QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLV, NUMBER 3
OCTOBER 1987, PAGES 503-514

A LINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATION FOR VISCOELASTIC RODS AND PLATES*

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Abstract. It is proved that the resolvent kernel of a certain Volterra integrodifferential equation in Hilbert space is absolutely integrable on $(0, \infty)$. Weaker assumptions on the convolution kernel appearing in the integral term are used than in existing results. The equation arises in the linear theory of isotropic viscoelastic rods and plates.

1. Introduction. We study the equation

$$y'(t) = -A * Ly(t) + g(t), \quad y(0) = y_0,\ t \geqslant 0 \left(' = \frac{d}{dt}\right), \tag{1.1}$$

in a Hilbert space H, where y_0 and $g(t)$ belong to H, A: $\mathbf{R}^+ \to \mathbf{R}$ is locally absolutely continuous, L denotes a self-adjoint linear operator defined on a dense domain D of H. We assume the spectrum of L is contained in $[1, \infty)$ and $*$ denotes the convolution

$$h_1 * h_2(t) = \int_0^t h_1(t-s) h_2(s)\, ds.$$

Let $\{E_\lambda\}$ be the spectral family corresponding to L. Define

$$U(t) = \int_1^\infty u(t, \lambda)\, dE_\lambda$$

where $u(t, \lambda)$ is the solution of

$$u'(t) = -\lambda A * u(t), \quad u(0) = 1,\ t \geqslant 0. \tag{1.2}$$

Existence, uniqueness, and representation results for (1.1) work out just as in [6]. In particular, the conclusions of Theorem 1.1 below imply that

$$U'(t)y = \frac{d}{dt}[U(t)y], \text{ if } L^{-1/2}y \in D; \tag{1.3}$$

* Received July 8, 1986.

moreover, if $y_0 \in D$, $g\colon \mathbf{R}^+ \to H$ is continuous with $g(t) \in D$ for all t, and $Lg\colon \mathbf{R}^+ \to H$ is locally Bochner-integrable, then the unique solution of (1.1) is given by

$$y(t) = U(t)y_0 + U * g(t). \tag{1.4}$$

The results (1.13), (1.18), and (1.19) in Theorem 1.1 imply, respectively,

$$\|U(t)\| \leqslant 1 \quad (t \geqslant 0), \tag{1.5}$$

$$\lim_{t \to \infty} \|tU(t)\| = 0, \tag{1.6}$$

$$\int_0^\infty \|U(t)\|\, dt < \infty. \tag{1.7}$$

Under various assumptions on g, (1.4)–(1.7) can be used to study the asymptotic behavior of $y(t)$ as $t \to \infty$.

Throughout this paper, the Fourier transform $\hat{h}$ is defined for a function h such that $h(t)e^{-\sigma t} \in L^1(\mathbf{R}^+)$ for all $\sigma > 0$ by the formula

$$\hat{h}(\tau) = \int_0^\infty e^{-i\tau t} h(t)\, dt \quad (\operatorname{Im}\tau < 0),$$

$$\hat{h}(\tau_0) = \lim_{\tau \to \tau_0,\, \operatorname{im}\tau < 0} \hat{h}(\tau) \quad (\tau_0 \in \mathbf{R}),$$

whenever the limit exists.

The assumptions on A are as follows:

$$\hat{A}(\tau) = \hat{a}(\tau) f(\tau) \quad (\operatorname{Im}\tau < 0), \tag{1.8}$$

where

$$f(\tau) = F(\hat{b}(\tau)/\hat{a}(\tau)), \qquad F(w) = \frac{w + m}{pw + q}, \tag{1.9}$$

$$m,\ p, \text{ and } q \text{ are nonnegative}, \quad p > 0,\ q > mp, \tag{1.10}$$

and

$$a(t) \text{ and } b(t) \text{ are continuous, nonnegative, nonincreasing, convex, and not constant on } \mathbf{R}^+, \text{ with } a(0) + b(0) < \infty. \tag{1.11}$$

In [6], the stronger assumption $-a'(0) - b'(0) < \infty$ is also used.

The equation for a dynamic problem in linear viscoelasticity [1] can be obtained by a "correspondence principle" from the corresponding problem in the purely elastic case. Formal differentiation of (1.1), with $L = \Delta^2$, $y(t) = y(\cdot, t)$, followed by an application of the Fourier transform (with $m = p = 1/2$, $q = 1$ in (1.9)) yields the transformed equation for transverse vibrations in a viscoelastic plate (see [6]). Similarly, the equations for waves in a rod come from (1.1), (1.8), (1.9) with $m = 0$, $p = 2$, $q = 1$; for longitudinal waves, $L = -\partial^2/\partial x^2$, while for bending waves, $L = \partial^2/\partial x^4$. In all cases we must take self-adjoint boundary conditions. There are indications that for certain viscoelastic materials the assumption that $A(0)$ or $A'(0)$ is finite may not be valid. (See [8] and references in [8]). Recently authors have studied (1.1) and equations related to (1.1) assuming that the convolution kernel or its derivative become infinite at the origin. (See [2], [3], [7], [8], [9], [11], and [12]).

Our first result is:

THEOREM 1.1. Suppose (1.10) and (1.11) hold.

(i) There exist $B \in L^2(\mathbf{R}^+)$ such that the function

$$A(t) \equiv f(\infty)a(t) + \int_0^t B(s)\,ds$$

satisfies (1.8).

(ii) For $\lambda > 0$, $u(t, \lambda)$ satisfies

$$\hat{u}(\tau, \lambda) = \left[i\tau + \lambda \hat{A}(\tau)\right]^{-1} \quad (\operatorname{Im} \tau < 0), \tag{1.12}$$

$$|u(t, \lambda)| \leqslant 1 \quad (t \geqslant 0). \tag{1.13}$$

(iii) If in addition,

$$-a' \text{ and } -b' \text{ are convex on } \mathbf{R}^+, \tag{1.14}$$

and

$$\limsup_{x \downarrow 0} \frac{x^5\left[C(x) + C_b(x)\right]}{\left[B(x) + B_b(x)\right]^3} < \infty, \tag{1.15}$$

where

$$C(x) = \int_0^x -a'(s)\,ds, \qquad C_b(x) = \int_0^x -b'(s)\,ds, \tag{1.16}$$

$$B(x) = \int_0^x -sa'(s)\,ds, \qquad B_b(x) = \int_0^x -sb'(s)\,ds, \tag{1.17}$$

then

$$\lim_{t \to \infty} \sup_{1 \leqslant \lambda < \infty} |tu(t, \lambda)| = 0, \tag{1.18}$$

$$\int_0^\infty \sup_{1 \leqslant \lambda < \infty} |u(t, \lambda)|\,dt < \infty. \tag{1.19}$$

In [6], Hannsgen proved the same result with the condition $a(0) + b(0) < \infty$ in (1.11) replaced by $-a'(0) - b'(0) < \infty$. Note that the condition $-a'(0) - b'(0) < \infty$ implies (1.15). The proof of Theorem 1.1 follows the proof of Hannsgen's result. The only difference is that we will use the estimates (2.1), (2.2), and (2.8), below, instead of the relations

$$\hat{a}(\tau) = -i\tau^{-1}a(0) + O(\tau^{-2}), \qquad \hat{b}(\tau) = -i\tau^{-1}b(0) + O(\tau^{-2}),$$
$$\hat{a}(\tau) = -i\tau^{-1}a(0) - a'(0)\tau^{-2} + o(\tau^{-2}),$$
$$\hat{b}(\tau) = -i\tau^{-1}b(0) - b'(0)\tau^{-2} + o(\tau^{-2}) \quad (\tau \to \infty,\ \tau \in S),$$

where $S \equiv \{\tau \in \mathbf{C}\colon \operatorname{Im} \tau < 0,\ \tau \neq 0\}$. To keep the paper more self-contained, we will include the entire proof. The part of the proof that differs from that in [6] will be found in the estimates from (2.26) to the end.

Remarks. (a) Regarding (i), it will be seen below in (2.2) that $f(\infty)$ exists and equals $F(b(0)/a(0))$.

(b) The assumption (1.15) is quite mild. If $-(a(t)+b'(t))$ has a singularity at $t=0$ like $t^{-\alpha}$, $0<\alpha<1$, $-\log t$, $\log(-\log t)$, or $t^{-1}(-\log t)^{-q}$, $q>1$, then an easy calculation shows that (1.15) holds. As already mentioned, if $-a'(0)-b'(0)<\infty$ then (1.15) holds.

(c) In Sec. 3 we give an example of a piecewise linear function $-a'$ (with any b) that satisfies (1.11) and (1.14) but not (1.15). I do not know if (1.19) holds for this example.

The condition $a(0)+b(0)<\infty$ is assumed in Theorem 1.1. The following result allows $b(0)=\infty$.

THEOREM 1.2. Assume that (1.10) holds. Let $b(t)=t^{-\beta}$, $0<\beta<1$, and assume that $a(t)$ satisfies (1.11) and (1.14). Then (1.13), (1.18), and (1.19) must hold.

Note. Our proof of these two theorems does not extend to $a(t)=t^{-\alpha}$, $0<\alpha<1$, even if $-b'(0)<\infty$.

2. Proof of Theorem 1.1. We first recall some consequences of (1.11). By [5], the functions

$$\hat{a}(\tau)\equiv\phi(\tau)-i\tau\theta(\tau)$$

and

$$\hat{b}(\tau)\equiv\phi_1(\tau)-i\tau\theta_1(\tau)$$

are analytic in $\{\operatorname{Im}\tau<0\}$ and continuous in S. Moreover, if $\operatorname{Im}\tau<0$ and $\operatorname{Re}\tau>0$, $\hat{a}(\tau)$ and $\hat{b}(\tau)$ lie in $\{-\pi/2<\arg w<0\}$: if (1.14) also holds, this conclusion remains true when $\operatorname{Im}\tau=0$, $\tau>0$. In this paper, $-\pi<\arg w\leqslant\pi$ ($w\in\mathbf{C}$).

Integration by parts and the Riemann–Lebesgue theorem show that

$$\hat{a}(\tau)=-i\tau^{-1}a(0)+o(\tau^{-1}),\qquad \hat{b}(\tau^{-1})=-i\tau^{-1}b(0)+o(\tau^{-1})\quad(\tau\to\infty). \tag{2.1}$$

As a consequence, it follows that

$$\begin{aligned}&f(\tau)=f(\infty)+o(1),\qquad \theta_1(\tau)=\tau^{-2}b(0)+o(\tau^{-2}),\\&\theta(\tau)=\tau^{-2}a(0)+o(\tau^{-2})\quad(\tau\to\infty),\end{aligned} \tag{2.2}$$

where $f(\infty)=F(b(0)/a(0))$.

From [12], we know that $\hat{a}(\tau)$ and $\hat{b}(\tau)$ are differentiable for $\tau>0$ and

$$2^{-3/2}\int_0^{1/\tau}a(t)\,dt\leqslant|\hat{a}(\tau)|\leqslant 4\int_0^{1/\tau}a(t)\,dt, \tag{2.3}$$

$$|\hat{a}'(\tau)|\leqslant 40\int_0^{1/\tau}ta(t)\,dt\quad(\tau>0), \tag{2.4}$$

$$\int_0^1\left(\frac{\int_0^{1/\tau}ta(t)\,dt}{\left(\int_0^{1/\tau}a(t)\,dt\right)^2}\right)d\tau<\infty, \tag{2.5}$$

with similar estimates for b. By [5, Lemma 2.2],

$$\theta'(\tau)<0\quad(\tau>0). \tag{2.6}$$

If (1.14) also holds, [2, Lemma 5.1] shows that $\hat{a}$ and $\hat{b}$ belong to $C^2(0,\infty)$ with

$$|\hat{a}''(\tau)|\leqslant 6000\int_0^{1/\tau}t^2a(t)\,dt, \tag{2.7}$$

and [7, p. 236] shows that

$$\begin{aligned} &CB(1/\tau) \leqslant \phi(\tau) \leqslant KB(1/\tau), \quad \tau > 0, \\ &CB_b(1/\tau) \leqslant \phi_1(\tau) \leqslant KB_b(1/\tau), \quad \tau > 0, \end{aligned} \tag{2.8}$$

for some positive constants C and K where

$$B(x) \equiv \int_0^x -sa'(s)\,ds \text{ and } B_b(x) \equiv \int_0^x -sb'(s)\,ds.$$

From (2.3)–(2.5) and (2.7) it follows that

$$\int_0^1 \frac{|\tau\hat{a}''(\tau)| + |\hat{a}'(\tau)|}{|\hat{a}(\tau)|^2}\,d\tau < \infty. \tag{2.9}$$

The formulas for b which are analogous to (2.3)–(2.7) and (2.9) also hold.

The fractional linear transformation F maps $\{\operatorname{Re} w \geqslant 0\}$ onto the disk with diameter $[m/q, 1/p]$. Moreover,

$$0 < \arg F(w) < \arg w \quad (0 < \arg w < \pi/2), \tag{2.10}$$

$$F(\bar{w}) = \overline{F(w)} \quad (w \in \mathbf{C}).$$

It follows that $\arg \hat{a}(\tau)f(\tau)$ lies between $\arg \hat{a}(\tau)$ and $\arg \hat{b}(\tau)$, and strictly between them when $\arg \hat{a}(\tau) \neq \arg \hat{b}(\tau)$ ($\tau \in S$). In particular, if we write

$$\hat{A}(\tau) = \Phi(\tau) - i\tau\Theta(\tau),$$

then in $\{\operatorname{Im}\tau < 0\}$ (and in S when (1.17) holds) we have,

$$\Phi(\tau) > 0. \tag{2.11}$$

For the remainder of the paper, M denotes a constant whose value may change each time that it appears.

To prove (i), define

$$G(\tau) \equiv (f(\tau) - f(\infty))i\tau\hat{a}(\tau).$$

For $0 < \tau < 1$, we have

$$\begin{aligned} |f(\tau) - f(\infty)| &= \left| \frac{(q - mp)(a(0)\hat{b}(\tau) - b(0)\hat{a}(\tau))}{(pb(0) + qa(0))(p\hat{b}(\tau) + q\hat{a}(\tau))} \right| \\ &\leqslant M|a(0)\hat{b}(\tau) - b(0)\hat{a}(\tau)| = o(\tau^{-1}) \quad \text{as } \tau \to \infty, \end{aligned}$$

where we have used the Riemann–Lebesgue lemma. Together with (2.1), the paragraph preceding (2.1), and the properties of F, G belongs to the Hardy class H^2 in $\{\operatorname{Im}\tau < 0\}$. Thus $G = \hat{B}$ for some B in $L^2(\mathbf{R}^+)$ and

$$\frac{1}{i\tau}G(\tau) + f(\infty)\hat{a}(\tau) = f(\tau)\hat{a}(\tau).$$

Therefore (1.8) holds with A as in (i), as asserted.

The proof of (ii) is the same as in [6]. The formulas (2.14)–(2.23) from [6] are valid. The verification of these is the same as in [6] except that the formulas

$$\hat{a}(\tau) = -i\tau^{-1}a(0) - a'(0)\tau^{-2} + o(\tau^{-2}),$$
$$\hat{b}(\tau) = i\tau^{-1}b(0) - b'(0)\tau^{-2} + o(\tau^{-2})$$

do not hold. In their place we use (2.1), (2.2), and (2.8). We list these results in (2.12)–(2.21) below:

$$\lim_{\substack{\tau \to 0 \\ \tau \in S}} \frac{1}{\hat{A}(\tau)} \text{ exists and is real and nonnegative.} \tag{2.12}$$

For each $\lambda > 0$

$$\hat{u}(\tau, \lambda) \text{ is continuous in } \{\operatorname{Im} \tau \leqslant 0\}. \tag{2.13}$$

Also, $\hat{a}$, $\hat{b} \in C^2(0, \infty)$ when (1.14) holds; therefore $\hat{A} \in C^2(0, \infty)$. We have

$$\int_0^1 \frac{|\tau \hat{A}''(\tau)| + |\hat{A}'(\tau)|}{|\hat{A}(\tau)|^2} \, d\tau < \infty, \tag{2.14}$$

$$\frac{|\tau \hat{A}'(\tau)| + |\tau^2 \hat{A}''(\tau)|}{|\hat{A}(\tau)|} = O(1) \quad (\tau \to 0+), \tag{2.15}$$

$$\tau^{j+1}\hat{A}^{(j)}(\tau) = O(1) \quad (\tau \to \infty,\ j = 0,1,2). \tag{2.16}$$

Furthermore, $\hat{u} \in C^2(0, \infty)$ as a function of τ, and

$$\frac{\partial^j \hat{u}(\tau, \lambda)}{\partial \tau^j} = O(\tau^{-j-1}) \quad (\tau \to \infty,\ j = 0,1,2,\ \lambda > 0), \tag{2.17}$$

$$\hat{u}(\tau, \lambda) = O(\tau^{-1}) \quad (\tau \to \infty,\ \tau \in S,\ \lambda > 0), \tag{2.18}$$

$$u(\cdot, \lambda) \in L^1(\mathbf{R}^+) \quad (0 < \lambda < \infty). \tag{2.19}$$

With $D(\tau, \lambda) \equiv \hat{A}(\tau) + i\tau\lambda^{-1}$, choose $\rho > 0$ such that

$$|\hat{A}(\tau)| \geqslant 2\tau \quad (0 < \tau \leqslant \rho). \tag{2.20}$$

By (2.12), ρ exists.

Finally, we have the representation

$$\pi u(t, \lambda) = \operatorname{Im}\{\lambda^{-1}u_1(t) + i\lambda^{-2}u_2(t) + \lambda^{-3}u_3(t) + u_4(t, \lambda) + u_5(t, \lambda)\}, \tag{2.21}$$

where

$$u_1(t) = \frac{1}{t}\int_0^\rho e^{i\tau t} \frac{\hat{A}'(\tau)}{\hat{A}^2(\tau)} \, d\tau,$$

$$u_2(t) = \frac{1}{t}\int_0^\rho \frac{e^{i\tau t}}{\hat{A}^2(\tau)}\left[1 - \frac{2\hat{A}'(\tau)}{\hat{A}(\tau)}\right] d\tau,$$

$$u_3(t) = \frac{1}{t}\int_0^\rho e^{i\tau t} \frac{2\tau \, d\tau}{\hat{A}^3(\tau)},$$

$$u_4(t, \lambda) = \frac{-1}{\lambda^3 t}\int_0^\rho e^{i\tau t} \frac{\tau^2 D_\tau(\tau, \lambda)}{\hat{A}^3(\tau) D(\tau, \lambda)}\left[\frac{2}{\hat{A}(\tau)} + \frac{1}{D(\tau, \lambda)}\right] d\tau,$$

$$u_5(t, \lambda) = \frac{1}{t\lambda}\int_\rho^\infty e^{i\tau t} \frac{D_\tau(\tau, \lambda)}{D^2(\tau, \lambda)} \, d\tau.$$

Following [6] we show that

$$|u_4(t,\lambda)| \leqslant Mt^{-2} \quad (\lambda, t \geqslant 1), \tag{2.22}$$

$$|u_5(t,\lambda)| \leqslant Mt^{-2} \quad (\lambda, t \geqslant 1). \tag{2.23}$$

Then by (2.21), (2.19), and (1.13), (1.19) follows. Moreover, (1.18) is a consequence of (2.21), (2.22), (2.23), and the Riemann–Lebesgue theorem.

For (2.22), integrate by parts in the definition of u_4 (using (2.15)) and then use (2.15) and (2.20).

For (2.23), integrate u_5 by parts (using (2.16)) and then use (2.11) and (2.16) to see that

$$\frac{t^2\lambda}{L}\left|u_5\left(t, \frac{\lambda}{L}\right)\right| \leqslant M\left[1 + \int_\rho^\infty \frac{\tau^{-3}}{|D^2(\tau,\lambda/L)|} + \frac{\tau^{-4}+\lambda^{-2}}{|D^3(\tau,\lambda/L)|}\right] d\tau, \tag{2.24}$$

where L is defined in (2.26) below. Define $\omega_1 = \omega_1(\lambda) > 0$ by $\theta(\omega_1) = \lambda^{-1}$. By (2.6), ω_1 is unique when it exists. Now let $\omega(\lambda) = \omega_1(\lambda)$ when ω_1 exists and $\omega_1 \geqslant \rho$, and let $\omega(\lambda) = \rho$ otherwise. As shown in [5], ω is continuous and nondecreasing, and by [5, Eq. (6.8)], there exists $Q > 0$ such that

$$\frac{|\omega(\lambda) - \tau|(\omega(\lambda)+\tau)}{\tau^2\lambda} \leqslant Q\left|\frac{1}{\lambda} - \theta(\tau)\right| \quad (\tau \geqslant \rho,\ \lambda \geqslant 1). \tag{2.25}$$

To proceed we need a lower bound on $\operatorname{Im} D(\tau, \lambda/L) = -\tau\Theta(\tau) + \tau\lambda^{-1}$. Writing $\Theta(\tau)$ explicitly in terms of ϕ, ϕ_1, θ, and θ_1, a straightforward calculation shows that

$$\lim_{\tau\to\infty} \frac{\Theta(\tau)}{\theta(\tau)} = L \equiv \frac{pb(0)^2 + (mp+q)a(0)b(0) + mpa(0)^2}{(pb(0) + qa(0))^2}, \tag{2.26}$$

where we have used (2.2) and (2.8).

Define the function

$$L_1(\tau) \equiv \frac{p(\tau\theta_1(\tau))^2 + (mp+q)\tau^2\theta(\tau)\theta_1(\tau) + mp\tau^2\theta^2(\tau)}{(p\tau\theta_1(\tau) + q\tau\theta(\tau))^2}.$$

We then have

$$\left|\Theta(\tau) - \frac{L}{\lambda}\right| \geqslant L\left|\theta(\tau) - \frac{1}{\lambda}\right| - \theta(\tau)|L_1(\tau) - L| - |\Theta(\tau) - L_1(\tau)\theta(\tau)|. \tag{2.27}$$

We find a common denominator and use (2.2) and (2.8) to estimate $|\Theta(\tau) - \theta(\tau)L_1(\tau)|$. The result of this straightforward but tedious calculation is

$$|\Phi(\tau) - L_1(\tau)\theta(\tau)| \leqslant M\left(B_b^2(\tau^{-1}) + B^2(\tau^{-1})\right) \quad (\tau \geqslant \rho_1), \tag{2.28}$$

where ρ_1 is a positive constant.

A similar calculation shows that for some $\rho_2 > 0$,

$$|L_1(\tau) - L| \leqslant M\tau^2|b(0)\theta(\tau) - a(0)\theta_1(\tau)| \quad (\tau \geqslant \rho_2).$$

Integration by parts yields

$$\theta(\tau) = a(0)\tau^{-2} - \tau^{-2}\operatorname{Re}\widehat{(-a')}(\tau),$$

and

$$\theta_1(\tau) = b(0)\tau^{-2} - \tau^{-2}\operatorname{Re}\widehat{(-b')}(\tau) \quad (\tau > 0).$$

Therefore,

$$\begin{aligned} |L_1(\tau) - L| &\leqslant M\left|\operatorname{Re}\widehat{(-a')}(\tau) - \operatorname{Re}\widehat{(-b')}(\tau)\right| \\ &\leqslant M\left|\widehat{(-a')}(\tau)\right| + \left|\widehat{(-b')}(\tau)\right| \\ &\leqslant M\left(\int_0^{1/\tau} -a'(s)\,ds + \int_0^{1/\tau} -b'(s)\,ds\right) \quad (\tau \geqslant \rho_2), \end{aligned} \tag{2.29}$$

where the last inequality is a consequence of [12, Lemma 1] applied to $-a'$ and $-b'$. There exist positive constants ρ_3, M_1, M_2 such that

$$\tau^2 B_b^2(1/\tau) \leqslant M_1\int_0^{1/\tau} -b'(s)\,ds \quad \text{and} \quad \tau^2 B^2(1/\tau) \leqslant M_2\int_0^{1/\tau} -a'(s)\,ds \quad (\tau > \rho_3)$$

because $B(x) \leqslant x\int_0^x -a'(s)\,ds$ and $\int_0^x -a'(s)\,ds \to 0$ as $x \to 0$. (Similarly for B_b).

Therefore (2.27), (2.25), (2.29), (2.2), and (2.28) imply that there is a constant $M_1 > 0$ such that

$$Q\left|\Theta(\tau) - \frac{L}{\lambda}\right| \geqslant \frac{L|\omega - \tau|(\omega + \tau)}{\tau^2\lambda} - \frac{M_1}{\tau^2}\int_0^{1/\tau} -a'(s) - b'(s)\,ds \tag{2.30}$$

for $\tau \geqslant \rho_4 \equiv \max\{\rho_1, \rho_2, \rho_3\}$.

Next, we assume that ρ_5 is so large that

(i) $$\int_0^{1/\tau} -a'(s) - b'(s)\,ds \leqslant \frac{La(0)}{8M_1}, \quad \tau \geqslant \rho_5, \tag{2.31}$$

(ii) $$\frac{\omega^2}{2a(0)} \leqslant \lambda \leqslant \frac{2\omega^2}{a(0)} \quad \text{for } \omega/2 \geqslant \rho_5$$

(a consequence of (2.2) and the definition of ω),

(iii) $$\varepsilon(\lambda) \leqslant \omega/2 \quad \text{for } \omega/2 \geqslant \rho_5,$$

where $\varepsilon(\lambda) = \varepsilon$ is defined below in (2.33). Define $\rho_6 = \max\{\rho_4, \rho_5\}$. In the following, we also redefine ω to be $\max\{2\rho_6, \omega\}$ and denote the new function by the same letter, ω.

For $\rho_6 \leqslant \tau \leqslant \omega/2$, we use (2.30) and (2.31) to obtain

$$\begin{aligned} Q\left|\Theta(\tau) - \frac{L}{\lambda}\right| &\geqslant \frac{L}{\tau^2\lambda}\left[(\omega - \tau)(\omega + \tau) - \frac{\lambda M_1}{L}\int_0^{1/\tau} -a'(s) - b'(s)\,ds\right] \\ &\geqslant \frac{L}{\tau^2\lambda}\left[\omega^2 - \frac{\omega^2}{4} - \frac{2M_1\omega^2}{a(0)L}\left(\frac{La(0)}{8M_1}\right)\right] \\ &= \frac{L\omega^2}{2\tau^2\lambda}. \end{aligned}$$

Therefore,

$$\frac{1}{\lambda}\int_{\rho_6}^{\omega/2} \frac{\tau^{-3}}{|D(\tau,\lambda/L)|^2} + \frac{\tau^{-4}+\lambda^2}{|D(\tau,\lambda/L)|}\,d\tau \tag{2.32}$$

$$\leqslant \frac{M}{\lambda}\int_{\rho_6}^{\omega/2} \frac{\tau^{-3}}{(\omega^2/\lambda\tau^2)} + \frac{\tau^{-4}+\lambda^{-2}}{(\omega^2/\lambda\tau)^3}\,d\tau$$

$$\leqslant M\left(\frac{\lambda}{\omega^3} + \frac{\lambda^2}{\omega^5} + \frac{\omega^4}{\omega^6}\right) \leqslant M.$$

Define $\varepsilon = \varepsilon(\lambda)$ by the formula

$$\varepsilon(\lambda) = \frac{8M_1}{a(0)L}\omega\int_0^{2/\omega} -a'(s) - b'(s)\,ds. \tag{2.33}$$

Note that $\varepsilon(\lambda) \to \infty$ as $\lambda \to \infty$ because $-a'(0+) - b'(0+) = \infty$ and $\omega \to \infty$ as $\lambda \to \infty$.

For $\rho_6 \leqslant \omega/2 < \tau < \omega - \varepsilon$ and $\omega + \varepsilon < \tau$ we use (2.30), (2.31), and (2.33) to obtain the estimate

$$Q\left|\Theta(\tau) - \frac{L}{\lambda}\right| \geqslant \frac{L|\omega-\tau|}{\tau\lambda}\left[\frac{\omega+\tau}{\tau} - \frac{M_1\lambda\int_0^{1/\tau} -a'(s) - b'(s)\,ds}{\tau L|\omega-\tau|}\right]$$

$$\geqslant \frac{L|\omega-\tau|}{\tau\lambda}\left[1 - \frac{M_1 4\omega^2\int_0^{1/\tau} -a'(s) - b'(s)\,ds}{a(0)\omega L\varepsilon}\right]$$

$$= \frac{L|\omega-\tau|}{\tau\lambda}\left[1 - \frac{1}{2}\right] = \frac{L|\omega-\tau|}{2\tau\lambda}.$$

Therefore,

$$\frac{1}{\lambda}\left(\int_{\omega/2}^{\omega-\varepsilon} + \int_{\omega+\varepsilon}^{\infty}\right)\left(\frac{\tau^{-3}}{|D(\tau,\lambda/L)|^2} + \frac{\tau^{-4}+\lambda^{-2}}{|D(\tau,\lambda/L)|^3}\right)d\tau$$

$$\leqslant \frac{M}{\lambda}\int_{\omega/2}^{\infty} \frac{\tau^{-3}\lambda^2}{|\omega-\tau|^2} + \frac{(\tau^{-4}+\lambda^{-2})\lambda^3}{|\omega-\tau|^3}\,d\tau$$

$$\leqslant M\left(\frac{\lambda}{\omega^3}\int_{\rho_1}^{\infty} x^{-2}\,dx + \int_{\rho_1}^{\infty} x^{-3}\,dx\right) \tag{2.34}$$

$$\leqslant M,$$

where we have used (2.31).

For $\omega - \varepsilon < \tau < \omega + \varepsilon$ we need a lower bound for $\operatorname{Re} D(\tau,\lambda) = \Phi(\tau)$. By writing Φ out in terms of ϕ, ϕ_1, θ, and θ_1, and using (2.8) we obtain

$$\frac{1}{\Phi} = \frac{N}{D} \leqslant M\left(\frac{1}{\phi(\tau)} + \frac{1}{\phi_1(\tau)}\right) \leqslant M\left(\frac{1}{B(\tau^{-1})} + \frac{1}{B_b(\tau^{-1})}\right), \tag{2.35}$$

for $\rho \leqslant \tau$, where

$$N = p^2\phi_1^2 + 2pq\phi_1\phi + q\phi^2 + p^2(\tau\theta_1)^2 + pq\tau^2\theta\theta_1 + q^2(\tau\theta_1)^2,$$

and

$$D = p\phi_1^2\phi + (q + mp)\phi_1\phi^2 + mq\phi^3 + p\phi(\tau\theta_1)^2 + 2mp\phi\tau\theta\tau\theta_1 + mq\phi(\tau\theta)^2 + (q - mp)\phi_1(\tau\theta)^2.$$

Therefore, by (2.31), (2.33), and (1.15) we have the estimate

$$\begin{aligned}\frac{1}{\lambda}\int_{\omega-\varepsilon}^{\omega+\varepsilon} &\frac{\tau^{-3}}{|D(\tau,\lambda)|^2} + \frac{\tau^{-4}+\lambda^{-2}}{|D(\tau,\lambda)|^3}\,d\tau \\ &\leqslant \frac{1}{\lambda}\int_{\omega-\varepsilon}^{\omega+\varepsilon} \frac{\tau^{-3}}{\Phi(\tau)^2} + \frac{\tau^{-4}+\lambda^{-2}}{\Phi(\tau)^3}\,d\tau \\ &\leqslant \frac{M\varepsilon}{\lambda(B(2/\omega)+B_b(2/\omega))^2\omega^3} + \frac{M\varepsilon}{\lambda^3(B(2/\omega)+B_b(2/\omega))^3} \\ &\leqslant M\left(\frac{x^4}{(B(x)+B_b(x))^2} + \frac{x^5\int_0^x -a'(s)-b'(s)\,ds}{(B(x)+B_b(x))^3}\right) \leqslant M,\end{aligned} \tag{2.36}$$

where $x = 2/\omega$.

For $\rho < \tau < \rho_6$, we use (2.35) to obtain

$$\frac{1}{\lambda}\int_{\rho}^{\rho_1} \frac{\tau^{-3}}{|D(\tau,\lambda/L)|^2} + \frac{\tau^{-4}+\lambda^{-2}}{|D(\tau,\lambda/L)|^3}\,d\tau \leqslant \frac{1}{\lambda}\int_{\rho}^{\rho_1} \frac{\tau^{-3}}{\Phi(\tau)^2} + \frac{\tau^{-4}+\lambda^{-2}}{\Phi(\tau)^3}\,d\tau \leqslant M. \tag{2.37}$$

The estimates (2.37), (2.36), (2.34), (2.32), and (2.24) show that (2.23) holds, finishing the proof of Theorem 1.1.

The proof of (1.13) in Theorem 1.2 is the same as the corresponding proof in [6]. The proof of (1.18) and (1.19) follows that of Theorem 1.1 exactly up to formula (2.24), except that instead of (2.16) and (2.24) we have, respectively,

$$\begin{aligned}\hat{A}(\tau) &= O\left(\int_0^{1/\tau} a(s)\,ds\right) \\ \hat{A}'''(\tau) &= O\left(\int_0^{1/\tau} sa(s) + sb(s)\,ds\right) \\ \hat{A}''(\tau) &= O\left(\tau^{-1}\int_0^{1/\tau} sa(s) + sb(s)\,ds\right) \quad (\text{as}\,\tau \to \infty),\end{aligned} \tag{2.38}$$

and

$$\frac{t^2\lambda}{L}\left|u_5\left(t,\frac{\lambda}{L}\right)\right| \leqslant M\left[1 + \int_{\rho}^{\infty}\left(\frac{\tau^{-1}\int_0^{1/\tau} sa(s)+sb(s)\,ds}{|D(\tau,\lambda/L)|^2} + \frac{\lambda^{-2} + \left(\int_0^{1/\tau} sa(s)+sb(s)\,ds\right)^2}{|D(\tau,\lambda/L)|^3}\right)d\tau\right] \tag{2.39}$$

(where we have used (2.4)).

A straightforward calculation shows that

$$\lim_{\tau \to \infty} \frac{\Theta(\tau)}{\theta(\tau)} = \frac{1}{p},$$

and there is a constant R such that

$$\left| \Theta(\tau) - \frac{1}{p} \theta(\tau) \right| \leqslant \frac{R}{\tau^3}, \quad \tau \geqslant \rho_1$$

for some $\rho_1 > \rho$. The rest of the argument follows as in [6], with $L = 1/p$ (see [6, formula (2.30) until the end of the proof]). We will therefore omit the details.

3. An example. Let $-a'(t) = \sum_{k=0}^{\infty} a_k (x_k - t) \chi_{[0, x_k)}(t)$ where $\chi_{[0,\alpha)}(t)$ equals one for $0 \leqslant t < \alpha$ and zero otherwise, and $a_k = 2^{2^{2^k}}$, $x_k = 2^{(-.75)2^{2^k}}$ for $k = 0, 1, 2, \ldots$. Let $b(t)$ be any function satisfying (1.11) and (1.14). The function $a(t)$ satisfies (1.11) and (1.14) by its definition and the calculation

$$\infty > \int_0^1 -a'(t)\, dt = \frac{1}{2} \sum_{k=0}^{\infty} a_k x_k^2 < \infty.$$

An easy calculation shows that for $x_{n+1} < x < x_n$,

$$C(x) \equiv \int_0^x -a'(s)\, ds = x \sum_{k=0}^{n} a_k (x_k - x/2) + \tfrac{1}{2} \sum_{k=n+1}^{\infty} a_k x_k^2,$$

and

$$B(x) \equiv \int_0^x -sa'(s)\, ds = x^2 \sum_{k=0}^{n} a_k (x_k/2 - x/3) + \tfrac{1}{6} \sum_{k=n+1}^{\infty} a_k x_k^3.$$

It is also easy to see that

$$C(x) \sim x a_n x_n, \qquad B(x) \sim x^2 a_n x_n \quad \text{as } n \to \infty,$$

where $F \sim G$ as $n \to \infty$ means $\lim_{n \to \infty} F/G = 1$. It follows that $x^5 C(x)/B(x)^3 \sim 1/(a_n x_n)^2$. Now the fact that $\lim_{n \to \infty} 1/(a_n x_n)^2 = \infty$ implies that (1.15) does not hold.

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