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PERIODIC SOLUTIONS OF THE EQUATION* $\ddot{x} + g(x) = E\cos t + \sigma h(t)\dot{x}$

By

LUIZ A. C. LADEIRA AND PLÁCIDO Z. TÁBOAS

(ICMSC-USP, São Carlos, SP-Brazil)

1. Introduction. Let g be an odd C^4-function with $xg(x) > 0$, $xg''(x) \geqslant 0$, for every $x \neq 0$, $1 < g'(0) < 4 < \lim_{x\to\infty} g'(x) < 9$. Loud [2] has proved that to each real number E, there corresponds a unique solution $x(t; E)$ of the equation

$$\ddot{x} + g(x) = E\cos t \tag{1.1E}$$

which is 2π-periodic, even and odd-harmonic, that is,

$$x(t+\pi, E) = -x(t; E), \quad t \in \mathbf{R}.$$

Let us consider the linear variational equation with respect to $x(t; E)$,

$$\ddot{y} + g'(x(t; E))y = 0, \tag{1.2E}$$

where a prime means derivative with respect to x, and denote by $p(t; E)$, $q(t; E)$ its solutions satisfying

$$\dot{p}(0; E) = q(0; E) = 0,$$
$$p(0; E) = \dot{q}(0; E) = 1.$$

The hypotheses on g imply p is an even function of t and q is odd. According to Loud [4], there are values $E_0, E_1 > 0$ of the parameter E such that

$$\dot{p}\left(\frac{\pi}{2}; E_0\right) = 0, \qquad q\left(\frac{\pi}{2}; E_1\right) = 0.$$

This means that $p(t; E_0)$ is an even, π-periodic, nontrivial solution of ($1.2E_0$) and $q(t; E_1)$ is an odd, π-periodic nontrivial solution of ($1.2E_1$).

Under the hypothesis that $q(\pi/2; E_0) \neq 0$, and hence $q(t; E_0)$ is nonperiodic, which we assume to hold throughout this paper, the existence of 2π-periodic non-odd-harmonic solutions of (1.1E), near $x(t; E_0)$ for E near E_0, has been guaranteed in [4].

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In this work we prove the existence of such solutions when (1.1E) is perturbed by a small damping term, $\sigma h(t)\dot{x}$, where $\sigma \in \mathbf{R}$ and h is a 2π-periodic continuous function satisfying some generic conditions. The very important case $h(t) = 1$ does not satisfy those generic conditions, and is solved separately. Following a device in [6], we show how the bifurcation diagram in the parameter's space "passes through" the nongeneric case, $h(t) = 1$, varying the function h conveniently.

The hypothesis $q(t; E_0)$ nonperiodic implies that the space of periodic solutions of ($1.2\mathrm{E}_0$) is one-dimensional. Indeed these solutions are proportional to $p(t; E_0)$. We believe that the two-dimensional case $q(\pi/2; E_0) = 0$ is exceptional. We are not concerned with the genericness of the condition $q(t; E_0)$ nonperiodic, but it will be clear from Sec. 4 below that an adaptation of a Loud's example [3] shows that it is not difficult to obtain concrete situations where this hypothesis is satisfied.

We denote by $\mathscr{P}$ the Banach space of the 2π-periodic continuous functions $f\colon \mathbf{R} \to \mathbf{R}$ with the supremum norm $|\cdot|_0$, and by $\mathscr{P}^{(2)}$ the vector subspace of $\mathscr{P}$ of the C^2-functions with a C^2 norm.

2. The nonautonomous perturbation. Suppose that E, σ are real parameters, $h \in \mathscr{P}$, g satisfy all the hypotheses assumed in Sec. 1, and consider the equation

$$\ddot{x} + \sigma h(t)\dot{x} + g(x) = E\cos t. \tag{2.1}$$

The aim of this paper is to characterize the number of 2π-periodic solutions of (2.1) which are near $x_0 = x(\cdot\,; E_0)$, for (E, σ) near $(E_0, 0)$. We make $\lambda = E - E_0$, $y = x - x_0$, and the problem reduces to the study of the existence of small 2π-periodic C^2 solutions of the equation

$$\ddot{y} + g'(x_0)y = \lambda\cos t - \sigma h\dot{x}_0 - \sigma h\dot{y} - G(y) \tag{2.2}$$

for small $|\lambda|$, $|\sigma|$, where

$$[G(y)](t) = g(x_0(t) + y(t)) - g(x_0(t)) - g'(x_0(t))y(t).$$

If $y \in \mathscr{P}^{(2)}$, it follows that $G(y) \in \mathscr{P}$ and $G(y) = O(|y|^2)$, as $|y| \to 0$. Here we mean the smallness of solutions in the sense of the $\mathscr{P}^{(2)}$ space.

Let $P\colon \mathscr{P}^{(2)} \to \mathscr{P}^{(2)}$ and $Q\colon \mathscr{P} \to \mathscr{P}$ be the projections given by

$$P\phi = \phi(0)p, \quad \phi \in \mathscr{P}^{(2)}, \tag{2.3}$$

$$Q\phi = \left(c\int_{-\pi}^{\pi} p\phi\right)p, \quad \phi \in \mathscr{P}, \quad c = \left(\int_{-\pi}^{\pi} p^2\right)^{-1}. \tag{2.4}$$

Thus, it is a consequence of the Fredholm alternative (see [1], page 146, for instance) that the equation

$$\ddot{y} + g'(x_0)y = f \tag{2.5}$$

with $f \in \mathscr{P}$, has a 2π-periodic solution if and only if $Qf = 0$. Furthermore, if $Qf = 0$, there is a unique solution $y = y(f) \in \mathscr{P}^{(2)}$ such that $Py = 0$, and it follows from the Closed Graph Theorem that the linear operator $K\colon f \in (I - Q)\mathscr{P} \to y(f) \in (I - P)\mathscr{P}^{(2)}$ is continuous.

Notice that if $f \in \mathscr{P}$, then $K(I - Q)f$ is the only 2π-periodic solution of the equation

$$\ddot{y} + g'(x_0)y = (I - Q)f \tag{2.6}$$

which satisfies $PK(I - Q)f = 0$, that is, $[K(I - Q)f](0) = 0$; and that the general 2π-periodic solution y of (2.6) is given by $y = ap + K(I - Q)f$, for some $a \in \mathbf{R}$.

We now apply the Liapunov–Schmidt reduction to the problem (2.2); that is, the variable y is replaced by $up + z$ in (2.2) and this equation is broken down into the supplementary subspaces $(I - Q)\mathscr{P}$ and $Q\mathscr{P}$. Taking into account that $Q(\cos) = 0$ it follows that y is a 2π-periodic solution of (2.2) if and only if $y = up + z$ where the pair (u, z), with $u \in \mathbf{R}$ and $z \in (I - P)\mathscr{P}^{(2)}$, is a solution of the system of equations

$$z = K(I - Q)\left[\lambda\cos - \sigma h\dot{x}_0 - \sigma uh\dot{p} - \sigma h\dot{z} - G(up + z)\right], \tag{2.7}$$

$$0 = Q\left[\sigma h\dot{x}_0 + \sigma uh\dot{p} + \sigma h\dot{z} + G(up + z)\right]. \tag{2.8}$$

LEMMA 2.1. There exist neighborhoods of the origin, $V \subset \mathbf{R}^3$, $U \subset (I - P)\mathscr{P}^{(2)}$ such that, for each $(u, \lambda, \sigma) \in V$, Eq. (2.7) has a unique solution $z = z^*(u, \lambda, \sigma) \in U$. Furthermore, $z^*\colon U \to V$ is a C^4 function and

$$z^*(u, \lambda, 0) = A_1\lambda + \tfrac{1}{2}A_2u^2 + \text{h.o.t.} \tag{2.9}$$

where $A_1 = K\cos$, $A_2 = -K[g''(x_0)p^2]$ are even, 2π-periodic, and h.o.t. indicates terms which are $O(|u|^3 + |u\lambda| + |\lambda|^2)$ as $u, \lambda \to 0$.

Proof. Let

$$H\colon (I - P)\mathscr{P}^{(2)} \times \mathbf{R}^3 \to (I - P)\mathscr{P}^{(2)}$$

given by

$$H(z, u, \lambda, \sigma) = z - K(I - Q)\left[\lambda\cos - \sigma h\dot{x}_0 - \sigma uh\dot{p} - \sigma h\dot{z} - G(up + z)\right],$$

$z \in (I - P)\mathscr{P}^2$; $u, \lambda, \sigma \in \mathbf{R}$. Thus, H is of class C^4, $H(0,0,0,0) = 0$, and $D_1H(0,0,0,0) = I$, the identity in $(I - P)\mathscr{P}^{(2)}$, where D_1 is the derivative with respect to z. The existence of U, V, z^* follows from the Implicit Function Theorem. The expansion (2.9) is accomplished by taking into account that $z^*(0,0,0) = 0$ and

$$z^*(u, \lambda, 0) = \lambda K(I - Q)\cos + K(I - Q)\left[-G\left(up + z^*(u, \lambda, 0)\right)\right],$$

$$G(w) = \tfrac{1}{2}g''(x_0)w^2 + O\left(|w|^3\right), \quad \text{as } |w| \to 0, \quad w \in \mathscr{P}^{(2)},$$

$$Q\cos = Q\left[g''(x_0)p^2\right] = 0.$$

The last condition follows from the fact that $g''(x_0)p^2$ is odd-harmonic. Since cos and $g''(x_0)p^2$ are even functions, it follows finally that $K\cos$ and $K[g''(x_0)p^2]$ are even, and the proof is complete.

Lemma 2.1 shows that Eq. (2.2) has a small solution $y \in \mathscr{P}^{(2)}$ if and only if $y = up + z^*(u, \lambda, \sigma)$, where (u, λ, σ) satisfies the bifurcation equation

$$F(u, \lambda, \sigma) = \int_{-\pi}^{\pi} p\left[\sigma h\dot{x}_0 + \sigma uh\dot{p} + \sigma h\dot{z}^*(u, \lambda, \sigma) + G\left(up + z^*(u, \lambda, \sigma)\right)\right] = 0. \tag{2.10}$$

F is a C^4 function and can be rewritten as

$$F(u, \lambda, \sigma) = B_1u\lambda + B_2u^3 + B_3\sigma + \text{h.o.t.} \tag{2.11}$$

with

$$B_1 = 4\int_0^{\pi/2} p^2(t) g''(x_0(t)) b_1(t)\, dt,$$

$$B_2 = 4\int_0^{\pi/2} p^2(t) g''(x_0(t)) b_2(t)\, dt + \frac{2}{3}\int_0^{\pi/2} p^4(t) g'''(x_0(t))\, dt,$$

where

$$b_j(t) = \tfrac{1}{2}\left[A_j(t) - A_j(t+\pi)\right],\ j = 1,2,$$

$$B_3 = \int_{-\pi}^{\pi} p(t)\dot{x}_0(t) h(t)\, dt;$$

and h.o.t. are terms of order

$$O(u^4 + |u^2\lambda| + \lambda^2 + \sigma^2 + |u\sigma| + |\lambda\sigma|), \quad \text{as } u, \lambda, \sigma \to 0.$$

THEOREM 2.1. Suppose the hypotheses assumed in Sec. 1 are satisfied and $B_1 B_2 B_3 \neq 0$. Then, there exist neighborhoods $V_1 \subset \mathscr{P}^{(2)}$ of x_0, $V_2 \subset \mathbf{R}_2$ of $(E, \sigma) = (E_0, 0)$ and a cusp $C \subset V_2$ given by the equations

$$\begin{aligned} E &= E_0 - 3(B_2/B_1)u^2 + O(|u|^3), \\ \sigma &= 2(B_2/B_3)u^3 + O(|u|^4) \end{aligned} \tag{2.12}$$

as $|u| \to 0$, which divides V_2 into two open, connected, disjoint sets O_1, O_2 such that, for $(E, \sigma) \in O_1$, Eq. (2.1) has a unique 2π-periodic solution in V_1; for $(E, \sigma) \in O_2$, Eq. (2.1) has precisely three solutions in V_1. For $(E, \sigma) \in C$, the equation has precisely two solutions in V_1 if $\sigma \neq 0$, and a unique solution if $\sigma = 0$.

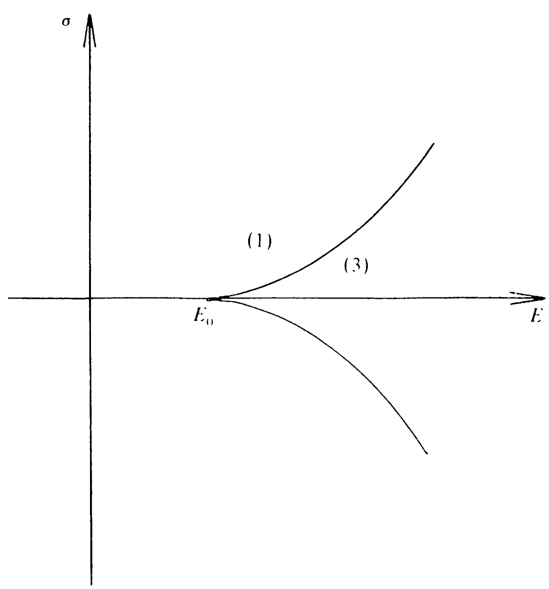

FIG. 1

Proof. The proof will be given for the case $B_1B_2 < 0$; the case $B_1B_2 > 0$ is similar. As a consequence of Lemma 2.1, the proof is reduced to examining the number of solutions of (2.11), as (E, σ) varies in V_2. If

$$v = B_2^{1/3}u, \qquad \lambda_1 = -B_1B_2^{-1/3}\lambda, \qquad \lambda_2 = B_3\sigma,$$

then Eq. (2.11) becomes

$$G(v, \lambda_1, \lambda_2) = v^3 - v\lambda_1 - \lambda_2 + G_1(v, \lambda_1, \lambda_2) = 0 \tag{2.13}$$

where

$$G_1(v, \lambda_1, \lambda_2) = O\left(v^4 + |v^2\lambda_1| + \lambda_1^2 + \lambda_2^2 + |v\lambda_2| + |\lambda_1\lambda_2|\right),$$

as $v, \lambda_1, \lambda_2 \to 0$.

The bifurcation curve is determined by the multiple solutions of (2.13), that is, the triples $(v, \lambda_1, \lambda_2)$ which are solutions of both (2.13) and the equation

$$D_1G(v, \lambda_1, \lambda_2) = 3v^2 - \lambda_1 + D_1G_1(v, \lambda_1, \lambda_2) = 0. \tag{2.14}$$

Since the Jacobian matrix

$$\frac{D(G, D_1G)}{D(\lambda_1, \lambda_2)}(0,0,0) = \begin{bmatrix} 0 & -1 \\ -1 & D_3D_1G(0,0,0) \end{bmatrix}$$

is nonsingular, the Implicit Function Theorem shows that (2.13) and (2.14) uniquely define λ_1, λ_2 as functions of v, with

$$\lambda_1 = 3v^2 + O(|v^3|), \qquad \lambda_2 = 2v^3 + O(v^4)$$

as $v \to 0$. Returning to the original variables, we obtain the parametrization (2.12).

To show that the number of solutions of (2.13) effectively changes when the cusp is crossed, it suffices to notice that $D_1^2G(v, \lambda_1, \lambda_2) \neq 0$ and $D_3G(v, \lambda_1, \lambda_2) \neq 0$ on the points of the cusp. This means that on the cusp, $v = 0$ is a quadratic singularity of $G(\cdot, \lambda_1, \lambda_2)$ and the values $G(v, \lambda_1, \lambda_2)$ vary monotonically with λ_2. This completes the proof.

As is pointed out in Loud [4], the numbers B_1, B_2 are generally nonzero. The case $B_1B_2 > 0$ only changes the bifurcation diagram by a reflection on a vertical axis passing through $(E_0, 0)$. One can see that condition $B_3 \neq 0$ is a generic hypothesis in h in the sense that $\{h \in \mathscr{P}\colon \int_{-\pi}^{\pi} ph\dot{x}_0 \neq 0\}$ is an open dense set in $\mathscr{P}$. Of great interest, however, is the case $h(t) = 1$ for which the symmetry properties of p and x_0 imply $B_3 = 0$. We examine this special case in the next section.

3. The autonomous perturbation. Consider the equation

$$\ddot{x} + g(x) = E\cos t - \dot{x}. \tag{3.1}$$

Repeating the above procedure and noticing that $Q\cos = Q\dot{x}_0 = Q\dot{p} = 0$, Eq. (3.1) becomes equivalent to the system:

$$z = K(I - Q)\left[\lambda\cos - \sigma\dot{x}_0 - \sigma u\dot{p} - \sigma\dot{z} - G(up + z)\right], \tag{3.2}$$

$$0 = Q\left[\sigma\dot{z} + G(up + z)\right], \tag{3.3}$$

where $z = x - x_0 - up$, $u \in \mathbf{R}$. Equation (3.2) can be "solved" for small $|u|$, $|\lambda|$, $|\sigma|$ by a function $z = z^*(u, \lambda, \sigma)$ which can be expanded as

$$z^*(u, \lambda, \sigma) = A_1\lambda + \tfrac{1}{2}A_2u^2 + A_3\sigma + \tfrac{1}{2}A_4\sigma^2 + A_5u\sigma + \text{h.o.t.},$$

where h.o.t. indicates terms of order

$$O\big(|u|^3 + \lambda^2 + |\sigma|^3 + |u\lambda| + |u^2\sigma| + |u\sigma^2|\big), \quad \text{as } u, \lambda, \sigma \to 0.$$

The functions A_1, A_2 are given in Lemma 2.1;

$$\begin{aligned} A_3 &= -K\dot{x}_0 \text{ is odd, } 2\pi\text{-periodic, and odd-harmonic;} \\ A_4 &= -K\left[2\dot{A}_3 + g''(x_0)A_3^2\right] \text{ is even and } 2\pi\text{-periodic;} \\ A_5 &= -K\left[\dot{p} + g''(x_0)pA_3\right] \text{ is odd and } \pi\text{-periodic.} \end{aligned}$$

According to these facts, Eq. (3.1) has a solution x in $\mathscr{P}^{(2)}$ near x_0 if and only if $x = x_0 + up + z^*(u, \lambda, \sigma)$, where (u, λ, σ) satisfies

$$F(u, \lambda, \sigma) = \int_{-\pi}^{\pi} p\left[\sigma\dot{z}^*(u, \lambda, \sigma) + G\big(up + z^*(u, \lambda, \sigma)\big)\right] = 0. \tag{3.4}$$

Expanding G and z^* as

$$\begin{aligned} G(w) &= \tfrac{1}{2}g''(x_0)w^2 + \tfrac{1}{6}g'''(x_0)w^3 + O\big(|w|^4\big) \\ z^*(u, \lambda, \sigma) &= A_1\lambda + \tfrac{1}{2}A_2u^2 + A_3\sigma + \tfrac{1}{2}A_4\sigma^2 + A_5u\sigma \\ &\quad + O\big(|u|^3 + \lambda^2 + |\sigma|^3 + |u^2\sigma| + |u\sigma^2|\big) \end{aligned}$$

as $u, \lambda, \sigma \to 0$, $w \to 0$ in $\mathscr{P}$, Eq. (3.4) can be rewritten as

$$\begin{aligned} F(u, \lambda, \sigma) = {} & B_1u\lambda + B_2u^3 + \int_{-\pi}^{\pi} pg''(x_0)\big\{u\sigma A_3p + \lambda\sigma A_1A_3 + \tfrac{1}{2}\sigma^2A_3^2 \\ & + \tfrac{1}{2}\sigma^3A_3A_4 + \tfrac{1}{4}u\sigma^2[pA_4 + A_3A_5] + \tfrac{1}{2}u^2\sigma[2pA_5 + A_2A_3]\big\} \\ & + \tfrac{1}{6}\int_{-\pi}^{\pi} pg'''(x_0)\left[3u^2\sigma p^2A_3 + 3u\sigma^2pA_3 + \sigma^3A^3\right] \\ & + \sigma\int_{-\pi}^{\pi} p\left[\lambda\dot{A}_1 + \tfrac{1}{2}u^2\dot{A}_2 + \sigma\dot{A}_3 + \tfrac{1}{2}\sigma^2\dot{A}_4 + u\sigma\dot{A}_5\right] \\ & + O\big(u^4 + \lambda^2 + \sigma^4 + |u^2\lambda| + |u\sigma|^2 + |u^3\sigma| + |u\sigma^3| + |u\lambda\sigma| + |\lambda\sigma^2|\big) \end{aligned}$$

which, by virtue of the symmetries of the functions p, $g''(x_0)$, A_j, $j = 1, \ldots, 5$, can be simplified and denoted in a compact form, in such a way that we have the following.

LEMMA 3.1. The function F has continuous derivatives up through order 4 and has the expansion:

$$F(u, \lambda, \sigma) = B_1u\lambda + B_2u^3 + B_4u\sigma^2 + \text{h.o.t.}$$

where B_1, B_2 are given in Sec. 2,

$$B_4 = \int_0^{\pi} p\left[g''(x_0)(A_4p + 2A_3A_5) + 2g'''(x_0)A_3^2\right] + 2\int_0^{\pi} p\dot{A}_5,$$

and h.o.t. indicates terms of order

$$O\big(u^4 + \lambda^2 + \sigma^4 + |u^2\lambda| + |u\sigma|^2 + |u^3\sigma| + |u\sigma^3| + |u\lambda\sigma| + |\lambda\sigma^2|\big) \quad \text{as } u, \lambda, \sigma \to 0.$$

Now, we are able to state the final result for Eq. (3.1).

THEOREM 3.1. Suppose that g satisfies all the hypotheses assumed in Sec. 1 and $B_1B_2B_4 \neq 0$. Then there are neighborhoods $W_1 \subset \mathscr{P}^{(2)}$, of x_0, $W_2 \subset \mathbf{R}^2$ of $(E,\sigma) = (E_0, 0)$, and a quadratic curve Γ, given by

$$E = E_0 - (B_4/B_1)\sigma^2 + O\left(|\sigma|^3\right)$$

as $\sigma \to 0$. The curve Γ divides W_2 into two open, connected, disjoint sets O_1, O_2 such that, for $(E,\sigma) \in O_1$, Eq. (3.1) has a unique solution in W_1, and for $(E,\sigma) \in O_2$, it has precisely three solutions in W_1. For $(E,\sigma) \in \Gamma$, Eq. (3.1) has precisely two solutions in W_1 if $\sigma \neq 0$, and a unique solution if $\sigma = 0$.

Proof. As before, we consider only the case $B_1B_2 < 0$. The bifurcation equation (3.4) is equivalent to:

$$H(v,\lambda_1,\lambda_2) = v^3 - v\lambda_1 + av\lambda_2^2 + H(v,\lambda_1,\lambda_2) = 0 \tag{3.5}$$

where

$$v = B_2^{1/3}u, \qquad \lambda_1 = B_1B_2^{-1/3}\lambda, \qquad \lambda_2 = \left|B_2^{-1/3}B_4\right|^{1/2}\sigma, \qquad a = \operatorname{sign} B_2B_4,$$

$$H_1(v,\lambda_1,\lambda_2) = O\left(v^4 + \lambda_1^2 + \lambda_2^4 + |v^2\lambda_1| + (v\lambda_2)^2 + |v^3\lambda_2| + |v\lambda_1\lambda_2| + |\lambda_1\lambda_2^2|\right),$$

$$\text{as } v, \lambda_1, \lambda_2 \to 0.$$

Since $H \in C^4$, $D_1^jH(0,0,0) = 0$, $j = 0,1,2$, and $D_1^3H(0,0,0) = 6$, it follows from the Implicit Function Theorem that there exist $r > 0$ and a neighborhood $W \subset \mathbf{R}^2$ of $(0,0)$ such that for each $(\lambda_1, \lambda_2) \in W$, there corresponds a unique

$$u = u(\lambda_1, \lambda_2) \in (-r, r),$$

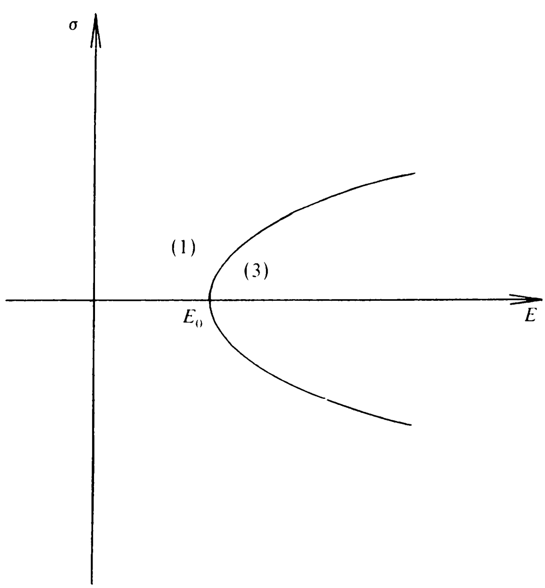

FIG. 2

with

$$D_1^2 H(u(\lambda_1, \lambda_2), \lambda_1, \lambda_2) = 0, \quad u \in C^2.$$

Therefore, the function $D_1 H(\cdot, \lambda_1, \lambda_2)$ has a minimum at $u = u(\lambda_1, \lambda_2)$. If

$$\alpha(\lambda_1, \lambda_2) = D_1 H(u(\lambda_1, \lambda_2), \lambda_1, \lambda_2),$$

we have

$$\alpha(\lambda_1, \lambda_2) = -\lambda_1 + a\lambda_2^2 + O\left(\lambda_1^2 + |\lambda_2^3|\right), \quad \text{as } \lambda_1, \lambda_2 \to 0.$$

Since

$$H(-r, 0, 0) < 0, \qquad H(r, 0, 0) > 0, \qquad D_1 H(\pm r, 0, 0) > 0,$$

by restricting further the diameter of W it follows from a continuity argument that

$$H(-r, \lambda_1, \lambda_2) < 0, \qquad H(r, \lambda_1, \lambda_2) > 0, \qquad D_1 H(\pm r, \lambda_1, \lambda_2) > 0$$

for $(\lambda_1, \lambda_2) \in W$. These relations imply that Eq. (3.5) has at least one solution $v(\lambda_1, \lambda_2) \in (-r, r)$, for each $(\lambda_1, \lambda_2) \in W$.

Let us define

$$A_1 = \{(\lambda_1, \lambda_2) \in W \colon \alpha(\lambda_1, \lambda_2) \geqslant 0\},$$
$$A_2 = \{(\lambda_1, \lambda_2) \in W \colon \alpha(\lambda_1, \lambda_2) < 0\}.$$

For $(\lambda_1, \lambda_2) \in A_1$, it follows from the definition of $u(\lambda_1, \lambda_2)$ that $H(\cdot, \lambda_1, \lambda_2)$ is a strictly increasing function in the interval $(-r, r)$. Thus, Eq. (3.5) has a unique solution v in this interval.

For $(\lambda_1, \lambda_2) \in A_2$, we have $\alpha(\lambda_1, \lambda_2) < 0$ and, therefore, $D_1 H(\cdot, \lambda_1, \lambda_2)$ must vanish precisely at two points $v_1(\lambda_1, \lambda_2)$, $v_2(\lambda_1, \lambda_2)$ such that

$$-r < v_1(\lambda_1, \lambda_2) < u(\lambda_1, \lambda_2) < v_2(\lambda_1, \lambda_2) < r,$$

$$v_1 = -\left[\frac{\lambda_1 - a\lambda_2}{3}\right]^{1/2} + \text{h.o.t.},$$

$$v_2 = \left|\frac{\lambda_1 - a\lambda_2}{3}\right|^{1/2} + \text{h.o.t.},$$

$$\text{h.o.t.} = o\left[\left(|\lambda_1| + \lambda_2^2\right)^{1/2}\right] \quad \text{as } \lambda_1, \lambda_2 \to 0.$$

At the point $v_1(\lambda_1, \lambda_2)$, the function $H(\cdot, \lambda_1, \lambda_2)$ assumes a local maximum value $\beta_1(\lambda_1, \lambda_2)$, and at $v_2(\lambda_1, \lambda_2)$, a local minimum value $\beta_2(\lambda_1, \lambda_2)$, so that if $\beta = \beta_1 \beta_2$, then $\beta(\lambda_1, \lambda_2) > 0$ implies that (3.5) has a unique solution; for $\beta(\lambda_1, \lambda_2) = 0$ there are precisely two solutions, and for $\beta(\lambda_1, \lambda_2) < 0$, precisely three solutions.

Moreover, it is easy to see that

$$\beta(\lambda_1, \lambda_2) = (4/27)\left(-\lambda_1 + a\lambda_2^2\right)^3 + o\left[\left(|\lambda_1| + \lambda_2^2\right)^3\right]$$

as $\lambda_1, \lambda_2 \to 0$, and that the equation $\beta(\lambda_1, \lambda_2) = 0$, returning to the original variables E, σ, describes the curve Γ. Notice that if $\lambda_2 = 0$ ($\sigma = 0$ in the original parameters) along the curve Γ: $\beta(\lambda_1, \lambda_2) = 0$, then λ_1 must vanish so that (3.5) becomes $v = 0$ and, therefore, the last assertion of the theorem is proved.

The bifurcation curves—the cusp of Theorem 2.1 and the quadratic curve Γ of Theorem 3.1—suggest that, when the periodic function h moves along a path in the space $\mathscr{P}$ passing through $h(t) = 1$, the cusp deforms and approximates Γ. In the sequel, this behavior is investigated.

We introduce a new parameter by considering the first-order term in the form $\sigma_1\dot{x} + \sigma_2 h(t)\dot{x}$, where h satisfies the condition $B_3 \neq 0$. Therefore, the case $\sigma_1 = 0$ is the situation of Theorem 2.1, and the case $\sigma_2 = 0$ is the situation of Theorem 3.1.

Consider the equation

$$\ddot{x} + g(x) = E\cos t - (\sigma_1 + \sigma_2 h(t))\dot{x} \tag{3.6}$$

with E, σ_1, σ_2 being real parameters, $h \in \mathscr{P}$ and g as above. Repeating the procedure of Sec. 2 one can reduce Eq. (3.6) to the bifurcation equation

$$F(u, \lambda, \sigma_1, \sigma_2) = B_1 u\lambda + B_2 u^3 + B_3\sigma_2 + B_4 u\sigma_1^2 + \text{h.o.t.} \tag{3.7}$$

where u is the coordinate of the solution in the null space of the operator

$$y \in \mathscr{P}^{(2)} \to \left[\ddot{y} + g'(x_0)y\right] \in \mathscr{P},$$

B_1, B_2, B_3, B_4 are as previously defined, and h.o.t. indicates terms of order

$$O\Big(u^4 + \lambda^2 + \sigma_1^4 + \sigma_2^2 + |u^2\lambda| + |u\sigma_1|^2 + |u\sigma_1^3| + |u^3\sigma_1| + |u\lambda\sigma_1| + |\lambda\sigma_1^2| + |u\sigma_2|$$
$$+ |\sigma_1\sigma_2| + |\lambda\sigma_2|\Big), \quad \text{as } u, \lambda, \sigma_1, \sigma_2 \to 0.$$

THEOREM 3.2. Suppose that g satisfies all the hypotheses of Sec. 1 and $B_1B_2B_3B_4 \neq 0$. Then, there are neighborhoods V of $(E, \sigma_1, \sigma_2) = (E_0, 0, 0)$, W of $x_0 \in \mathscr{P}^{(2)}$, and a surface $S \subset V$ given by

$$E - E_0 = -(1/B_1)\left[B_4\sigma_1^2 + \left(\frac{27B_2B_3}{4}\right)^{1/3}\sigma_2^{2/3}\right] + \text{h.o.t.} \tag{3.8}$$

where h.o.t. means higher order terms in σ_1, σ_2, which divides V into two open, connected, disjoint sets A, B such that for $(E, \sigma_1, \sigma_2) \in A$, Eq. (3.6) has a unique solution in W; for $(E, \sigma_1, \sigma_2) \in B$, Eq. (3.6) has precisely three solutions in W and for $(E, \sigma_1, \sigma_2) \in S$, Eq. (3.6) has precisely two solutions in W if $E \neq E_0$, and a unique solution when $E = E_0$.

Proof. It is the same as that of Theorem 2.1, with obvious adjustments.

Figure 3 sketches the shape of surface S, for $B_1B_2 < 0$, $B_2B_4 > 0$. Notice that the intersection of that surface with the plane $\sigma_1 = 0$ is the cusp described in Theorem 2.1, and the intersection with the plane $\sigma_2 = 0$ is the quadratic curve Γ of Theorem 3.1.

Theorem 3.2 shows how the cusp of Theorem 2.1 continuously transforms into the parabolic curve of Theorem 3.1, as illustrated by the following analysis.

Let us rewrite Eq. (3.6), for $\sigma_1 \neq 0$, as:

$$\ddot{x} + g(x) = E\cos t - \sigma_1(1 + \varepsilon h(t))\dot{x} \tag{3.9}$$

with $\varepsilon = \sigma_2/\sigma_1$. We are concerned with studying (3.9) for small $|\varepsilon|$.

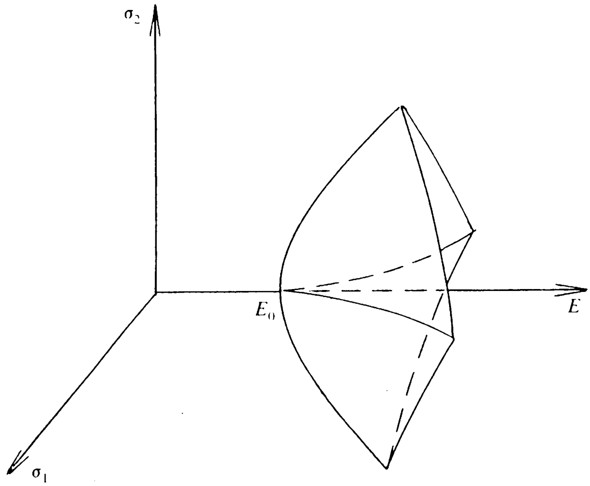

FIG. 3

According to Theorem 3.2, the bifurcation surface for (3.9) is given by

$$E - E_0 = -(1/B_1)\sigma_1^{2/3}\left(B_4\sigma_1^{4/3} + \left(\frac{27B_2B_3^2}{4}\right)^{1/3}\varepsilon^{2/3} + \cdots\right). \tag{3.10}$$

For a fixed $\varepsilon = \varepsilon_0$, let

$$r(\varepsilon_0) = \varepsilon_0^{1/2}\left(\frac{27|B_2B_3^2|}{4|B_4^3|}\right)^{1/4}.$$

Equation (3.10) shows that, for $|\sigma_1| > r(\varepsilon_0)$, we have

$$\left(\frac{27B_2B_3^2}{4}\right)^{1/3}\varepsilon_0\sigma_1^2 < |B_4|\sigma_1^2, \tag{3.11}$$

and, for $|\sigma_1| < r(\varepsilon_0)$, the inequality (3.11) inverts so that, when σ_1 is in the interval $(-r(\varepsilon_0), r(\varepsilon_0))$ the curve should be seen as a cusp plus a "parabolic" perturbation, while for σ_1 out of this interval, (3.10) describes a parabola plus a "cuspidal" perturbation. Since $r(\varepsilon_0) \to 0$, as $\varepsilon_0 \to 0$, it follows that we have a cusp for $0 < \sigma_1 < r(\varepsilon_0)$ which deforms into a parabola in the limiting case $\varepsilon_0 = 0$.

Remark 1. Similar results can be obtained for E near E_1 instead of E_0. Moreover, the Loud's existence theorem for the solution $x(\cdot, E)$ is stated in [4] under the following weaker assumption on g.

There exists an integer $n \geqslant 1$ such that either

$$(2n-1)^2 < g'(0) < 4n^2 < \lim_{x\to\infty} g'(x) < (2n+1)^2$$

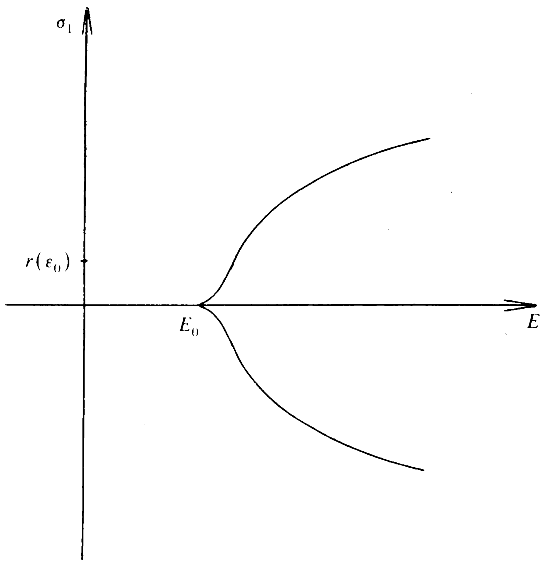

FIG. 4

or

$$(2n-1)^2 < \lim_{x\to\infty} g'(x) < 4n^2 < g'(0) < (2n+1)^2.$$

All earlier results are also valid under this hypothesis. The proofs are merely adaptations of those we have given.

Remark 2. Loud's results on the number of periodic solutions of (1.1) can be obtained from Theorems 2.1 and 3.1 (or 3.2) by considering $\sigma = 0$ (or $\sigma_1 = \sigma_2 = 0$).

4. An example. In this section we give an example for which all the hypotheses assumed with respect to g are satisfied. Since these assumptions are open, it follows that any function g_1 which is uniformly near the g of this example also satisfies the hypotheses.

This example is inspired by another one given in [3] where piecewise linear restoring forces are allowed. The numerical computations were done with a PDP 11/45 computer.

Let v: $\mathbf{R} \to \mathbf{R}$ be given by

$$v(t) = \exp\left(\tfrac{3}{4} - t\right)^{-1}\left(\tfrac{5}{4} - t\right)^{-1}, \quad \text{if } \tfrac{3}{4} < t < \tfrac{5}{4},$$

and $v(t) = 0$, otherwise. Let us define the function g by

$$g(x) = \tfrac{9}{4}x + a\int_0^x \int_0^t v(s)\,ds\,dt \quad \text{if } x \geqslant 0$$

$$g(x) = -g(-x) \quad \text{if } x < 0$$

where

$$a = 4\left(\int_0^2 v(s)\,ds\right)^{-1}.$$

We have immediately that the odd function g is of class C^{∞}, and simple calculations show that

$$xg(x) > 0, \qquad xg''(x) \geqslant 0, \quad \text{for } x \neq 0, \qquad g'(0) = 9/4,$$

and

$$\lim_{x \to \infty} g'(x) = 25/4.$$

Furthermore, it can be shown by means of more extensive numerical computations (using Adams' methods—see [5], for example) that, for $E = 9.152820$, we have

$$\dot{p}(\pi/2) \cong 0, \qquad q(\pi/2) \cong -0.26018.$$

Therefore $E_0 \cong 9.152820$ and $E_0 \neq E_1$.

The coefficients of the expansion (3.5) and (3.7) are given by

$$B_1 \cong 0.356, \qquad B_2 \cong 3.836, \qquad B_4 \cong 10.367.$$

So, all of our hypotheses are true for this function g.

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