in Y. Thus,

$$A \simeq A^*(Y) = A^*(Y + b), \qquad \kappa_i \simeq \kappa_i^*(Y) = \kappa_i^*(Y + b),$$

with corresponding distortions of each cross section of the form

$$u_i \simeq u_i^*(Y_\alpha, Y) = u_i^*(Y_\alpha, Y + b).$$

We anticipate that this approximation should be relevant when b (like F and $\hat{M}$) varies significantly only over distances $\gg D$.

As suggested by Ericksen [3], analysis for bodies having helicoidal reference configurations introduces few difficulties. We treat hyperelastic materials with strain-energy density

$$W = W(x_{j:K}, \mathbf{X}) = W(H_{ij}x_{j:K}, \mathbf{X}), \tag{3.5}$$

which is invariant under rotations represented by the arbitrary proper orthogonal matrix $\mathbf{H}$ ($\mathbf{H}\mathbf{H}^T = \mathbf{I}$, $\det \mathbf{H} = +1$). The components $x_{j:K}$ of the deformation gradient are defined by

$$\frac{\partial \mathbf{x}}{\partial X_K} = \frac{\partial x_j}{\partial X_K}\mathbf{I}_j = x_{j:K}\mathbf{I}_j,$$

and the corresponding Piola–Kirchhoff stress components are

$$T_{Kj} = \frac{\partial W}{\partial x_{j:K}}. \tag{3.6}$$

Integration of the momentum equations (3.1) over an arbitrary material region $\mathscr{R}$ then gives

$$\frac{d}{dt} \iiint_{\mathscr{R}} \rho v_j \, d\mathbf{X} = \iint_{\partial\mathscr{R}} T_{Kj} N_K \, dS, \tag{3.7}$$

where $d\mathbf{X} \equiv dX_1 dX_2 dX_3$.

For application to the twisted rod (2.1)–(2.5), it is advantageous to replace the Lagrangian coordinates X_K by the helicoidal material coordinates Y_J using (2.1) and (2.4) to define the functions

$$X_K = X_K(Y_J), \qquad Y_J = Y_J(X_K), \tag{3.8}$$

which are invertible for $Y_J \in \mathscr{D} \times (0, L)$. The appropriate stress measure, motivated by consideration of (3.7) as in [3], is τ_{Rj} where

$$\tau_{Rj} = J Y_{R:K} T_{Kj}, \qquad T_{Kj} = J^{-1} X_{K,R} \tau_{Rj}, \tag{3.9}$$
$$J \equiv \det X_{L,p} = \left(\det Y_{R:K}\right)^{-1},$$

and for any scalars f and g

$$f_{,K} \equiv \frac{\partial f}{\partial Y_K}, \qquad g_{:J} \equiv \frac{\partial g}{\partial X_J}.$$

Note that $\tau_{Rj}\mathbf{I}_j$ is *not* the force per unit area of the material surface $Y_R =$ constant, but that $\mathbf{I}_j \tau_{Rj} dY_P dY_Q$ is the traction on the coordinate element $dY_P dY_Q$, whenever $e_{RPQ} = 1$. Inclusion of the factors J and J^{-1} in (3.9) and definition of the surface element $\overline{N}_R \, d\Sigma$ by

$$\overline{N}_R \, d\Sigma = J^{-1} X_{K,R} N_K \, dS, \qquad N_K \, dS = J Y_{R:K} \overline{N}_R \, d\Sigma, \qquad \overline{N}_R \overline{N}_R = 1$$

means that the traction on any surface element may be written as

$$\mathbf{I}_j T_{kj} N_K \, dS = \mathbf{I}_j \tau_{Rj} \overline{N}_R \, d\Sigma.$$

Equation (3.7) may then be expressed as

$$\frac{d}{dt} \iiint_{\mathscr{R}} \bar{\rho}(\mathbf{Y}) v_j \, d\mathbf{Y} = \iint_{\partial\mathscr{R}} \tau_{Rj} \overline{N}_R \, d\Sigma = \iiint_{\mathscr{R}} \tau_{Rj,R} \, d\mathbf{Y}, \tag{3.10}$$

or, equivalently,

$$\tau_{Rj,R} = \bar{\rho} \dot{v}_j, \tag{3.11}$$

where $\bar{\rho}(\mathbf{Y}) = \rho(\mathbf{X}) J$ and $d\mathbf{Y} = dY_1 dY_2 dY_3$. Moreover, following Ericksen [3], it can readily be shown that

$$\tau_{Rj} = \frac{\partial \overline{W}}{\partial x_{j,R}}, \tag{3.12}$$

where

$$x_{j,R} = \frac{\partial x_j}{\partial Y_R}, \qquad \overline{W}(x_{j,R}, \mathbf{Y}) \equiv J W(x_{p:K}, \mathbf{X}). \tag{3.13}$$

Here $\overline{W} d\mathbf{Y}$ is the strain energy in the material element $d\mathbf{Y}$ and, following from (3.5), possesses the invariance property

$$\overline{W}(x_{j,R}, \mathbf{Y}) = \overline{W}(H_{jk} x_{k,R}, \mathbf{Y}) \quad \text{for all } H_{ij} H_{kj} = \delta_{ik}.$$

As remarked by Ericksen [3], introduction of the factors J in (3.9) and (3.13) avoids the use of metric tensors and yields simple forms for (3.11) and (3.12), so allowing us to "discard one piece of luggage."

In (3.13) the derivatives $x_{j,R}$ are found from (2.6), (2.7), and (2.9), which give

$$\mathbf{I}_j x_{j,R} = \mathbf{x}_{,R} = u_{i,R}\mathbf{e}_i + (A\mathbf{e}_3 + u_i \boldsymbol{\kappa} \wedge \mathbf{e}_i)\delta_{R3}.$$

Thus,

$$x_{j,R} = M_{jk} p_{kR},$$

where

$$p_{k\alpha} = u_{k,\alpha}, \qquad p_{k3} = u_{k,3} + A\delta_{k3} + e_{klm}\kappa_l u_m \tag{3.14}$$

and where $M_{jk} = \mathbf{e}_k \cdot \mathbf{I}_j$ are the elements of the proper orthogonal matrix $\mathbf{M}(Y, t)$ for which $\mathbf{e}_k = M_{jk}\mathbf{I}_j$ and $\mathbf{I}_j = M_{jk}\mathbf{e}_k$. The invariance of $\overline{W}$ under rotations gives

$$\overline{W}(x_{j,R}, \mathbf{Y}) = \overline{W}(p_{kR}, \mathbf{Y}),$$

so that, when the traction $\tau_{Rj}\mathbf{I}_j$ is resolved along the directions of the local triad $\mathbf{e}_k(Y, t)$ $(k = 1, 2, 3)$ as

$$\tau_{Rj}\mathbf{I}_j = \sigma_{Rk}\mathbf{e}_k, \qquad \tau_{Rj} = M_{jk}\sigma_{Rk},$$

the stress components σ_{Rk} are given by

$$\sigma_{Rk} \equiv \frac{\partial \overline{W}}{\partial p_{kR}} = M_{jk}\frac{\partial \overline{W}}{\partial x_{j,R}} = M_{jk}\tau_{Rj}. \tag{3.15}$$

Expressing the momentum equation (3.11) as

$$(\tau_{Rj}\mathbf{I}_j)_{,R} = (\sigma_{Rk}\mathbf{e}_k)_{,R} = \bar{\rho}\dot{\mathbf{v}}, \tag{3.16}$$

and then resolving into components along the directions $\mathbf{e}_k$ yields

$$\sigma_{Rj,R} + e_{jlk}\kappa_l\sigma_{3k} = \bar{\rho} a_j = \bar{\rho}(\dot{v}_j + e_{jmn}\omega_m v_n), \tag{3.17}$$

where the components v_l, a_l of velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are given by

$$\mathbf{v} = \dot{\mathbf{x}} = v_l\mathbf{e}_l = \mathbf{V}(Y, t) + (\dot{u}_l + e_{lmn}\omega_m u_n)\mathbf{e}_l,$$
$$\mathbf{a} = \dot{\mathbf{v}} = a_l\mathbf{e}_l = \dot{v}_l\mathbf{e}_l + \boldsymbol{\omega} \wedge \mathbf{v}.$$

The corresponding boundary condition stating that the traction $\mathbf{I}_j T_{Kj} N_K$ vanishes on the lateral boundary is

$$\sigma_{Rj}\overline{N}_R = 0 = \sigma_{\alpha j}\overline{N}_\alpha \quad \text{on } \partial\mathscr{D} \times (0, L). \tag{3.18}$$

Integration of (3.16) over a cross section $Y =$ constant and use of (3.18) then gives (3.2) and (3.3) for a naturally helical body, provided that the definitions (3.4) are amended to

$$\mathbf{F}(Y,t) \equiv \iint_{\mathscr{D}} \sigma_{3j}\mathbf{e}_j\,dS, \qquad \mathbf{M}(Y,t) \equiv \iint_{\mathscr{D}} \{\mathbf{x} - \mathbf{r}(Y,t)\} \wedge \sigma_{3j}\mathbf{e}_j\,dS,$$
$$\mathbf{q} \equiv \iint_{\mathscr{D}} \bar{\rho}\mathbf{v}\,dS, \qquad \boldsymbol{\pi} \equiv \iint_{\mathscr{D}} \{\mathbf{x} - \mathbf{r}(Y,t)\} \wedge (\bar{\rho}\mathbf{v})\,dS. \tag{3.19}$$

As is normal in rod and beam theories, detailed boundary conditions over $X_3 = 0, L$ are left unspecified at this stage. ·

4. Canonical distortions. To associate distortions $u_i(\mathbf{Y},t)$ with the stretch $A(Y,t)$, curvatures $\kappa_\alpha(Y,t)$, and torsion $\kappa_3(Y,t)$ at a typical cross section, we take as leading-order approximations the distortions governed by (3.14)–(3.17) in which u_i, A, and κ_i are independent of t and periodic in Y, with fundamental period $b(>0)$. Like the corresponding problem for helical deformations of initially straight prisms (Ericksen [1]) this problem has a variational formulation. Indeed, the appropriate specializations of (3.17) and (3.19) are, respectively, the Euler-Lagrange equations and natural boundary conditions associated with stationary values of the *b-canonical energy*[2]

$$E_b(u_i, u_{i,R};\, A(Y), \kappa_l(Y)) \equiv \int_0^b \iint_{\mathscr{D}} \overline{W}(p_{jR}, Y_\alpha)\,dS\,dY, \tag{4.1}$$

for specified $A(Y)$, $\kappa_l(Y)$ satisfying $A(b) = A(0)$, $\kappa_l(b) = \kappa_l(0)$, and with p_{jR} given by (3.14) subject to $u_j(Y_\alpha, b) = u_j(Y_\alpha, 0)$, $p_{jR}(Y_\alpha, b) = p_{jR}(Y_\alpha, 0)$.

In the "periodic configurations," deformed cross sections at Y and $Y + b$ are congruent, but do not, in general, have the same orientations. The locations of the two cross sections differ by a translation along a certain "twist axis" and a certain rotation about that axis. The special case, $u_3 = 0$, $A =$ constant, $\kappa_i =$ constant, describes "helical configurations" in which all cross sections are congruent. These have distortions $u_i = u_i(Y_\alpha)$ corresponding to stationary values of the *cross-sectional energy*

$$\tilde{E}(u_i, u_{i,\alpha};\, A, \kappa_l) \equiv \iint_{\mathscr{D}} \overline{W}(p_{jR}, Y_\alpha)\,dS, \quad dS \equiv dY_1\,dY_2, \tag{4.2}$$

where (3.14) takes the simplified form

$$p_{j\alpha} = u_{j,\alpha}, \qquad p_{j3} = A\delta_{j3} + e_{jkl}\kappa_k u_l. \tag{4.3}$$

Indeed, the Euler-Lagrange equations and natural boundary conditions become

$$\sigma_{\alpha j,\alpha} + e_{jkl}\kappa_l\sigma_{3k} = 0 \quad \text{over } \mathscr{D}, \tag{4.4}$$

$$\sigma_{\alpha j}\overline{N}_\alpha = 0 \quad \text{over } \partial\mathscr{D}, \tag{4.5}$$

where σ_{Rj} is given by (3.15).

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[2] This describes deformations of a "canonical body" for which $\overline{W}(\mathbf{p}, \mathbf{Y})$ does not depend explicitly on Y_3. For this body, any inhomogeneity and anisotropy on a cross section at Y_3 is congruent to the inhomogeneity and anisotropy on $Y_3 = 0$.

Another special case is the limit $b \to \infty$. This is analogous to the static solutions for an elastica in which the elastica is asymptotic to a straight line parallel to the stress-resultant $\mathbf{F}$.

The variational problems based on (4.1) and (4.2) do not naturally incorporate the kinematic constraints (2.8)–(2.10) determining appropriate b-periodic functions $A(Y)$ and $\kappa_i(Y)$ (or constants A, κ_i for (4.2)), so it is best to analyze periodic configurations by adapting the representation in [7] as

$$\mathbf{x}(\mathbf{Y}) = \mathbf{x}_0 + w_i(Y_\alpha, Y)\mathbf{j}_i(Y) + aY\mathbf{j}_3, \qquad w_i(Y_\alpha, Y + b) = w_i(Y_\alpha, Y), \tag{4.6}$$

where

$$\mathbf{j}_i'(Y) = \bar{\boldsymbol{\kappa}} \wedge \mathbf{j}_i = \bar{\kappa}\mathbf{j}_3 \wedge \mathbf{j}_i. \tag{4.7}$$

Equation (4.7) shows that the orthonormal triad of vectors $\{\mathbf{j}_i(Y)\}$ rotates uniformly about the member $\mathbf{J}_3$ of a fixed orthonormal triad $\{\mathbf{J}_i\}$, with period $2\pi/\bar{\kappa}$ in Y, according to

$$\mathbf{j}_1(Y) = \mathbf{J}_1 \cos\bar{\kappa}Y + \mathbf{J}_2 \sin\bar{\kappa}Y, \quad \mathbf{j}_2(Y) = -\mathbf{J}_1 \sin\bar{\kappa}Y + \mathbf{J}_2 \cos\bar{\kappa}Y, \quad \mathbf{j}_3 = \mathbf{J}_3. \tag{4.8}$$

Thus, w_i are displacement components relative to the "twist axis" $\mathbf{x} = \mathbf{x}_0 + aY\mathbf{J}_3$. Since, for any fixed constants χ_i, the curve $\mathbf{x} = \mathbf{x}_0 + \chi_i\mathbf{j}_i(Y) + aY\mathbf{j}_3$ defines a helix having pitch $2\pi a/\bar{\kappa}$ and having the twist axis as axis, any "b-periodic" material configuration (4.6)–(4.8) describes a coiling relative to the helices. The cross section at $Y + b$ differs from that at Y only by a displacement $ba\mathbf{J}_3$ and a rotation through an angle $b\bar{\kappa}$ about the twist axis.

Differentiating (4.6) and resolving $\mathbf{x}_{,R}$ into components along $\mathbf{j}_k(Y) = \overline{M}_{jk}\mathbf{I}_j$ gives

$$x_{j,R} = \overline{M}_{ji}\bar{p}_{iR}, \qquad \overline{M}_{ji} = \mathbf{j}_i(Y)\cdot\mathbf{I}_j,$$

where

$$\bar{p}_{i\alpha} = w_{i,\alpha}, \qquad \bar{p}_{i3} = w_{i,3} + a\delta_{i3} + \bar{\kappa}e_{i3l}w_l. \tag{4.9}$$

Then, replacement of p_{jR} in (4.1) by $\bar{p}_{jR}$ simplifies the *b-canonical energy* to

$$\overline{E}(w_i, w_{i,R};\, a, \bar{\kappa}, b) \equiv \int_0^b \iint_{\mathscr{D}} \overline{W}(\bar{p}_{jR}, Y_\alpha)\, dS\, dY. \tag{4.10}$$

Stationary values of $\overline{E}$ amongst $w_i(Y_\alpha, Y)$ having $w_i(Y_\alpha, b) = w_i(Y_\alpha, 0)$, $w_{i,3}(Y_\alpha, b) = w_i(Y_\alpha, 0)$ is equivalent to solution of the canonical boundary value problem (C):

$$\bar{\sigma}_{Rj,R} + e_{j3k}\bar{\kappa}\bar{\sigma}_{3k} = 0 \qquad \text{over } \mathscr{D}\times(-\infty,\infty), \tag{4.11}$$

$$\bar{\sigma}_{\alpha j}\overline{N}_\alpha = 0 \qquad \text{over } \partial\mathscr{D}\times(-\infty,\infty), \tag{4.12}$$

$$\bar{\sigma}_{Rj} = \frac{\partial \overline{W}}{\partial \bar{p}_{jR}} \tag{4.13}$$

together with (4.9) and the periodicity condition

$$w_j(Y_\alpha, Y + b) = w_j(Y_\alpha, Y).$$

Before confirming and extending Ericksen's result [7] that the stress-resultant must have the form $\mathbf{F} = F\mathbf{J}_3$ and, except in the case $F = 0$, $b\bar{\kappa}/2\pi =$ integer, the moment $\hat{\mathbf{M}}$ of the tractions about a point of the twist axis must also be parallel to $\mathbf{J}_3$, we note that the problem (C) possesses some invariances. Neither a coordinate change $Y \to Y + c$, nor a rigid body displacement

$$w_1 \to w_1 \cos\phi - w_2 \sin\phi, \qquad w_2 \to w_1 \sin\phi + w_2 \cos\phi, \qquad w_3 \to w_3 + d$$

representing a rotation ϕ and a translation d along the twist axis, changes the form of (C). Consequently, functions w_i corresponding to any stationary value $\bar{E} = \hat{E}(a, \bar{\kappa}, b)$ of (4.10) contain the three parameters ϕ, c, d. To fix ideas, we use this arbitrariness to simplify the interpretation of the curve of centers

$$\mathbf{x} = \mathbf{x}_0 + w_i(0, Y)\mathbf{j}_i(Y) + aY\mathbf{j}_3 \equiv \mathbf{r}^*(Y). \tag{4.14}$$

By choosing c appropriately, we make the radial distance

$$\left|(\mathbf{r}^* - \mathbf{x}_0) \wedge \mathbf{j}_3\right| = \left\{w_1^2(0, Y) + w_2^2(0, Y)\right\}^{1/2}$$

from the twist axis have a maximum at $Y = 0$. Then suitable choices of ϕ and d allow us to set

$$w_2(0,0) = 0 = w_3(0,0), \qquad w_1(0,0) \geqslant 0.$$

The corresponding configurations are denoted by

$$w_i = w_i^*(Y_\alpha, Y, a, \bar{\kappa}, b), \tag{4.15}$$

$$p_{i\alpha}^* = w_{i,\alpha}^*, \qquad p_{i3}^* = w_{i,3}^* + a\delta_{i3} + \bar{\kappa}e_{i3l}w_l^*,$$

where w_i^*, p_{iR}^* and all derivatives except those with respect to b are periodic in Y with period $= b$.

Corresponding to each Y, the stress-resultant and stress-couple are readily characterized by considering the *cross-sectional energy* $E^*(Y, a, \bar{\kappa}, b)$ defined by

$$E^*(Y, a, \bar{\kappa}, b) = \iint_{\mathscr{D}} \overline{W}\left(p_{jR}^*, Y_\alpha\right) dS, \tag{4.16}$$

where $\bar{p}_{jR} = p_{jR}^*$ gives a stationary value to $\bar{E}$ in (4.10). Then, resolving $\mathbf{F}$ and $\hat{\mathbf{M}}$ into components along the local triad $\{\mathbf{j}_i(Y)\}$ as $\mathbf{F} = F_k^*\mathbf{j}_k$, $\hat{\mathbf{M}} = M_k^*\mathbf{j}_k$ gives

$$F_k^* = \iint_{\mathscr{D}} \sigma_{3k}^*\, dS, \qquad M_k^* = e_{klm} \iint_{\mathscr{D}} w_l^* \sigma_{3m}^*\, dS,$$

where $\sigma_{3j}^* = \partial\overline{W}/\partial p_{j3}^*$. Since all components F_k^*, M_k^* are periodic in Y with period b, while the vectors $\mathbf{j}_\alpha(Y)$ have period $2\pi/\bar{\kappa}$, it follows immediately from the integrated form of the equilibrium equations

$$F_k^*(Y + b)\mathbf{j}_k(Y + b) = \mathbf{F}(Y + b) = \mathbf{F}(Y) = F_k^*(Y)\mathbf{j}_k(Y)$$

that $F_\alpha^*(Y) = 0$, unless $\bar{\kappa}b/2\pi$ is an integer. However, even with $\mathbf{j}_k(Y + b) = \mathbf{j}_k(Y)$ the moment equation

$$\begin{aligned} M_k^*(Y + b)\mathbf{j}_k(Y + b) &= \hat{\mathbf{M}}(Y + b) = \hat{\mathbf{M}}(Y) - ba\mathbf{j} \wedge \mathbf{F}(Y) \\ &= M_k^*(Y)\mathbf{j}_k(Y) - bae_{3lk}F_l^*(Y)\mathbf{j}_k(Y) \end{aligned}$$

gives $aF_\alpha^*(Y) = 0$. Consequently, $\mathbf{F} = F_3^*\mathbf{J}_3$ unless $a = 0$ so that the portion $0 \leqslant Y \leqslant b$ of the rod is formed into a ring. Also, this moment equation gives

$$M_\alpha^*(Y)\{\mathbf{j}_\alpha(Y + b) - \mathbf{j}_\alpha(Y)\} = \mathbf{0},$$

so that

$$\mathbf{F} = F\mathbf{J}_3, \qquad \hat{\mathbf{M}} = M\mathbf{J}_3, \tag{4.17}$$

except possibly when $\bar{\kappa}b/2\pi$ = integer, in which case congruent cross sections at Y and $Y + b$ have the same orientations.

Now, from (4.15) and (4.16),

$$\frac{\partial E^*}{\partial a} = \iint_{\mathscr{D}} \sigma^*_{Rj} \frac{\partial p^*_{jR}}{\partial a}\, dS = \iint_{\mathscr{D}} \sigma^*_{33}\, dS = F^*_3 = F(a, \bar{\kappa}, b),$$

and similarly

$$\frac{\partial E^*}{\partial \bar{\kappa}} = \iint_{\mathscr{D}} \sigma^*_{3j} e_{j3l} w^*_l\, dS = e_{3lj} \iint_{\mathscr{D}} w^*_l \sigma^*_{3j}\, dS = M^*_3 = M(a, \bar{\kappa}, b).$$

Using the definition

$$\hat{E}(a, \bar{\kappa}, b) = b^{-1} \int_0^b E^*(Y, a, \bar{\kappa}, b)\, dY,$$

these give

$$\frac{\partial \hat{E}}{\partial a} = \frac{\partial E^*}{\partial a}, \qquad \frac{\partial \hat{E}}{\partial \bar{\kappa}} = \frac{\partial E^*}{\partial \bar{\kappa}}. \tag{4.18}$$

Hence, except possibly in the case $\bar{\kappa}b/2\pi$ = integer, the stress system over *each* cross section in a canonical deformation is equipollent to

$$\mathbf{F} = \frac{\partial \hat{E}}{\partial a}\mathbf{J}_3, \qquad \hat{\mathbf{M}} = \frac{\partial \hat{E}}{\partial \bar{\kappa}}\mathbf{J}_3. \tag{4.19}$$

Thus, the magnitudes F and M of $\mathbf{F}$ and $\hat{\mathbf{M}}$ are derived directly from the stationary value $b\hat{E}(a, \bar{\kappa}, b)$ of (4.10), without calculation of E^* at each value of Y. To determine components F_i and M_i of the stress- and couple-resultants at each cross section, it is necessary also only to find the deformed geometry of the curve of centers in a canonical deformation. This is treated in the following section. Nevertheless, in a practical computation of $\hat{E}(a, \bar{\kappa}, b)$ and w^*_j the identities (4.18) may provide a useful check. Another is provided by the result

$$\begin{aligned} \frac{\partial E^*}{\partial Y} &= \iint_{\mathscr{D}} \sigma^*_{Rj} p^*_{jR,3}\, dS \\ &= \iint_{\mathscr{D}} \left\{ \left(\sigma^*_{\alpha j} w^*_{j,3}\right)_{,\alpha} + \left(\sigma^*_{3j} w^*_{j,3}\right)_{,3} - \sigma^*_{Rj,R} w^*_{j,3} + \bar{\kappa}\sigma^*_{3j} e_{j3l} w^*_{l,3} \right\} dS, \end{aligned}$$

which reduces to

$$\frac{\partial E^*}{\partial Y} = \frac{\partial}{\partial Y} \iint_{\mathscr{D}} \sigma^*_{3j} w^*_{j,3}\, dS, \tag{4.20}$$

in view of the lateral boundary condition (4.11) and the equilibrium equations (4.10). A further identity

$$\frac{\partial E^*}{\partial b} = \frac{\partial}{\partial Y} \iint_{\mathscr{D}} \sigma_{3j}^* \frac{\partial w_j^*}{\partial b}\, dS \tag{4.21}$$

is derived by a similar sequence of steps.

5. The canonical constitutive laws. In a canonical configuration (4.14), (4.15) having $w_i = w_i^*(Y_\alpha, Y, a, \bar{\kappa}, b)$, the curve of centers is given by

$$\mathbf{x} = \mathbf{r}^*(Y) = \mathbf{x}_0 + \mathbf{w}^*(Y) + aY\mathbf{j}_3, \tag{5.1}$$

where

$$\mathbf{w}^*(Y) = w_i^*(0, Y, a, \bar{\kappa}, b)\mathbf{j}_i(Y).$$

Consequently, the stress-couple $\mathbf{M}$ associated with the cross section Y is given by

$$\mathbf{M}(Y) = \hat{\mathbf{M}} - \mathbf{w}^* \wedge \mathbf{F} = M\mathbf{J}_3 + F\mathbf{J}_3 \wedge \mathbf{w}^*(Y).$$

The orthonormal triad $\{\mathbf{e}_i^*(Y)\}$ characterizing the local rod orientation is defined in terms of the vectors

$$\boldsymbol{\mu}_3(Y) = p_{i3}^*(0, Y, a, \bar{\kappa}, b)\mathbf{j}_i(Y), \qquad \boldsymbol{\mu}_1(Y) = p_{i1}^*(0, Y, a, \bar{\kappa}, b)\mathbf{j}_i(Y),$$

which are respectively the material derivatives $\mathbf{x}_{,3}$ and $\mathbf{x}_{,1}$ at points of the curve of centers. Thus,

$$\mathbf{e}_3^* \equiv \frac{\boldsymbol{\mu}_3}{|\boldsymbol{\mu}_3|}, \qquad \mathbf{e}_1^* \equiv \frac{\boldsymbol{\mu}_1 - (\boldsymbol{\mu}_1 \cdot \mathbf{e}_3^*)\mathbf{e}_3^*}{|\boldsymbol{\mu}_1 - (\boldsymbol{\mu}_1 \cdot \mathbf{e}_3^*)\mathbf{e}_3^*|}, \qquad \mathbf{e}_2^* \equiv \mathbf{e}_3^* \wedge \mathbf{e}_1^*.$$

Derivatives of these define the curvatures κ_α^* and the torsion κ_3^* by

$$\frac{d\mathbf{e}_i^*}{dY} = \boldsymbol{\kappa}^* \wedge \mathbf{e}_i^* = e_{ijk}\kappa_j^*(Y)\mathbf{e}_k^*, \qquad \kappa_j^* = \boldsymbol{\kappa}^* \cdot \mathbf{e}_j, \tag{5.2}$$

while the corresponding stretch is $A^*(Y) = |\boldsymbol{\mu}_3(Y)|$. (These functions $\kappa_i^*(Y)$, $A^*(Y)$ provide appropriate functions for the variational problem (4.1).)

For each set of values $(a, \bar{\kappa}, b)$, the functions $\kappa_i^*(Y)$, $A^*(Y)$ and the components $M_i = \mathbf{M} \cdot \mathbf{e}_i^*$, $F_i = \mathbf{F} \cdot \mathbf{e}_i^*$ of the resultants are periodic functions of $\theta \equiv 2\pi Y/b$. By analogy with Kirchhoff theory for slender rods (where the components M_i are linearly related to the κ_i *via* Saint-Venant's semi-inverse solutions and where F_3 is proportional to $A - 1$, see [4]), we propose to relate M_i and F implicitly to κ_i and A by expressing all of these quantities in terms of the four parameters a, $\bar{\kappa}$, b, θ. Additionally we evaluate the momenta $\mathbf{q}$ and $\boldsymbol{\pi}$ defined in (3.19) as though the configuration was

$$\mathbf{x} = \mathbf{r}(Y, t) + u_i^*\mathbf{e}_i(Y, t),$$

where

$$u_i^*(Y_\alpha, a, \bar{\kappa}, b, \theta) \equiv \{w_k^*(Y_\alpha, Y, a, \bar{\kappa}, b) - w_k^*(0, Y, a, \bar{\kappa}, b)\}\mathbf{j}_k \cdot \mathbf{e}_i^*$$

gives the four parameter set of distortions of cross sections occurring in canonical deformations. Then, in the momentum equations (3.2) and (3.3) we write

$$\mathbf{M} = M_i(a, \bar{\kappa}, b, \theta)\mathbf{e}_i, \qquad \mathbf{F} \cdot \mathbf{e}_3 = F_3(a, \bar{\kappa}, b, \theta), \tag{5.3}$$

$$\mathbf{q} = m\mathbf{V}(Y, t) + m\frac{\partial}{\partial t}\{\mathbf{e}_i g_i(a, \bar{\kappa}, b, \theta)\}, \tag{5.4}$$

$$\boldsymbol{\pi} = mg_j(a, \bar{\kappa}, b, \theta)\mathbf{e}_j \wedge \mathbf{V} + I_{jk}(a, \bar{\kappa}, b, \theta)\mathbf{e}_j \wedge \frac{\partial \mathbf{e}_k}{\partial t} + e_{ijk}\mathbf{e}_i \iint_{\mathscr{D}} \bar{\rho} u_j^* \frac{\partial u_k^*}{\partial t}\, dS, \tag{5.5}$$

where

$$\mathbf{V} = \frac{\partial \mathbf{r}}{\partial t}, \qquad mg_i(a, \bar{\kappa}, b, \theta) \equiv \iint_{\mathscr{D}} \bar{\rho} u_i^*\, dS, \qquad m = \iint_{\mathscr{D}} \bar{\rho}\, dS$$

$$I_{jk}(a, \bar{\kappa}, b, \theta) = \iint_{\mathscr{D}} \bar{\rho} u_j^* u_k^*\, dS = I_{kj}.$$

Together with the kinematic relations

$$\frac{\partial \mathbf{e}_i}{\partial Y} = \kappa_j(a, \bar{\kappa}, b, \theta)\mathbf{e}_j \wedge \mathbf{e}_i, \qquad \frac{\partial \mathbf{r}}{\partial Y} = A(a, \bar{\kappa}, b, \theta)\mathbf{e}_3, \tag{5.6}$$

these form a complete set of equations governing $a, \bar{\kappa}, b, \theta, \mathbf{r}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, F_1$, and F_2.

The system (3.2), (3.3), (5.3)–(5.6) defines a special "constrained theory" of rods [14, see p. 647], one that is based on cross-sectional distortions which make the traction vanish on the lateral boundary $\partial\mathscr{D}$, which take into account the material composition and natural (twisted) shape of the rod and which are consistent with the field equations in certain (static) deformations. It also bears some relationship to a "coarse theory" derived from a fine theory (three-dimensional elastodynamics) by the process described by Muncaster [15]. The present coarse theory concerns the evolution of the class of "coarse states" $p \equiv \{\kappa_\alpha(Y, t),\ \tau(Y, t),\ A(Y, t),\ \mathbf{r}(Y, t),\ \mathbf{e}_i(Y, t)\}$. For general motions of the rod, Eqs. (2.6)–(2.10) provide the mapping which relates p to the full deformation given by $\mathbf{x}(Y, t)$. Muncaster outlines a general (but complicated) procedure characterizing the "constitutive" terms required in the equations for the evolution of the states p (here, the rod equations (3.2) and (3.3)). As he states, it is not usually tractable to determine all fine details (here $u_i(Y_\alpha, Y, t)$) corresponding to general states p. To derive a "closed" system of evolution equations, it is necessary to use approximations or asymptotic methods. Examples suggest that the first approximation to the fine details is given by treating certain parameters in p as constants. This is exactly the case in the present theory, where the cross-sectional configurations corresponding to κ_α, τ, and A are taken as $u_i^*(Y_\alpha, a, \bar{\kappa}, b, \theta)$ with the parameters $a, \bar{\kappa}, b, \theta$ arising in periodic configurations chosen to give the correct local values of κ_α, τ, and A. Thus, the constitutive response is chosen to be an *exact* consequence of three-dimensional elasticity in a small class of (static) deformations.

6. Rods having axial symmetry. For rods and tubes which are axially symmetric in their straight reference configuration the periodic configurations are relatively simple to categorize. In this case, the full four-parameter family of canonical deformations can be represented in terms of stationary values of a family of "cross-sectional energies".

The natural material coordinate system is the Lagrangian system $\{X_K\}$, with $\mathscr{D}$ being the circular region $X_\alpha X_\alpha < R^2$ (or an annulus $r^2 < X_\alpha X_\alpha < R^2$). Axial symmetry implies that the strain energy function (3.5) has the invariance property $W(\partial x_j/\partial X_K, X_K) = W(\partial x_j/\partial \overline{X}_L, \overline{X}_L)$ for all coordinate systems

$$\overline{X}_L = H_{LK}X_K, \qquad H_{LR}H_{LK} = \delta_{RK}, \qquad \overline{X}_3 = X_3$$

obtained from $\{X_K\}$ by a rotation about the $\mathbf{I}_3$ axis. Thus, for *any* value of b, any deformation given parametrically by

$$\mathbf{x}(\mathbf{X}) = \mathbf{x}_0 + \dot{w}_i(Y_\beta)\mathbf{j}_i(Y) + aY\mathbf{j}_3, \tag{6.1}$$

$$Y_1 = X_1\cos\theta - X_2\sin\theta, \qquad Y_2 = X_1\sin\theta + X_2\cos\theta, \qquad \theta = 2\pi Y/b, \tag{6.2}$$

$$\mathbf{j}_i'(Y) = \bar{\kappa}\mathbf{j}_3 \wedge \mathbf{j}_i, \qquad Y \equiv X_3$$

deforms each material helix Y_β = constant into a helix of period $2\pi/\bar{\kappa}$. With respect to the natural material coordinates $\{X_K\}$ it must be viewed as a "b-periodic deformation," yet in terms of the appropriate coordinates $\{Y_J\}$ it is simply a helical deformation. The cylindrical outer surface deforms into a helicoid and all cross sections have congruent distortions.

Since (4.9) simplifies to

$$\bar{p}_{i\alpha} = w_{i,\alpha}, \qquad \bar{p}_{i3} = a\delta_{i3} + \bar{\kappa}e_{i3l}w_l, \tag{6.3}$$

making $\bar{p}_{jR}$ independent of Y, the corresponding stationary values of the integral in (4.10) are given by stationary values $E^*(a, \bar{\kappa}, b)$ of the *cross-sectional energy*

$$\iint_{\mathscr{D}} \overline{W}(\bar{p}_{jR}, Y_\alpha)\, dS.$$

It follows immediately from (4.17)–(4.19) that at each cross section

$$\mathbf{F} = \frac{\partial E^*}{\partial a}\mathbf{J}_3, \qquad \hat{\mathbf{M}} = \frac{\partial E^*}{\partial \bar{\kappa}}\mathbf{J}_3. \tag{6.4}$$

As in (4.14), we choose the corresponding minimizers

$$w_i^* = w_i^*(Y_\alpha, a, \bar{\kappa}, b) \quad \text{to have} \quad w_2^* = w_3^* = 0, \quad w_1^* \geqslant 0 \quad \text{at } Y_\alpha = 0.$$

The curve of centers $\mathbf{x} = \mathbf{x}_0 + w_1^*(0, a, \bar{\kappa}, b)\mathbf{j}_1(Y) + aY\mathbf{j}_3$ is a helix of radius $w^* \equiv w_1^*(0, a, \bar{\kappa}, b)$, but the functions w_i^* are still determined only to within a transformation $Y_\alpha \to H_{\alpha\beta}Y_\beta$, $H_{\alpha\beta}H_{\beta\gamma} = \delta_{\beta\gamma}$, which corresponds to a change in the origin of θ, or Y. Except in the case $w_1^*(0, a, \bar{\kappa}, b) = 0$ of pure extension and torsion, this arbitrariness is removed by selecting $Y_1 = R$, $Y_2 = 0$ as the locus of maximum $w_\alpha^* w_\alpha^*$, so that the material curve $Y_1 = R$, $Y_2 = 0$ is the helix of maximum radius.

A reference triad $\{\mathbf{e}_i^*(Y)\}$ can be defined in terms of derivatives of the configuration (6.1) with respect to X_3 and X_1, in a manner similar to that in Sec. 5. However, this is not the simplest choice. The unit tangent to the curve of centers is

$$\mathbf{e}_3^*(Y) = \mathbf{j}_2(Y)\cos\beta + \mathbf{j}_3\sin\beta,$$

where the deformed helical angle β satisfies

$$\sin\beta = a/A, \qquad \cos\beta = \bar{\kappa}w^*/A, \quad A \equiv \left\{a^2 + (\bar{\kappa}w^*)^2\right\}^{1/2}.$$

For each 2π increase in θ the material element $\partial\mathbf{x}/\partial X_1$ at $X_\alpha = 0$ advances one turn around the curve of centers relative to the orthogonal triad $\mathbf{e}_3^*$, $\mathbf{j}_1$, and $\mathbf{e}_3^* \wedge \mathbf{j}_1$ (which itself rotates uniformly about the direction $\mathbf{j}_3$). However, $\partial\mathbf{x}/\partial X_1$ does not rotate uniformly with θ. A simpler choice of material triad contains the vectors $\mathbf{e}_3^*$ and $\mathbf{j}_1\cos\theta + \mathbf{e}_3^* \wedge \mathbf{j}_1\sin\theta$. Thus, in terms of the triad of vectors

$$\begin{aligned}
\mathbf{e}_1^* &= \mathbf{j}_1(Y)\cos\theta + \{\mathbf{j}_2(Y)\sin\beta - \mathbf{j}_3(Y)\cos\beta\}\sin\theta,\\
\mathbf{e}_2^* &= -\mathbf{j}_1(Y)\sin\theta + \{\mathbf{j}_2(Y)\sin\beta - \mathbf{j}_3(Y)\cos\beta\}\cos\theta,\\
\mathbf{e}_3^* &= \mathbf{j}_2(Y)\cos\beta + \mathbf{j}_3(Y)\sin\beta,
\end{aligned}$$

the curvatures $\kappa_\alpha = \bar{\boldsymbol{\kappa}} \cdot \mathbf{e}_\alpha^*$ and torsion $\tau = \bar{\boldsymbol{\kappa}} \cdot \mathbf{e}_3^*$ have the forms

$$\kappa_1 = -w^*\bar{\kappa}^2A^{-1}\sin\theta, \qquad \kappa_2 = -w^*\bar{\kappa}^2A^{-1}\cos\theta, \qquad \tau = a\bar{\kappa}A^{-1} = \tau(a,\bar{\kappa},b), \tag{6.5}$$

and the canonical distortions are

$$u_i^* = \left\{w_l^*(Y_\beta, a, \bar{\kappa}, b)\mathbf{j}_l - w^*\mathbf{j}_i\right\} \cdot \mathbf{e}_i^*, \tag{6.6}$$

which are b-periodic functions of Y. The corresponding stress- and couple-resultants

$$\mathbf{F} = \frac{\partial E^*}{\partial a}\mathbf{j}_3, \qquad \mathbf{M} = \frac{\partial E^*}{\partial\bar{\kappa}}\mathbf{j}_3 + w^*\frac{\partial E^*}{\partial a}\mathbf{j}_2$$

have components $M_i = \mathbf{M} \cdot \mathbf{e}_i^*$, $F_3 = \mathbf{F} \cdot \mathbf{e}_3^*$ which may be expressed as

$$\begin{aligned}
M_\alpha &= B(a,\bar{\kappa},b)\kappa_\alpha, \qquad M_3 = aA^{-1}\left(\frac{\partial E^*}{\partial\bar{\kappa}} + \frac{\bar{\kappa}w^{*2}}{a}\frac{\partial E^*}{\partial a}\right),\\
F_3 &= aA^{-1}\frac{\partial E^*}{\partial a}, \qquad B(a,\bar{\kappa},b) = \frac{1}{\bar{\kappa}^2}\left(\bar{\kappa}\frac{\partial E^*}{\partial\bar{\kappa}} - a\frac{\partial E^*}{\partial a}\right).
\end{aligned} \tag{6.7}$$

The constitutive laws (6.7) are derived from stationary values of the cross-sectional energy, defined over a single cross section and depending on the three parameters $(a, \bar{\kappa}, b)$. As might be anticipated, the "bending rigidity" $B(a, \bar{\kappa}, b)$ like the torque $M_3(a, \bar{\kappa}, b)$ and tension $F_3(a, \bar{\kappa}, b)$ is independent of the ratio κ_2/κ_1. Equations (6.7) give a specific, simplified form for (5.3). Equations (5.4) and (5.5) likewise have a simplified form. For example, the deformed location of the mass center of the cross section is computed as

$$mg_i\mathbf{e}_i = \iint_{\mathcal{D}} \bar{\rho}\left(w_l^*\mathbf{j}_l - w^*\mathbf{j}_1\right) dS \equiv mG_k(a,\bar{\kappa},b)\mathbf{j}_k,$$

so that the dependence on θ takes a specific trigonometric form. In this, as in other quantities in (5.5), θ may be eliminated in favor of κ_1, κ_2, a, $\bar{\kappa}$, and b, where $\kappa_1^2 + \kappa_2^2 = (w^*\bar{\kappa}/A)^2$. This yields a four-parameter set of constitutive laws and associated momenta appropriate for solution of (3.2) and (3.3) for naturally straight rods or tubes having axial symmetry.

We conclude by remarking that the canonical deformations illustrate the ambiguity of twist [8] which may arise in analysis of the stationary values of (4.1). For given $(a, \bar{\kappa}, b)$ there may be a strictly helical deformation in which cross sections at Y_0 and $Y_0 + b$ are congruent, but differ only by a displacement and a rotation through an angle $\bar{\kappa} b$. This deformation should be associated with $b = 0$, since the "periodicity" may be arbitrarily small. A truly "b-periodic" configuration may be distinguished from the helical deformation by the fact that the material curve $X_1 = R$, $X_2 = 0$, $Y \in [x_0, x_0 + b]$ links the surface spanned by the curve of centers $\mathbf{x} = \mathbf{r}(Y)$ and the axis $\mathbf{x} = \mathbf{x}_0 + aY\mathbf{j}_3$ exactly once in the former configuration, but zero times in the latter.

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