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WEIGHTED LIAPUNOV FUNCTIONS FOR A CLASS OF THIRD-ORDER AUTONOMOUS DIFFERENTIAL EQUATIONS*

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1. Introduction. In 1963, Walter Leighton published a paper [7] in which he provided Liapunov functions for general classes of second- and third-order differential equations. In a subsequent paper [3], the current author and Walter Leighton considered Liapunov functions for second-order systems that were more general than those given in Leighton's 1963 paper. Further, this latter paper gave a *class* of weighted Liapunov functions for certain second-order systems. Further work utilizing weighted Liapunov functions for second-order equations was published by the current author in [1] and by S. Duchich and the current author in [2]. A separate paper by A. Skidmore [8] extended Leighton's 1963 work on third-order equations to fourth-order equations, but no "weighting" was considered in either paper.

The purpose of this paper is to extend Leighton's work (in [7]) on third-order systems by providing a class of weighted Liapunov functions for an equation

$$\dddot{x} + \phi(x, \dot{x}, \ddot{x}) = 0$$

and the associated system

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = -\phi(x, y, z). \tag{1.1}$$

This system is a bit more general than third-order systems considered by Leighton in [7]. Moreover, for third-order systems considered by Leighton in [7], our weight functions V_w (w = weight) reduce to those given by Leighton when $w = 1$. In turn, as pointed out by Leighton in [7], his Liapunov functions are "sharp" in that they provide stability criteria in linear cases when the Routh-Hurwitz condition holds. Thus our class of functions provides breadth and sensitivity.

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2. A class of Liapunov functions. We assume here that ϕ is of class C'' in an open set containing the origin and that the origin is an isolated critical point of (1.1).

We will consider weight functions $w(x, y, z)$ which are of class C'' neighboring the origin and are positive in a deleted neighborhood of the origin.

For such a function w and a constant α, we define

$$\begin{aligned} V_w(x, y, z) = &\int_0^z vw(x,0,v)\,dv + \int_0^y w(x,u,0)\phi(x,u,0)\,du \\ &+\alpha\Big\{zyw(x, y,0) + \int_0^x w(t,0,0)\phi(t,0,0)\,dt \\ &+ \int_0^y u\left[w(x,u,0)\phi_z(x,u,0) + w_z(x,u,0)\phi(x,u,0)\right] du\Big\}. \end{aligned}$$

If $w \equiv 1$ and if $\phi(x, y, z) = z\psi(x, y) + \theta(x, y)$, then this reduces to the function considered by Leighton in [7].

In the sequel it will be convenient to define

$$A(x, y) = w(x, y,0)\phi(x, y,0)$$

and

$$B(x, y) = yw(x, y,0)\phi_z(x, y,0) + yw_z(x, y,0)\phi(x, y,0).$$

To compute $\dot{V}_w$ we first compute the various first partial derivatives. We obtain

$$\begin{aligned} \frac{\partial V_w}{\partial x} &= \int_0^z vw_x(x,0,v)\,dv + \int_0^y (A_x(x,u) + \alpha B_x(x,u))\,du \\ &\quad+\alpha w(x,0,0)\theta(x,0,0) + \alpha zyw_x(x, y,0), \\ \frac{\partial V_w}{\partial y} &= A(x, y) + \alpha w(x, y,0)z + \alpha zyw_y(x, y,0) + \alpha B(x, y), \\ \frac{\partial V_w}{\partial z} &= zw(x,0, z) + \alpha yw(x, y,0). \end{aligned}$$

We then have

$$\begin{aligned} \dot{V}_w = &\, y\int_0^z vw_x(x,0,v)\,dv + y\int_0^y (A_x(x,u) + \alpha B_x(x,u))\,du \\ &+\alpha yw(x,0,0)\phi(x,0,0) + \alpha zy^2w_x(x, y,0) \\ &+z\left[w(x, y,0)\phi(x, y,0) + \alpha w(x, y,0)z + \alpha zyw_y(x, y,0) + \alpha B(x, y)\right] \\ &-\phi(x, y, z)\left[zw(x,0,z) + \alpha yw(x, y,0)\right] \\ = &\, y^2\left[\frac{1}{y}\int_0^y (A_x(x,u) + \alpha B_x(x,u))\,du\right] + z^2\left[\alpha w(x, y,0) + \alpha yw_y(x, y,0)\right] \\ &+zy\left(\frac{\alpha B(x, y)}{y} + \frac{1}{z}\int_0^z vw_x(x,0,v)\,dv + \alpha yw_x(x, y,0)\right) \\ &+zw(x, y,0)\phi(x, y,0) + \alpha yw(x,0,0)\phi(x,0,0) \\ &-z\phi(x, y, z)w(x,0,z) - \alpha yw(x, y,0)\phi(x, y, z) \\ = &\, Cy^2 + Dzy + Ez^2, \end{aligned} \tag{2.1}$$

where C, D, and E are given by

$$C = \frac{1}{y}\int_0^y \left(A_x(x,u) + \alpha B_x(x,u) - A_y(x,u)\right) du,$$

$$D = \frac{\alpha B(x,y)}{y} + \frac{1}{z}\int_0^z \left(z w_x(x,0,v) - \alpha w(x,y,0)\phi_z(x,y,v)\right) dv$$

$$+\alpha y w_x(x,y,0) + \phi(x,y,z)\frac{1}{y}\int_0^y w_y(x,u,v)\, du,$$

$$E = \alpha w(x,y,0) + \alpha y w_y(x,y,0) - \frac{1}{z}\int_0^z \frac{\partial(\phi w)}{\partial z}(x,y,v)\, dv.$$

We obtain the following result.

THEOREM 1. If there exists a weight function w and a constant α such that V_w is positive definite neighboring the origin and such that $E < 0$ and $D^2 - 4EC < 0$, then the origin is an asymptotically stable critical point of (1.1) and V_w is a Liapunov function for (1.1).

Here we assume that ϕ and w satisfy the regularity conditions given at the beginning of Sec. 2. Further, the theorem allows us to conclude *asymptotic* stability since the set $\dot{V}_w = 0$ is the x axis, which is an invariant set (see [4]).

Example 1. We let $w = 1$ and find sufficient conditions for the asymptotic stability of the origin. If $\phi(x,y,z) = z\psi(x,y) + \theta(x,y)$, these conditions agree with those given in [7].

We first find conditions which insure that V_w ($w = 1$) will be locally positive definite. Consider the matrix

$$(a_{ij}) = \left(\frac{\partial^2 V_w}{\partial x_i \partial x_j}\right), \qquad 1 \leqslant i \leqslant 3, \quad 1 \leqslant j \leqslant 3.$$

It is easily shown that

$$\det(a_{ij})^0 = \begin{vmatrix} \alpha\phi_x^0 & \phi_x^0 & 0 \\ \phi_x^0 & \phi_y^0 + \alpha\phi_z^0 & \alpha \\ 0 & \alpha & 1 \end{vmatrix},$$

where the superscript denotes evaluation at the origin. If we assume that

$$\phi_z^0 > \alpha, \qquad \phi_x^0 > 0, \qquad \phi_y^0 > 0 \tag{2.2}$$

and let $\alpha = (\phi_x^0)(\phi_y^0)^{-1}$, then the principal minors of the above determinant will be positive and hence V will be locally positive definite.

The discriminant $D^2 - 4EC$ of (2.1) evaluated at the origin is given by

$$(D^2 - 4EC)^0 = (-4EC)^0 = -4\left(\alpha - \phi_z^0\right)\left(\phi_x^0 - \phi_y^0\right),$$

which will be negative neighboring the origin if $\phi_x^0 < \phi_y^0$. Since $E^0 = \alpha - \phi_z^0$, it follows that $V_w(w = 1)$ is a Liapunov function for (1.1) if (2.2) holds and if $\phi_x^0 < \phi_y^0$ (here $\alpha = (\phi_x^0)(\phi_y^0)^{-1}$).

If the above conditions hold and ϕ contains higher-order terms in z, V_w $(w = 1)$ is a Liapunov function. For instance, V_w is a Liapunov function when $\phi(x, y, z) = 4z + z^3 + x + 2y$. Such cases weren't considered by Leighton in [7].

Now assume that w is any weight function satisfying the regularity conditions given at the beginning of Sec. 2. Further, assume that $w^0 > 0$ and that (2.2) holds and that $\phi_x^0 < \phi_y^0$. It can be shown that

$$\det\left(\frac{\partial^2 V_w^0}{\partial x_i \partial x_j}\right) = \begin{vmatrix} \alpha w^0 \phi_x^0 & w^0 \phi_x^0 & 0 \\ w^0 \phi_x^0 & w^0 \phi_y^0 + \alpha w^0 \phi_z^0 & \alpha w^0 \\ 0 & \alpha w^0 & w^0 \end{vmatrix}.$$

Also,

$$(D^2 - 4EC)^0 = -4\left(\alpha w^0 - w^0 \phi_z^0\right)\left(w^0 \phi_x^0 - w^0 \phi_y^0\right).$$

We obtain the following result, which in certain cases provides a large class of Liapunov functions for (1.1).

THEOREM 2. Let $w_0 > 0$ and assume that ϕ satisfies (2.2) and that $\phi_x^0 < \phi_y^0$. Then V_w is a Liapunov function for the system (1.1), where $\alpha = (\phi_x^0)(\phi_y^0)^{-1}$.

In some earlier papers (see [1], [2], [3]) concerning second-order systems, particular attention was given to considering estimates of regions of asymptotic stability obtained from V_w by varying the weight function w. For certain weight functions and certain second-order systems, one obtains better estimates than with the single estimate provided by V_w with $w = 1$. In certain cases (see [1], [2]) optimal estimates over certain subclasses of weight functions may be found.

The following example illustrates that some of these methods carry over to the third-order case.

Example 2. In the following example we consider a system (see [5, p. 228])

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = -y - bz - f(x). \tag{2.3}$$

Under appropriate conditions, the equation $V = V(P_0)$ will be an estimate of the region of asymptotic stability, where V is the function above with $w = 1$ for the system (2.3) and $V(P_0)$ is the smallest positive critical value of V. We will show how to choose a weight function w such that the manifold $V_w = V_w(P_0)$ bounds a subregion of the region of asymptotic stability of the origin and wholly contains the region $V = V(P_0)$ (see, in particular, [2]).

We assume that $f \in C'(-\infty, \infty)$, and that f vanishes only at $x = 0$ and at some $x_0 > 0$. We assume further that $0 \leqslant f'(x) < m < 1 < b$ for all x, that $xf(x) \geqslant 0$ and $f'(0) > 0$, and that

$$\lim_{x \to \infty} f(x) = L_1 \quad \text{and} \quad \lim_{x \to -\infty} f(x) = L_2,$$

where L_1 and L_2 are finite.

With $w = 1$ and $\alpha = 1$ we have

$$V = \frac{z^2}{2} + \frac{y^2}{2} + f(x)y + zy + \int_0^x f(u)\,du + \frac{by^2}{2},$$

and with $w = 1 + Bx^2$ we have

$$V_w = \frac{z^2}{2}(1 + Bx^2) + \frac{y^2}{2}(1 + Bx^2) + (1 + Bx^2)f(x)y$$
$$+ zy(1 + Bx^2) + \int_0^x (1 + Bu^2)f(u)\,du + \frac{by^2}{2}(1 + Bx^2).$$

We summarize several facts concerning V and V_w and omit the computational details.

(i) V and V_w are locally positive definite.

(ii) For B sufficiently small and positive and for $B = 0$, the points $(x_0, 0, 0)$ and $(0, 0, 0)$ are the only finite critical points of V_w.

(iii) For all $B \geqslant 0$, V_w has no infinite critical points, which follows from the fact that there is no sequence $\{X_n\}$ in R^3 such that

$$|X_n| \to \infty \quad \text{and} \quad \frac{\partial V_w}{\partial x_i}(X_n) \to 0 \qquad \text{as } n \to \infty \ (1 \leqslant i \leqslant 3).$$

(iv) From the above and [6], it follows that for B small and positive

$$V(x, y, z) = V(x_0, 0, 0) \tag{2.4}$$

and

$$V_w(x, y, z) = V_w(x_0, 0, 0) \tag{2.5}$$

form closed 2 manifolds containing the origin, both of which lie in a subset of the region of asymptotic stability of the origin.

We now show that the region (2.5) wholly contains (2.4) and that for B sufficiently small and positive, $\dot{V}_w \leqslant 0$ inside (2.5).

We first verify the latter. It may be shown that

$$\dot{V}_w = z^2\big(xyB + (1 + Bx^2)(1 - b)\big)$$
$$+ y^2\big(xyB + 2xf(x)B + 2xzB + xybB + [f'(x) - 1][1 + Bx^2]\big).$$

Since the region (2.5) is compact, it follows that for B sufficiently small, $\dot{V}_w \leqslant 0$ inside this region.

To verify that (2.5) contains (2.4), we solve for z in terms of x and y. For the surface (2.5), we have that

$$z = -y \pm 2^{1/2}\left[\frac{\int_0^{x_0}(1 + Bu^2)f(u)\,du}{1 + Bx^2} - \frac{y^2 b}{2} - f(x)y - \frac{\int_0^x (1 + Bu^2)f(u)\,du}{1 + Bx^2}\right]^{1/2}, \tag{2.6}$$

and for (2.4) we have that

$$z = -y \pm 2^{1/2}\left[\int_0^{x_0} f(u)\,du - \int_0^{x} f(u)\,du - \frac{by^2}{2} - f(x)y\right]^{1/2}$$

$$= -y \pm 2^{1/2}b^{-1/2}\left[\int_0^{x_0} f(u)[b - f'(u)]\,du - \frac{b^2}{2}\left(y + \frac{f(x)}{b}\right)^2 - \int_0^{x} f(u)[b - f'(u)]\,du\right]^{1/2}. \quad (2.7)$$

If we set the quantity in square brackets in (2.7) equal to zero, it may be seen that this is a closed curve bounding the domain of z in (2.7). The projection of this curve on the x axis is the interval $[x_1, x_0]$, where x_1 is the unique negative value such that

$$\int_0^{x_0} f(u)(b - f'(u))\,du = \int_0^{x_1} f(u)(b - f'(u))\,du.$$

To prove our assertion, it suffices to prove that the quantity in square brackets in (2.6) exceeds the quantity in square brackets in (2.7). This reduces to proving that

$$\int_x^{x_0} u^2 f(u)\,du \geqslant x^2\int_x^{x_0} f(u)\,du, \qquad x_1 \leqslant x \leqslant x_0.$$

If we put

$$G(x) = \int_x^{x_0} u^2 f(u)\,du - x^2\int_x^{x_0} f(u)\,du$$

and note that

$$G'(x) = -2x\int_x^{x_0} f(u)\,du, \qquad x_1 \leqslant x \leqslant x_0,$$

the inequality (which is strict in (x_1, x_0)) follows at once.

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