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A NOTE ON THE STOKES' PARADOX*

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Abstract. A particular solution of the Stokes' flow equations is presented which shows a nonuniformity of limits between the near and far flow fields which relates to the Stokes' paradox.

1. Introduction. The Stokes' paradox represents the nonexistence of bounded solutions to the Stokes' equations for the two-dimensional flow past a finite body; the velocities are found to grow logarithmically with the distance from the body. The basic resolution to the paradox for such flows was presented by Oseen (1910), though it wasn't until the singular perturbation analyses of Kaplun and Lagerstrom (1957), and Proudman and Pearson (1957) that the nature of the paradox was fully understood. The solution of the Stokes' equations is an inner solution in the neighborhood of the body, and the uniform stream at infinity can only be recovered from the solution of the Oseen equation, which includes a convective term. The singular solution, called a stokeslet, with velocities which grow as the logarithm of the radial distance, then forms the inner boundary condition for an Oseen flow.

In this brief note, an example of a two-dimensional Stokes flow is presented where a uniform stream and a stokeslet are different, nonuniform limits for the overlap region between the near and far flow fields. Although there is an element of artificiality in the particular situation, the ability of the biharmonic equation of Stokes flow to display this feature is of definite mathematical interest. The equivalent problem in three dimensions does not possess this nonuniformity.

2. Two-dimensional model. The specific model considered represents the flow between two concentric circular cylinders with radii a and λa ($\lambda > 1$). If ar is the radial distance from the center, and θ is the angular measure, then the radial and angular velocities can be written as $U_0 u(r, \theta)$, $U_0 v(r, \theta)$ respectively, where U_0 is the velocity scale. When the

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stream function $\psi(r,\theta)$ is defined by $u = r^{-1}\psi_\theta$, $v = -\psi_r$, then the Stokes' equation is just

$$\nabla^4\psi = 0. \tag{1}$$

We now set the boundary conditions:

$$u = \sin\theta,\ v = 0 \quad \text{on } r = 1; \qquad u = v = 0 \quad \text{on } r = \lambda. \tag{2}$$

This indicates a simple outflow and inflow from the surface of the inner cylinder, with no net increase for the mass flux. The solution to this problem is straightforward, being given by $\psi(r,\theta) = \phi(r)\cos\theta$, where ϕ is the sum of terms proportional to r^3, $r\ln r$, r, r^{-1}. Satisfying the boundary conditions implies

$$\psi = -\frac{\cos\theta}{4(\lambda^2-1)\{(\lambda^2+1)\ln\lambda-(\lambda^2-1)\}}\Big[(\lambda^2-1+2\ln\lambda)r^3-2(\lambda^2-1)(\lambda^2+3)r\ln r$$
$$+\{2(\lambda^4-3)\ln\lambda-(\lambda^2-1)(\lambda^2-3)\}r+\{2\lambda^4\ln\lambda-3(\lambda^2-1)\}r^{-1}\Big]. \tag{3}$$

Now, when $\lambda \gg 1$ and $r = O(1)$, the expression (3) is given by

$$\psi = -\frac{1}{2}(r+r^{-1})\cos\theta + O\{(\ln\lambda)^{-1}\}; \tag{4}$$

it is noted that this could have been found directly as the solution to (1) subject to the conditions (2) on $r = 1$ alone through excluding the terms r^3 and $r\ln r$ which have the fastest growth as $r \to \infty$. Hence, if there is no outer cylinder, then the effect of this inflow and outflow from the surface of the cylinder $r = 1$ is to produce a uniform stream with magnitude $\frac{1}{2}U_0$ at infinity in the direction corresponding to $\theta = \frac{1}{2}\pi$. A similar phenomenon was discovered by Jeffery (1922) for the flow due to two equal counter-rotating circular cylinders, and perhaps (4) represents the simplest example of what has come to be known as Jeffery's paradox (cf. Dorrepaal, O'Neill, and Ranger (1984)). When the outer cylinder is present, but has a large radius, then (4) represents the dominant part of the solution where $r = O(1)$.

Next, we write $r = \lambda\rho$, and then take $\lambda \gg 1$ with $\rho = O(1)$; the approximation to (3) is now given by

$$\psi = -\frac{\lambda}{4\ln\lambda}\Big[(\rho^3-\rho-2\rho\ln\rho)\cos\theta + O\{(\ln\lambda)^{-1}\}\Big]. \tag{5}$$

Here we note that the expression (5) could have been obtained (up to a multiplicative constant) by solving (1) with just the boundary conditions (2) on $r = \lambda$, which also has the weakest singularity as $\rho \to 0$. When, in fact, we let $\rho \to 0$, it follows that $\psi \simeq \lambda\rho\ln\rho\cos\theta/(2\ln\lambda)$, which represents a stokeslet at the origin, with velocities proportional to $\ln\rho$. Under normal circumstances, it would be expected, when λ is large, that the separate limits for ψ, with $r \to \infty$ and $\rho \to 0$, are the same, and it is the nonuniformity of the limits here which is of interest. Further, as $r \to \infty$ the stream function is that for a uniform stream, whereas as $\rho \to 0$ the stream function is that for a stokeslet, which is often taken to represent the effect of a uniform stream past a finite body in two dimensions; the resolution is completely within the Stokes' equation.

It is observed that the vorticity derived from (5) is $\omega \cong -(2\rho-\rho^{-1})(\lambda\ln\lambda)^{-1}\cos\theta$, which is zero for all $\rho = 2^{-1/2} \cong 0.707$.

If the boundary conditions (2) are changed to $u = \operatorname{sgn}\theta$ $(-\pi < \theta \leqslant \pi)$, $v = 0$ on $r = 1$, $u = v = 0$ on $r = \lambda$, $\lambda \gg 1$, then a uniform stream is still present for large r, now with velocity $2U_0\pi^{-1}$, and the stokeslet follows as $\rho \to 0$. In fact, the situation is similar for each odd function $u(1,\theta)$ which has a nonzero term in $\sin\theta$ for its Fourier series expansion, and satisfies $[u(1,\theta)]_{-\pi}^{\pi} = 0$.

When the analogous three-dimensional axisymmetric problem, with a balanced inflow and outflow out of the sphere $r = 1$, and no slip on $r = \lambda$, is considered, the nonuniformity is absent. The two limits $r \to \infty$ and $\rho \to 0$ are equivalent and represent the stream function for a (three-dimensional) stokeslet.

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