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ASYMPTOTIC SOLUTIONS FOR FINITE DEFORMATION OF THIN SHELLS OF REVOLUTION WITH A SMALL CIRCULAR HOLE*

By

HUBERTUS J. WEINITSCHKE (*University of Erlangen, West Germany*)

AND

CHARLES G. LANGE (*University of California, Los Angeles*)

Abstract. The method of matched asymptotic expansions is used to describe the finite deformation of thin shells of revolution with a small circular hole at the apex. The loading is assumed to be a rotationally symmetric, smoothly varying normal pressure. The mathematical problem is of singular perturbation type characterized by a boundary layer region at the inner edge of the small hole. The analytical results are compared with numerical approximations, and formulas for the stress concentration factors at the hole are presented.

1. Introduction. In the field of linear and nonlinear elasticity of thin structures, many important problems whose solutions are of considerable complexity can be analyzed by boundary layer methods. This involves a small parameter δ, which is usually related to the ratio of the shell thickness h to a shell length L such as the radius of a spherical shell. In the present paper, we discuss nonlinear shell problems where the small parameter ε is the ratio of the radius of a small hole at the apex of a shell of revolution to the radius of the outer edge of the shell. Accordingly, we find that the boundary layer structure is quite different from the one encountered in problems where the small parameter is given by $\delta = h/L$.

Suppose a shell of revolution is subjected to a rotationally symmetric, smoothly varying normal pressure, so that the solution, without the hole, would generally be slowly varying throughout the shell, except possibly near the outer edge (depending on the type of edge support). However, with a small hole at the apex, assumed free of radial edge traction and bending moment, the stress will generally change sharply near the hole, while away from the hole one would expect the solution to be close to the solution without a hole. This

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behavior is indeed confirmed by numerical solutions of the nonlinear shell equations, which are increasingly difficult and costly to obtain as ε becomes small.

We shall analyze this problem by singular perturbation techniques, which yield asymptotic solutions for small ε with little computing effort. In fact, the only nonlinear boundary value problem which must be solved numerically is the corresponding problem without the hole.

In a previous paper [1], the simpler problem of a flat circular elastic membrane with a small hole at the center was solved by a similar asymptotic analysis. In the case of bending of a thin circular plate of thickness h, one might expect the behavior of the membrane stresses near a small hole to be similar to that found for the limit case of a membrane ($h \to 0$). However, it will be seen that this is true only for the leading order term of the radial stress. The solution behavior near the hole is more complicated in plates and shells because of the presence of terms $\varepsilon^m \log^n \varepsilon$, m and n integers, in the asymptotic expansion, which do not appear in the membrane problem. Apart from the paper just quoted, the possibility of an asymptotic analysis of nonlinear shell problems involving small holes does not seem to have been noted in previous work.

It was observed in [1] that the present problems are different from the layer problems for membranes and shells. The boundary layer here is associated with the singular behavior of the solution at the inner edge when $\varepsilon = 0$. There is no reduction of order of the differential equations in the limit case $\varepsilon = 0$ in our problem.

2. Formulation of the problem. The asymptotic integration technique to be described applies to axisymmetric finite deformation of general thin shells of revolution with a small circular hole at the apex, provided the corresponding shell without a hole has a horizontal tangent at the apex. The basic equations under the assumption of small strain were formulated by Reissner in [2]. In order to explain the asymptotic integration technique in the simplest setting, and to exhibit the essential structure of the inner and outer solution, it will be sufficient to restrict our analysis to a shallow spherical shell, retaining only quadratic nonlinear terms in the basic equations. The extension to arbitrary shells of revolution and large rotation will then be seen to be quite straightforward, as long as buckling is excluded. This point will be discussed in the last section.

Consider a shallow spherical shell with a hole at the apex, subject to a normal surface load $p(r)$. The central circular hole of radius r_i is assumed to be free of traction and moment; that is, both the stress resultant N_r and the bending moment M_r vanish at $r = r_i$. At the outer edge $r = a$ various boundary conditions may be prescribed, such as clamped or simply supported edge conditions, or a radial tension and/or a radial moment. The basic equations given in [2] may be reduced to a set of two coupled second-order nonlinear differential equations relating to each other a dimensionless midsurface slope f and a dimensionless radial stress resultant g. They can be written in the form [3]

$$Lf = -\mu^2 g + fg + 2\gamma R(x, \varepsilon), \qquad x = r/a,$$
$$Lg = \mu^2 f - \frac{1}{2} f^2, \qquad Ly := y'' + (3/x) y', \tag{2.1}$$

where

$$\mu^2 = 2m\frac{H}{h}, \qquad m^2 = 12(1-\nu^2), \qquad \gamma = \frac{m^3 a^4}{4Eh^4}p_0, \qquad R(x,\varepsilon) = \frac{2}{x^2}\int_\varepsilon^x t\bar{p}(t)\,dt,$$

$$p_0 = \max_{[0,a]}|p(r)|, \qquad p(r) = p_0\bar{p}(x), \qquad |\bar{p}(x)| \leqslant 1,$$

h is the shell thickness, E is Young's modulus, ν is Poisson's ratio, and H is the height of the shell center (apex) above a horizontal plane through the edge.

The boundary conditions at the hole $r = r_i$ are

$$\varepsilon f'(\varepsilon) + (1+\nu)f(\varepsilon) = 0, \qquad g(\varepsilon) = 0, \qquad \varepsilon = r_1/a. \tag{2.2}$$

At a clamped outer edge $r = a$, we have

$$f(1) = 0, \qquad g'(1) + (1-\nu)g(1) = 0. \tag{2.3}$$

For a simply supported edge the boundary conditions are

$$f'(1) + (1+\nu)f(1) = 0, \qquad g'(1) + (1-\nu)g(1) = 0. \tag{2.4}$$

The condition on f may be replaced by $f'(1) + (1+\nu)f(1) = m_r$ if a radial moment is prescribed; similarly, one has $g(1) = n_r$ if a radial traction is applied at the edge $r = a$. In the case of uniform load $p = \text{const}$, we have $\bar{p} = 1$ and therefore

$$R(x,\varepsilon) = 1 - \frac{\varepsilon^2}{x^2}. \tag{2.5}$$

We remark that in the corresponding shell problem without hole the conditions (2.2) are absent. In this case we have (2.1) with $R(x,0)$, (2.3) or (2.4), and the regularity (symmetry) conditions

$$f'(0) = 0, \qquad g'(0) = 0. \tag{2.6}$$

The existence of solutions of this boundary value problem has been proved by Wagner [4] for arbitrary γ and μ, if $\bar{p}(x)$ is piecewise continuous in $[0,1]$. On the other hand, no existence or uniqueness results are known to the authors for the boundary value problem (2.1)–(2.3) (or (2.4)) for $\varepsilon > 0$, which is considered in this paper. Henceforth we shall refer to this problem as the annular shell problem, briefly problem A.

For small ε, numerical solutions of problem A show that $f(x)$ and $g(x)$ rise from their values at $x = \varepsilon$ to significantly larger values within a layer of order ε, with steep gradients in the layer, but varying slowly in the remaining part of the x-interval except for possible boundary layers at the outer edge $x = 1$. The latter will not concern us here. In fact, outside the layer the solutions of problem A for sufficiently small ε are close to the solution of the problem without a hole, subject to the same loading and outer edge conditions. The solution behavior near $x = \varepsilon$ is apparently due to the form of $R(x,\varepsilon)$. For instance, if R is given by (2.5), R rises from 0 to $1-\varepsilon$ in the interval $[\varepsilon, \sqrt{\varepsilon}]$, outside of which $R = O(1)$. Clearly, numerical solutions of problem A become increasingly difficult and costly for decreasing values of ε, because a large number of mesh points are needed in any discrete approximation to f and g in order to cope with the large gradients in the boundary layer. Our asymptotic analysis deals with this situation in a very simple way.

It will be seen that the main features of the boundary layer solution can be derived by studying in some detail the case of a flat annular plate $\mu = 0$, subjected to a uniform load, so that $R = 1 - (\varepsilon/x)^2$. Furthermore, we shall consider the clamped edge boundary conditions (2.3). The extension to more general cases will be discussed in the last section.

3. An exactly solvable linear problem. In order to find the correct form of the inner and outer expansions and to see the nature of their interrelation, we treat a simple model problem which will show the essential features of the asymptotic solution for problem A. The problem to be considered is

$$y'' + \frac{3}{x}y' = 2\gamma\left(1 - \frac{\varepsilon^2}{x^2}\right), \qquad y(\varepsilon) = 0,\ y(1) = 1, \tag{3.1}$$

which has the exact solution

$$y(x) = c_1 + \frac{c_2}{x^2} + \frac{\gamma}{4}x^2 + \gamma\varepsilon^2\left(\frac{1}{2} - \log x\right),$$

$$c_1 = 1 - c_2 - \gamma\left(\frac{1}{4} + \frac{1}{2}\varepsilon^2\right) = 1 - \frac{\gamma}{4} + O(\varepsilon^2), \qquad c_2 = \varepsilon^2\left(\frac{\gamma}{4} - \frac{1 - \gamma\varepsilon^2\log\varepsilon}{1 - \varepsilon^2}\right) = O(\varepsilon^2). \tag{3.2}$$

In the limit case $\varepsilon = 0$, we have $xy'' + 3y' = 2\gamma x$, $y(0) = 0$, $y(1) = 1$. However, the limit of the exact solution for $\varepsilon \to 0$ is

$$y(x) = 1 + (\gamma/4)(x^2 - 1), \tag{3.3}$$

which does not satisfy $y(0) = 0$, except in the special case $\gamma = 4$. Thus, in general, we have a boundary layer at $x = \varepsilon$. The behavior of the solution in the boundary layer near $x = \varepsilon$ is analyzed by the stretching transformation $s = (x/\varepsilon) - 1$, transforming (3.1) into a differential equation for $Y(s) := y(\varepsilon(1 + s))$, with $Y(0) = 0$. Assuming an inner solution $Y(s, \varepsilon) = Y_0(s) + \varepsilon^2 Y_1(s) + \cdots$, one easily finds

$$Y(s, \varepsilon) = C_0 h(s) + \varepsilon^2\left\{C_1 h(s) - \frac{3\gamma}{4(1 + s)^2} + \frac{\gamma}{4}(1 + s)^2 + \gamma\left[\frac{1}{2} - \log(1 + s)\right]\right\} + \cdots, \tag{3.4}$$

where $h(s) = 1 - (s + 1)^{-2}$ and C_0, C_1 are constants to be determined by matching $Y(s, \varepsilon)$ with an outer solution $y(x, \varepsilon) = y_0(x) + \varepsilon^2 y_1(x) + \cdots$. Substituting that expansion into (3.1), y_0 and y_1 can be computed, satisfying $y_0(1) = 1$, $y_1(1) = 0$. The result is

$$y(x, \varepsilon) = C\left(\frac{1}{x^2} - 1\right) + 1 + \frac{\gamma}{4}(x^2 - 1) + \varepsilon^2\left\{D\left(\frac{1}{x^2} - 1\right) - \gamma\log x\right\} + \cdots. \tag{3.5}$$

The inner and outer solutions (3.4) and (3.5) are now matched by the intermediate variable method [5]. We set $x = \eta\delta(\varepsilon)$ with $\delta \to 0$ and $\varepsilon/\delta \to 0$ as $\varepsilon \to 0$ and express $Y(s,\varepsilon)$ and $y(x, \varepsilon)$ in terms of η. Inspection shows that all terms can be matched by setting $C = 0$, $C_0 = 1 - \gamma/4$, $D = -C_0$, and $C_1 = 1 - 3\gamma/4$, except the term $\gamma\varepsilon^2\log\varepsilon$ (terms involving $\log\eta$ and terms of order $O(\delta^2)$ and $O(\varepsilon^2\log\delta)$ cancel). This term, which comes from $Y_1(s)$, can be matched if we modify the inner expansion by including a term $(\varepsilon^2 \log \varepsilon)\overline{Y}(s)$, which yields $\overline{Y}(s) = \overline{C}h(s)$. Setting $\overline{C} + \gamma = 0$ will then cancel the term in question. In this way, all terms in (3.4) and (3.5) match, except terms of order ε^4/δ^2 and $(\varepsilon^4/\delta^2)\log\varepsilon$, which can be shown to match with higher-order terms not displayed in the above formulas.

Inserting $\overline{Y}(s)$ and the constants C, D, and C_i, found by the above matching, into (3.4) and (3.5) and returning to the original variable x, it is seen that $y(x, \varepsilon) = Y(s, \varepsilon)$ up to terms of order $O(\varepsilon^4)$ and that $y(x, \varepsilon)$ is identical with the exact solution (3.2) up to $O(\varepsilon^4)$ terms, when c_1 and c_2 are expanded in powers of ε^2. Furthermore, we observe that there should be no term of order $\varepsilon^2 \log \varepsilon$ in the outer solution, because this would give rise to a term $\bar{c}(1 - x^{-2})\varepsilon^2 \log \varepsilon$ in $y(x, \varepsilon)$, which would not match with the inner solution, as there is no term of order $(\varepsilon^2/\delta^2)\log\varepsilon$ in $Y(s, \varepsilon)$. On the other hand, there will be a term of order $\varepsilon^4 \log \varepsilon$ in the outer expansion, as can be seen from the term c_2/x^2 of the exact solution.

4. The inner solution. We now return to problem A. Introducing the stretching transformation $s = (x/\varepsilon) - 1$, the differential equations (2.1) for $F(s) := f(\varepsilon(1 + s))$, $G(s) := g(\varepsilon(1 + s))$ become

$$\ddot{F} + \frac{3}{1+s}\dot{F} = \varepsilon^2\left\{ FG + 2\gamma\left(1 - \frac{1}{(1+s)^2}\right)\right\},$$

$$\ddot{G} + \frac{3}{1+s}\dot{G} = -\frac{1}{2}\varepsilon^2 F^2, \tag{4.1}$$

where we have set $\mu = 0$ and $R = 1 - (\varepsilon/x)^2$, the dot denoting d/ds. The boundary conditions (2.2) at the hole transform into

$$\dot{F}(0) + (1 + \nu)F(0) = 0, \qquad G(0) = 0. \tag{4.2}$$

We find that the correct inner asymptotic expansion is of the same form as in the linear model problem of Sec. 3. Therefore, we have for $Z := (F, G)$,

$$Z(s, \varepsilon) = Z_0(s) + \varepsilon^2(\log\varepsilon)Z_1(s) + \varepsilon^2 Z_2(s) + \varepsilon^4(\log\varepsilon)^2 Z_3(s)$$
$$+ \varepsilon^4(\log\varepsilon)Z_4(s) + \varepsilon^4 Z_5(s) + \cdots. \tag{4.3}$$

Substitution of (4.3) into (4.1) and (4.2) yields

$$KF_0 := \ddot{F}_0 + \frac{3}{1+s}\dot{F}_0 = 0, \qquad KG_0 = KF_1 = KG_1 = 0,$$

$$KF_2 = F_0 G_0 + 2\gamma\left(1 - \frac{1}{(1+s)^2}\right), \qquad KG_2 = -\frac{1}{2}F_0^2, \cdots, \tag{4.4}$$

$$\dot{F}_j(0) + (1 + \nu)F_j(0) = G_j(0) = 0, \qquad j = 0, 1, 2, \ldots. \tag{4.5}$$

Thus we obtain

$$F_0(s) = B_0\left(h_0(s) - \frac{2}{1+\nu}\right), \qquad G_0(s) = C_0 h_0(s), \qquad h_0(s) := 1 - \frac{1}{(1+s)^2},$$

$$F_1(s) = B_1\left(h_0(s) - \frac{2}{1+\nu}\right), \qquad G_1(s) = C_1 h_0(s). \tag{4.6}$$

The constants B_0, C_0 and B_1, C_1 are to be found by matching $F(s)$, $G(s)$ with an outer expansion of f, g. Next the solution of the (inhomogeneous) equations for F_2, G_2 is determined. We find, up to constants of integration B_2, C_2,

$$F_2(s) = B_2\left(h_0(s) - \frac{2}{1+\nu}\right) + p_0 + p_1 h_2(s) + p_2 h_3(s), \tag{4.7}$$

$$G_2(s) = C_2 h_0(s) + p_3 h_1(s) + p_4 h_2(s) + p_5 h_3(s) - \tfrac{3}{4}(p_4 + p_5), \tag{4.8}$$

where (4.5) has been used for $j = 2$, and where

$$\begin{aligned} h_1(s) &:= (1+s)^2 - 1/(1+s)^2, \\ h_2(s) &:= \tfrac{1}{4}(1+s)^2 + \tfrac{1}{2} - \log(1+s), \\ h_3(s) &:= \tfrac{1}{4}(1+s)^2 + 1 - 2\log(1+s) - \frac{1}{(1+s)^2}\left[\tfrac{1}{2} + \log(1+s)\right], \end{aligned} \tag{4.9}$$

$$p_1 = \gamma - \frac{1}{1+\nu} B_0 C_0, \qquad p_2 = \tfrac{1}{2} B_0 C_0, \qquad p_3 = -\left[\frac{B_0}{2(1+\nu)}\right]^2,$$

$$p_4 = \frac{1}{1+\nu} B_0^2, \qquad p_5 = -\tfrac{1}{4} B_0^2, \qquad p_0 = \frac{p_1 + 3p_2}{2(1+\nu)} - \tfrac{3}{4}(p_1 + p_2).$$

We observe a significant difference between F_2, G_2 and the corresponding terms of order $O(\varepsilon^2)$ in the solution for the annular membrane problem [1]: there are no logarithmic terms in the latter problem. In fact, it is precisely these terms that force us to include terms of order $\varepsilon^2 \log \varepsilon$ in the inner expansion, as explained in the previous section. Similarly, $\log^2(1+s)$ terms in $F_5(s)$ and $G_5(s)$ give rise to the terms of orders $\varepsilon^4 \log^2 \varepsilon$ and $\varepsilon^4 \log \varepsilon$ in (4.3).

It should be noted that the inner solution does not decay exponentially. However, all algebraically decaying terms can be matched.

5. The outer solution. Recalling the remarks at the end of Sec. 3 concerning the model problem, we assume an outer solution for $z = (f, g)$, valid away from the boundary layer, in the form

$$z(x, \varepsilon) = z_0(x) + \varepsilon^2 z_1(x) + \varepsilon^4(\log\varepsilon) z_2(x) + \varepsilon^4 z_3(x) + \cdots. \tag{5.1}$$

Substitution of (5.1) into (2.1) yields for the first few terms of the series, recalling that we take $\mu = 0$ and $R = 1 - (\varepsilon/x)^2$,

$$Lf_0 = f_0 g_0 + 2\gamma, \qquad Lg_0 = -\tfrac{1}{2} f_0^2, \tag{5.2}$$

$$Lf_1 - f_0 g_1 - g_0 f_1 = -(2/x^2)\gamma, \qquad Lg_1 + f_0 g_1 = 0 \tag{5.3}$$

$$Lf_2 = Lg_2 = 0, \tag{5.4}$$

$$Lf_3 - f_0 g_3 - g_0 f_3 = f_1 g_1, \qquad Lg_3 + f_0 f_3 = -\tfrac{1}{2} f_1^2. \tag{5.5}$$

All functions f_k, g_k must satisfy the boundary conditions (2.3). Equations (5.2) and (2.3) describe the *circular* plate problem (without hole), clamped at the edge $x = 1$, provided the boundary conditions at $x = 0$ turn out to be (2.6). On physical grounds, we should certainly expect this, as the effect of a small hole on the stress distribution should be a local one. In the limit case of a membrane this has been verified in [1]. Anticipating the same situation for the plate problem, the nonlinear boundary value problem for $f_0(x)$, $g_0(x)$ can then be solved numerically. As the solution is slowly varying for $x \in [0, 1]$, at least for moderately large values of γ, there are no numerical difficulties getting a sufficiently accurate solution f_0, g_0 by any standard computer software such as COLSYS [6].

In the remaining part of this section, we construct the solution $f_1(x)$, $g_1(x)$ satisfying (5.3) and (2.3). This solution is singular at $x = 0$, but the appropriate type of singularity is not difficult to find, as f_1 and g_1 must match with the inner solution in an overlap domain.

A particular solution of (5.3) should be of the form $A_1(x)\log x + A_2(x)$, where $A_i(x)$ are regular at $x = 0$, and hence can be written as power series in the form $\Sigma_0^\infty a_n x^{2n}$. Calculating a few coefficients indicates that a particular solution y_p, z_p of (5.3) should be sought by the ansatz

$$\begin{aligned} y_p(x) &= -\gamma(\log x)\left[1 + \tfrac{1}{8}x^2\tilde{y}_1(x)\right] - \gamma x^2\tilde{y}_2(x), \\ z_p(x) &= \tfrac{1}{8}\gamma x^2(\log x)\tilde{z}_1(x) + \gamma x^2\tilde{z}_2(x), \end{aligned} \tag{5.6}$$

where $\tilde{y}_i$ and $\tilde{z}_i$ are analytic in [0, 1], satisfying

$$\tilde{y}_1'(0) = \tilde{y}_2'(0) = \tilde{z}_1'(0) = \tilde{z}_2'(0) = 0. \tag{5.7}$$

Substituting (5.6) into (5.3), we get

$$L_1\tilde{y}_1 = \frac{8}{x^2}g_0(x) + g_0(x)\tilde{y}_1 - f_0(x)\tilde{z}_1, \qquad L_1\tilde{z}_1 = \frac{8}{x^2}f_0(x) + f_0(x)\tilde{y}_1, \tag{5.8}$$

$$L_1\tilde{y}_2 = -L_2\tilde{y}_1 + g_0(x)\tilde{y}_2 - f_0(x)\tilde{z}_2, \qquad L_1\tilde{z}_2 = -L_2\tilde{z}_1 + f_0(x)\tilde{y}_2, \tag{5.9}$$

where

$$L_1 y := y'' + \frac{7}{x}y' + \frac{8}{x^2}y, \qquad L_2 y := \frac{1}{4x}\left(y' + \frac{3}{x}y\right).$$

Regularity at $x = 0$ requires that $(\tilde{y}_1 - g_0)/x^2$ and $(\tilde{z}_1 - f_0)/x^2$ in (5.8) are bounded in [0, 1], which implies $\tilde{y}_1(0) = g_0(0)$, $\tilde{z}_1(0) = f_0(0)$. In (5.9), the terms $(8/x^2)\tilde{y}_2$ and $(8/x^2)\tilde{z}_2$ must cancel the terms $-(3/4x^2)\tilde{y}_1$ and $-(3/4x^2)\tilde{z}_1$, respectively. Hence, we have the initial conditions

$$\begin{gathered} \tilde{y}_1(0) = g_0(0), \qquad \tilde{z}_1(0) = f_0(0), \qquad \tilde{y}_2(0) = -\tfrac{3}{32}g_0(0), \\ \tilde{z}_2(0) = -\tfrac{3}{32}f_0(0), \end{gathered} \tag{5.10}$$

in addition to (5.7). The *linear* initial value problem (5.8), (5.9), (5.7), and (5.10) for $\tilde{y}_1$, $\tilde{z}_1$, $\tilde{y}_2$, $\tilde{z}_2$ is easily solved numerically by standard computer software. Alternatively, it may be solved exactly by power series, but we omit writing down the recurrence relations. The latter approach assumes that $f_0(x)$, $g_0(x)$ have also been computed in terms of power series. Note that $\tilde{y}_1$, $\tilde{z}_1$ may be obtained from (5.8) without reference to $\tilde{y}_2$, $\tilde{z}_2$.

We proceed to construct the general solution y_h, z_h of the homogeneous equations obtained from (5.3) by dropping the term $-2\gamma/x^2$. This solution will basically have the same form as y_p, z_p except that, as in (3.5), additional terms of order $1/x^2$ are needed in order to match with the inner solution. Thus we seek $y_h(x)$, $z_h(x)$ by superposition

$$\begin{pmatrix} y_h(x) \\ z_h(x) \end{pmatrix} = \sum_{i=1}^{4} \hat{c}_i \begin{pmatrix} y_i(x) \\ z_i(x) \end{pmatrix}, \tag{5.11}$$

where the functions y_i and z_i are of the form

$$y = A(x)\log x + \frac{\alpha}{x^2} + C(x), \qquad z = B(x)\log x + \frac{\beta}{x^2} + D(x). \tag{5.12}$$

Two of the constants $\hat{c}_i$ are to be determined from the boundary conditions at $x = 1$, the remaining two in the matching process. The functions A, B, C, D are assumed to be analytic in [0, 1]. From the symmetry of the differential equations we again have

$$A'(0) = B'(0) = C'(0) = D'(0) = 0. \tag{5.13}$$

Substitution of (5.12) into the homogeneous equations of (5.3) gives

$$LA = g_0(x)A + f_0(x)B, \qquad LB = -f_0(x)A, \tag{5.14}$$

$$\begin{gathered} LC = g_0(x)C + f_0(x)D - 2\left(\frac{A'}{x} + \frac{A}{x^2}\right) + \frac{1}{x^2}\left[\alpha g_0(x) + \beta f_0(x)\right], \\ LD = -f_0(x)C - 2\left(\frac{B'}{x} + \frac{B}{x^2}\right) - \frac{\alpha}{x^2} f_0(x). \end{gathered} \tag{5.15}$$

For reasons of regularity of the right-hand sides of (5.15) at $x = 0$, we must stipulate

$$\lim_{x\to 0}\left(-2A(x) + \alpha g_0(x) + \beta f_0(x)\right) = 0, \qquad \lim_{x\to 0}\left(-2B(x) - \alpha f_0(x)\right) = 0,$$

which determines α and β as follows:

$$\alpha = -\frac{2}{f_0(0)}B(0), \qquad \beta = \frac{2}{(f_0(0))^2}\left[A(0)f_0(0) + B(0)g_0(0)\right], \tag{5.16}$$

if $f_0(0) \neq 0$. A solution basis $(y_i(x), z_i(x))$, $i = 1, \ldots, 4$, is obtained from (5.13)–(5.16) by choosing $(A(0), B(0), C(0), D(0))$ to be the standard unit base vectors $\mathbf{e}_i$ of $\mathbf{R}^4$, that is, $\mathbf{e}_1 = (1, 0, 0, 0), \ldots, \mathbf{e}_4 = (0, 0, 0, 1)$. We shall denote the solutions of these four linear initial value problems by $A_i(x), \ldots, D_i(x)$, with corresponding values α_i, β_i, $i = 1, 2, 3, 4$. From (5.16) we have

$$\begin{gathered} \alpha_1 = 0, \qquad \beta_1 = \frac{2}{f_0(0)}, \qquad \alpha_2 = -\beta_1, \qquad \beta_2 = \frac{2g_0(0)}{(f_0(0))^2}, \\ \alpha_3 = \beta_3 = \alpha_4 = \beta_4 = 0. \end{gathered} \tag{5.17}$$

Furthermore, we have $A_j(x) = B_j(x) \equiv 0$ for $j = 3$ and $j = 4$, which implies, together with (5.17),

$$C_3(x) = A_1(x), \qquad D_3(x) = B_1(x), \qquad C_4(x) = A_2(x), \qquad D_4(x) = B_2(x),$$

which amounts to a substantial reduction in the calculations. Imposing the boundary condition (2.3) on $f_1 = y_p + y_h$ and $g_1 = z_p + z_h$ yields

$$\begin{gathered} \hat{c}_3 y_3(1) + \hat{c}_4 y_4(1) = \gamma y_2(1) - \hat{c}_1 \tilde{y}_1(1) - \hat{c}\tilde{y}_2(0), \\ \hat{c}_3(Mz_3)(1) + \hat{c}_4(Mz_4)(1) = -(Mz_p)(1) - \hat{c}_1(Mz_1)(1) - \hat{c}_2(Mz_2)(1), \end{gathered} \tag{5.18}$$

where

$$y_j(1) = \alpha_j + C_j(1), \qquad z_j(1) = \beta_j + D_j(1),$$

$$(Mz)(x) = z'(x) + (1-\nu)z(x), \qquad z_j'(1) = B_j(1) - 2\beta_j + D_j'(1).$$

The constants $\hat{c}_1$, $\hat{c}_2$ will be determined in the next section. Once they are known, the outer solution can be completed by solving (5.18) for $\hat{c}_3$ and $\hat{c}_4$.

6. Matching of inner and outer solutions. As in Sec. 3, we introduce an intermediate variable $\eta = x/\delta(\varepsilon)$, where $\eta = O(1)$, $\delta(\varepsilon) \to 0$, and $\varepsilon/\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. The inner and outer solutions will now be matched in a common domain of validity. As the method was described in Sec. 3, we shall not carry out the procedure here in detail. Replacing x by $\eta\delta$ in the outer solution and Taylor expanding the regular functions f_0, g_0, $\tilde{y}_i$, $\tilde{z}_i$, etc. at the origin, we obtain

$$\begin{aligned} f(x,\varepsilon) = {} & f_0(0) + \delta\eta f_0'(0) + \tfrac{1}{2}\delta^2\eta^2 f_0''(0) + \cdots \\ & + \varepsilon^2\left\{\gamma\ln(\delta\eta)\left[-1 - \tfrac{1}{8}\delta^2\eta^2\tilde{y}_1(0) - \tfrac{1}{16}\delta^4\eta^4\tilde{y}_1''(0) - \cdots\right]\right. \\ & \left. - \gamma\delta^2\eta^2\tilde{y}_2(0) - \tfrac{1}{2}\gamma\delta^4\eta^4\tilde{y}_2''(0) - \cdots\right\} \\ & + \varepsilon^2\left\{\sum_{i=1}^{4}\hat{c}_i\left[(\log\delta\eta)\left(A_i(0) + \tfrac{1}{2}\delta^2\eta^2A_i''(0) + \cdots\right) + \frac{\alpha_i}{\delta^2\eta^2}\right.\right. \\ & \left.\left. + C_i(0) + \tfrac{1}{2}\delta^2\eta^2C_i''(0) + \cdots\right]\right\} + \cdots \end{aligned} \tag{6.1f}$$

and a similar expression (6.1g) for $g(x,\varepsilon)$. Next substitute $s+1 = \delta\eta/\varepsilon$ into

$$\begin{pmatrix} F(s,\varepsilon) \\ G(s,\varepsilon) \end{pmatrix} = \begin{pmatrix} F_0(s) \\ G_0(s) \end{pmatrix} + (\varepsilon^2\log\varepsilon)\begin{pmatrix} F_1(s) \\ G_1(s) \end{pmatrix} + \varepsilon^2\begin{pmatrix} F_2(s) \\ G_2(s) \end{pmatrix} + \cdots. \tag{6.2}$$

As the formulas for $F_0(s), \ldots, G_2(s)$ are given explicitly in (4.6)–(4.9), we do not rewrite them here in terms of η. Equating the terms of order unity, δ, δ^2, ε^2/δ^2, etc. in (6.1) and (6.2), we find, in succession

$$O(1): \quad f_0(0) = B_0\frac{\nu-1}{\nu+1}, \qquad g_0(0) = C_0, \tag{6.3}$$

$$O(\delta): \quad f_0'(0) = 0, \qquad g_0'(0) = 0, \tag{6.4}$$

$$O(\delta^2): \quad f_0''(0) = \tfrac{1}{2}(p_1 + p_2), \quad g_0''(0) = 2p_3 + \tfrac{1}{2}(p_4 + p_5), \tag{6.5}$$

$$O(\varepsilon^2/\delta^2): \quad \sum_{i=1}^{4}\hat{c}_i\alpha_i = -\frac{2}{f_0(0)}\hat{c}_2 = -B_0, \tag{6.6}$$

$$\sum_{i=1}^{4}\hat{c}_i\beta_i = \frac{2}{(f_0(0))^2}\left(f_0(0)\hat{c}_1 + g_0(0)\hat{c}_2\right) = -C_0,$$

$$O(\varepsilon^2\log\delta): \quad -\gamma + \sum_{i=1}^{4}\hat{c}_iA_i(0) = -\gamma + \hat{c}_1 = -p_1 - 2p_2, \tag{6.7}$$

$$\sum_{i=1}^{4}\hat{c}_iB_i(0) = \hat{c}_2 = -p_4 - 2p_5,$$

$$O(\varepsilon^2): \qquad (\log\eta)\left[-\gamma + \sum_{i=1}^{4} \hat{c}_i A_i(0)\right] + \sum_{i=1}^{4} \hat{c}_i C_i(0)$$
$$= B_2\frac{\nu-1}{\nu+1} + p_1\left(\frac{1}{2} - \log\eta\right) + p_2(1 - 2\log\eta) + p_0, \tag{6.8}$$
$$(\log\eta)\left[\sum_{i=1}^{4} \hat{c}_i B_i(0)\right] + \sum_{i=1}^{4} \hat{c}_i D_i(0)$$
$$= C_2 + p_4\left(\frac{1}{2} - \log\eta\right) + p_5(1 - 2\log\eta) - \frac{3}{4}(p_4 + p_5),$$
$$O(\varepsilon^2\log\varepsilon): \qquad p_1 + 2p_2 + B_1\frac{\nu-1}{\nu+1} = 0, \qquad p_4 + 2p_5 + C_1 = 0. \tag{6.9}$$

The coefficients B_0, C_0 are determined from f_0, g_0 by (6.3), while (6.4) confirms conditions (2.6) for $f_0(x)$, $g_0(x)$. From (6.6) the coefficients $\hat{c}_1$, $\hat{c}_2$ are determined; the result is

$$\hat{c}_1 = -\frac{\nu}{\nu+1}B_0C_0 = \frac{\nu}{1-\nu}f_0(0)g_0(0), \qquad \hat{c}_2 = \frac{1}{2}B_0^2\frac{\nu-1}{\nu+1} = \frac{1}{2}f_0(0)^2\frac{\nu+1}{\nu-1}. \tag{6.10}$$

With this, $\hat{c}_3$ and $\hat{c}_4$ can be calculated from (5.19). At this stage, $f_0(x)$, $g_0(x)$, $F_0(s)$, $G_0(s)$ and $f_1(x)$, $g_1(x)$ are completely determined. The constants B_2, C_2 can now be found from (6.8) provided all terms involving $\log\eta$ cancel. From (6.8), (6.10), and (3.9) we find indeed

$$\begin{aligned} -\gamma + \hat{c}_1 &= -\gamma - \frac{\nu}{1+\nu}B_0C_0 = -p_1 - 2p_2, \\ \hat{c}_2 &= \frac{1}{2}B_0^2\frac{\nu-1}{\nu+1} = -p_4 - 2p_5, \end{aligned} \tag{6.11}$$

which, at the same time, verifies (6.7). Hence we find from (6.8)

$$B_2 = \frac{1+\nu}{1-\nu}\left(p_0 + p_2 + \frac{1}{2}p_1 - \hat{c}_3\right), \qquad C_2 = \hat{c}_4 + \left(\frac{B_0}{4}\right)^2\frac{5+\nu}{1+\nu}. \tag{6.12}$$

The constants B_1, C_1 are now determined from (6.9) as

$$B_1 = \frac{1}{1-\nu}[\gamma(1+\nu) + \nu B_0C_0], \qquad C_1 = \frac{1}{2}\frac{\nu-1}{\nu+1}B_0^2 = \hat{c}_2. \tag{6.13}$$

It remains to show that (6.5) is satisfied. From (5.2), (3.9), and (2.6) we have, in the limit $x \to 0$,

$$4f_0''(0) = f_0(0)g_0(0) + 2\gamma = B_0C_0\frac{\nu-1}{\nu+1} + 2\gamma = 2(p_1 + p_2),$$
$$4g_0''(0) = -\frac{1}{2}f_0(0)^2 = -\frac{1}{2}\left(B_0\frac{\nu-1}{\nu+1}\right)^2 = 8p_3 + 2(p_4 + p_5).$$

This completes the discussion of the matching relations (6.3) to (6.9). (The reader may wish to write down the inner and outer expansions in detail and convince himself that all terms have been matched up to the order given above.) In order to match terms of order

ε^4, $(\varepsilon^4/\delta^2)\log\varepsilon$, etc. in the expressions for $F_2(s)$ and $G_2(s)$, the calculation of higher-order terms in the asymptotic expansions of both f, g and F, G must be continued beyond the point carried out in this paper.

7. Stress concentration factors. Some quantities of particular interest in applications are the stress resultant and bending moment concentration factors at the hole $r = r_i$. Let $N_{r,0}$ and $M_{r,0}$ denote the stress resultant and moment, respectively, at the apex $r = 0$ of the shell, referring to solutions of the problem without a hole under the same load conditions. In the present context of a uniformly loaded clamped annular plate, $N_{r,0}(= N_{\theta,0})$ and $M_{r,0}(= M_{\theta,0})$ simply derive from the solution $f_0(x)$, $g_0(x)$ and hence are given by $g_0(0)$ and $(1+\nu)f_0(0)$, respectively, apart from scale factors. The stress and moment concentration factors are now defined by

$$S_N = N_\theta(r_i)/N_{\theta,0}, \qquad S_M = M_\theta(r_i)/M_{\theta,0}. \tag{7.1}$$

For sufficiently small ε we may use the inner asymptotic expansion (4.3), valid in a layer of order $O(\varepsilon)$ near $x = \varepsilon$. Taking the expressions for N_θ and M_θ

$$N_\theta = \omega_1[g(x) + xg'(x)], \qquad M_\theta = \omega_2[(1+\nu)f(x) + \nu x f'(x)],$$

given in [3], with certain constants ω_i, we obtain

$$S_N = \frac{\dot G(0,\varepsilon)}{g_0(0)} = \frac{1}{g_0(0)}\left[\dot G_0(0) + \varepsilon^2(\log\varepsilon)\dot G_1(0) + \varepsilon^2\dot G_2(0) + \cdots\right],$$

$$S_M = \frac{(1+\nu)F(0,\varepsilon) + \nu\dot F(0,\varepsilon)}{(1+\nu)f_0(0)} = \frac{1-\nu}{f_0(0)}\left[F_0(0) + \varepsilon^2(\log\varepsilon)F_1(0)\right.$$
$$\left. + \varepsilon^2 F_2(0) + \cdots\right].$$

Inserting the appropriate expressions from the inner solution and simplifying, we get

$$S_N(\varepsilon) = 2\left\{1 + \frac{1}{2}(\varepsilon^2\log\varepsilon)\frac{B_0^2}{C_0}\frac{\nu-1}{\nu+1} + \frac{\varepsilon^2}{C_0}\left[C_2 + \frac{B_0^2}{2}\left(\frac{3}{8} - \frac{3+\nu}{2(1+\nu)^2}\right)\right] + \cdots\right\}, \tag{7.2}$$

$$S_M(\varepsilon) = 2\left\{1 + (\varepsilon^2\log\varepsilon)\frac{1+\nu}{1-\nu}\left(\frac{\gamma}{B_0} - C_0\frac{\nu}{1+\nu}\right)\right.$$
$$\left. + \frac{\varepsilon^2}{B_0}\left[B_2 - \frac{1}{4}\left(\gamma + B_0C_0\left(\frac{3}{2} - \frac{1}{1+\nu}\right)\right)\right] + \cdots\right\}. \tag{7.3}$$

The leading term of the stress factor S_N is the same as that derived for the annular membrane in [1]. Note that the limits of both S_N and S_M for $\varepsilon \to 0$ are independent of ν and γ. The above formulas supply explicit correction terms for small ε. In remarkable contrast to the annular membrane, the dominant $O(\varepsilon^2\log\varepsilon)$ part of these correction terms depends only on the solution f_0, g_0 for the plate problem without a hole. It is only in the $O(\varepsilon^2)$ terms that the solution f_1, g_1 is needed (as $\hat c_3$, $\hat c_4$ enter into the calculation of B_2, C_2). In the case of a membrane, a rigorous proof for $\lim_{\varepsilon\to 0} S_N(\varepsilon) = 2$ has been obtained in [7]. We have not attempted to extend the method of [7] to the present problem.

The validity of the asymptotic results obtained in this paper is limited to moderately large values of γ. From (7.2) and (7.3), this statement can be made more precise. Following [8] we set $\delta := \gamma^{-1/3}$, $I = \delta f(x)$, $J = \delta^2 g(x)$; then Eqs. (5.2) can be written as

$$\delta^2 LI = IJ + 2, \qquad LJ = -\frac{1}{2} I^2.$$

The limit case $\delta \to 0$ is the circular membrane problem, for which $IJ + 2 = 0$, whence $I(x)$ and $J(x)$ are of order $O(1)$. This will also be true for sufficiently small δ, where a bending boundary layer is found at $x = 1$ (see [8]). We conclude that $f = O(\gamma^{1/3})$ and $g = O(\gamma^{2/3})$ for large γ. In view of (6.3) this means $B_0 = O(\gamma^{1/3})$, $C_0 = O(\gamma^{2/3})$; thus the factors of the $\varepsilon^2 \log \varepsilon$ terms in (7.2) and (7.3) are $\gamma/B_0 = O(\gamma^{2/3})$ and $B_0^2/C_0 = O(1)$, respectively. Hence, the second term in (7.3) is a relatively small correction term only if we require γ and ε to satisfy

$$\gamma^{2/3}\varepsilon^2 \log \varepsilon = (\varepsilon/\delta)^2 \log \varepsilon \ll 1, \tag{7.4}$$

while it is sufficient to require $\varepsilon^2 \log \varepsilon \ll 1$ in (7.2). In other words, for fixed γ (no matter how large) our results are valid for sufficiently small values of ε, where the layer solution $G(s)$ is much more accurate than $F(s)$ (see Table 3). Since (7.4) is based on $\delta \ll 1$, it may be overly restrictive. As a numerical solution for $f_0(x)$, $g_0(x)$, for given γ, is a necessary first step in our analysis, the quantity $\gamma^{2/3}$ in (7.4) may be replaced by the more precise term $\gamma/f_0(0)$ in (7.3), and in this way it can be checked a priori whether $(\gamma/f_0(0))\varepsilon^2 \log \varepsilon$ is small enough for the asymptotic results to be sufficiently accurate for both f and g.

8. Discussion of results and generalizations. The asymptotic solution is now compared with exact (high accuracy) numerical solutions of problem A, obtained by the general boundary value problem solver COLSYS [6]. We take two representative values of γ and ε, $\gamma = 10^2$ and 10^3, $\varepsilon = 10^{-1}$ and 10^{-2}. The inner and outer solutions are calculated up to and including terms of order $O(\varepsilon^2)$, according to Sec. 4–6 ($\nu = 1/3$). Table 1 compares the numerical solution $f(x)$ with the leading term asymptotic solutions $f_0(x)$, $F_0(s)$ and the more accurate asymptotic solutions which include terms of order $O(\varepsilon^2)$, with $\varepsilon = 0.1$. Tables 2, 3, and 4 compare the exact solutions $f(x)$, $g(x)$ with the $O(\varepsilon^2)$ asymptotic

Table 1. Comparison of numerical solution $f(x)$, for $\gamma = 100$, $\varepsilon = 0.1$, with leading term (f_0, F_0) and corrected to $O(\varepsilon^2)$ outer and inner asymptotic solutions.

$\gamma = 10^2$	Numerical	Outer solution		Inner solution	
$\varepsilon = 0.1$	$f(x)$	$f_0(x)$	$f_0 + \varepsilon^2 f_1$	$F_0(s)$	$F_0 + \varepsilon^2 \log \varepsilon F_1 + \varepsilon^2 F_2$
x					
.10	-31.373	-12.214	-33.430	-36.793	-31.753
.15	-19.785	-12.148	-20.434	-23.166	-20.032
.20	-15.788	-12.053	-16.045	-18.397	-16.014
.25	-13.951	-11.923	-14.061	-16.189	-14.207
.3	-12.921	-11.754	-12.966	-14.990	-13.244
.4	-11.689	-11.272	-11.688	-13.798	-12.246
.5	-10.688	-10.544	-10.676	-13.246	-11.650
.6	-9.525	-9.488	-9.511	—	—
.8	-5.991	-6.011	-5.984	—	—
1.0	0.	0.	0.	—	—

Table 2. Comparison between numerical and asymptotic solutions for f and g, $\gamma = 100$, $\varepsilon = 0.01$.

$\gamma = 10^2$ $\varepsilon = 0.01$	Numerical soltuion		Outer solution		Inner solution	
x	f	g	f	g	F	G
.01	−36.6187	0.	−36.7190	0.0583	−36.6197	0.
.02	−18.3107	9.8171	−18.3333	9.8340	−18.3112	9.8220
.03	−14.9218	11.6256	−14.9312	11.6360	−14.9222	11.6313
.04	−13.7354	12.2513	−13.7405	12.2596	−13.7358	12.2571
.05	−13.1848	12.5341	−13.1881	12.5416	−13.1853	12.5399
.1	−12.4254	12.8464	−12.4264	12.8529	−12.4268	12.8517
.2	−12.0926	12.6574	−12.0934	12.6637	—	—
.4	−11.2765	11.6146	−11.2782	11.6206	—	—
.6	−9.4885	10.0639	−9.4911	10.0694	—	—
.8	−6.0109	8.4106	−6.0131	8.4154	—	—
1.0	0.	7.1541	0.	7.1582	—	—

Table 3. Same as in Table 2 except that $\gamma = 1000$, $\varepsilon = 0.1$.

$\gamma = 10^3$ $\varepsilon = 0.1$	Numerical solution		Outer solution		Inner solution	
x	f	g	f	g	F	G
.10	−50.588	0.	−50.073	3.220	(−87.709)	0.
.15	−32.231	51.501	−31.165	52.599	(−55.857)	50.995
.20	−26.634	68.656	−25.990	69.140	(−46.319)	67.358
.25	−24.630	75.953	−24.271	76.195	(−43.350)	73.918
.3	−23.905	79.335	−23.705	79.466	(−42.879)	76.650
.4	−23.738	81.182	−23.670	81.224	(−44.647)	77.453
.5	−24.020	80.001	−23.993	80.014	—	—
.6	−24.140	77.155	−24.128	77.159	—	—
.8	−20.888	68.304	−20.885	68.302	—	—
1.0	0.	58.752	0.	58.751	—	—

Table 4. Same as in Table 2 except that $\gamma = 1000$, $\varepsilon = 0.01$.

$\gamma = 10^3$ $\varepsilon = 0.01$	Numerical solution		Outer solution		Inner solution	
x	f	g	f	g	F	G
.01	−65.779	0.	−65.954	1.905	−64.518	0.
.02	−32.914	69.167	−32.605	70.962	−32.283	70.493
.03	−26.867	81.945	−26.486	83.724	−26.356	83.516
.04	−24.779	86.394	−24.375	88.168	−24.312	88.049
.05	−23.831	88.431	−23.419	90.204	−23.386	90.125
.1	−22.674	90.934	−22.252	92.713	−22.276	92.676
.2	−22.643	90.659	−22.225	92.473	—	—
.4	−23.379	86.924	−23.025	88.877	—	—
.6	−23.948	80.109	−23.900	82.251	—	—
.8	−20.776	70.399	−21.368	72.592	—	—
1.0	0.	60.509	0.	62.458	—	—

Table 5. Stress concentration factors: comparison between numerical and asymptotic solutions (maximal error is about 2%, excluding the value S_M for $\gamma = 10^3$, $\varepsilon = 0.1$).

		$\gamma = 10^2$, $\varepsilon = 0.1$	$\gamma = 10^2$, $\varepsilon = 0.01$	$\gamma = 10^3$, $\varepsilon = 0.1$	$\gamma = 10^3$, $\varepsilon = 0.01$
Numerical solution	S_N	2.1951	2.00780	2.0423	2.0029
	S_M	1.7054	1.99052	1.5105	1.9641
Asymptotic solution	S_N	2.2529	2.00783	2.0449	2.0029
	S_M	1.7261	1.99051	(2.6189)	1.9644

solutions for $\gamma = 10^2$ and $\gamma = 10^3$. It is seen that the asymptotic results increase in accuracy as ε becomes smaller. The agreement for $\gamma = 100$, $\varepsilon = 0.01$ is impressive. But even for $\varepsilon = 0.1$ there is remarkably good agreement between the asymptotic and the numerical solution for $\gamma = 100$. In the case $\gamma = 1000$, $\varepsilon = 0.1$ the inequality (7.4) is not satisfied as $\gamma/|f_0(0)| = 45.4$, implying that the inner solution $F(s)$ should not be used. This is borne out by the numbers for $F(x)$ in Table 3, which shows that $G(s)$ as well as the outer solution is still a good approximation. Table 5 shows the stress concentration factors discussed in Sec. 7. Again, the case $\gamma = 10^3$, $\varepsilon = 0.1$ shows a large discrepancy between numerical and asymptotic solutions, as to be expected from the preceding remarks.

The results are also displayed in Figs. 1–5. We note that the outer solution is very close to the exact solution in most cases where the asymptotic solution applies, even inside part of the boundary layer. For $\varepsilon = 0.1$, $\gamma = 100$, the contribution of the various terms of the inner solution is displayed in Fig. 2, where, in terms of s, $Z^* = (F^*, G^*) = Z_0 + (\varepsilon^2 \log \varepsilon) Z_1$.

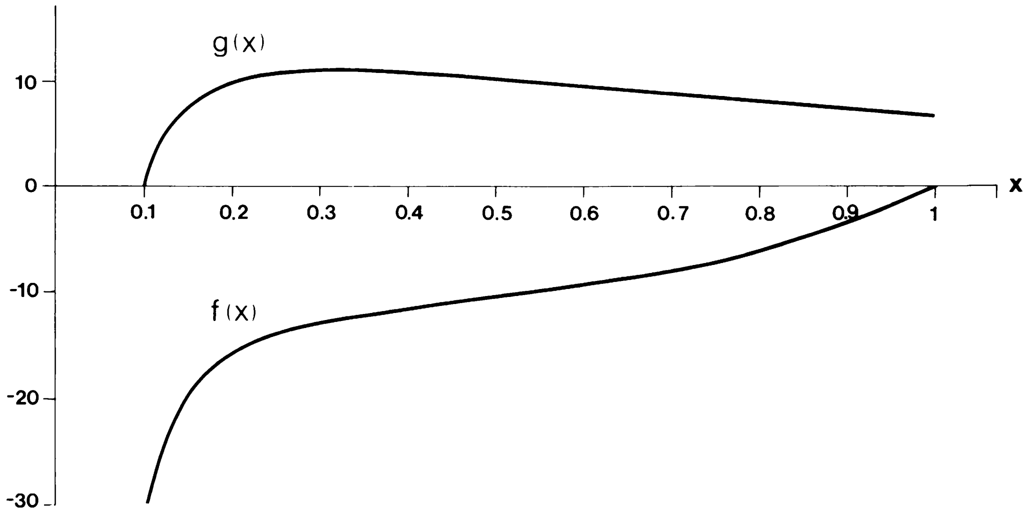

FIG. 1. Numerical solution with $\gamma = 100$, $\varepsilon = 0.1$.

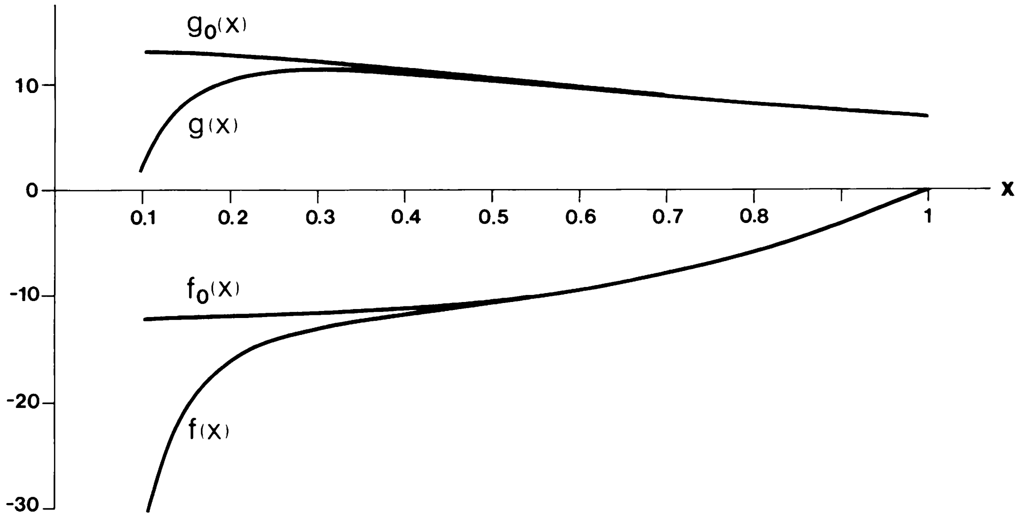

FIG. 2. Outer solution with $\gamma = 100$, $\varepsilon = 0.1$.

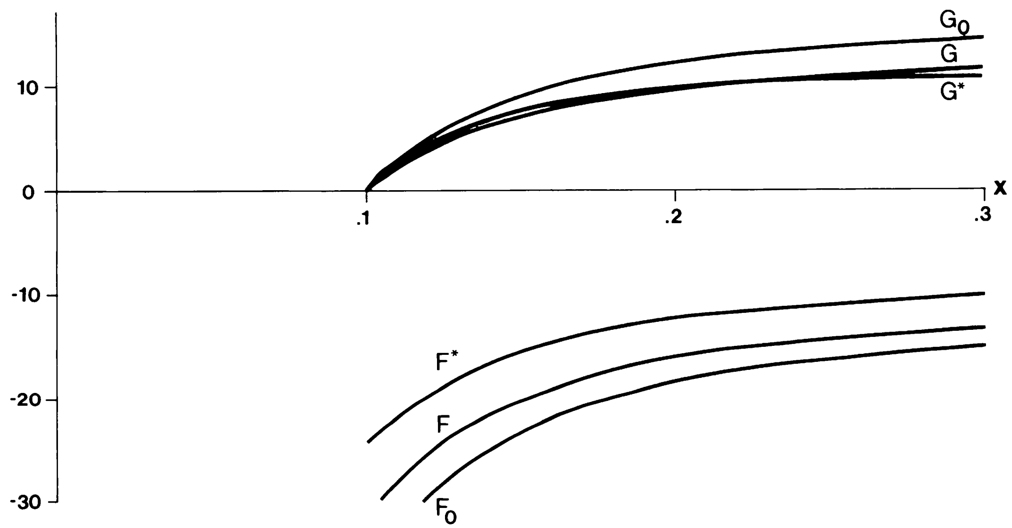

FIG. 3. Inner solution with $\gamma = 100$, $\varepsilon = 0.1$.

We next discuss briefly the modifications necessary for including spherical shells in the analysis. The terms $-\mu^2 g$ and $\mu^2 f$ in (2.1) imply the following changes in (4.4):

$$KF_2 = F_0 G_0 + \left(2\gamma - C_0\mu^2\right) h_0(s), \qquad KG_2 = -\frac{1}{2}F_0^2 + \mu^2 B_0\left(h_0(s) - \frac{2}{1+\nu}\right).$$

Hence the only change in $F_2(s)$ given by (4.7) is that γ in the coefficient p_1 must be replaced by $\gamma - \frac{1}{2}C_0\mu^2$. Similarly, the coefficients p_3, p_4 in $G_2(s)$ given by (4.8) change as follows:

$$p_3 = -\left(\frac{B_0}{2(1+\nu)}\right)^2\left[1 + \frac{\mu^2}{B_0}(1+\nu)\right], \qquad p_4 = B_0^2\left(\frac{1}{1+\nu} + \frac{\mu^2}{2B_0}\right).$$

There is no change in the functions F_0, G_0 and F_1, G_1. It is easy to see that inclusion of the terms $-\mu^2 f$, $\mu^2 g$ in the outer solution will only modify the regular parts of f_1, g_1, that is, $\tilde{y}_i$, $\tilde{z}_i$ in (5.6), and A_i, B_i, C_i, D_i in (5.11)–(5.16). The nature of the singularities ($\log x$ and x^{-2}) remains unchanged, implying some algebraic changes in the equations

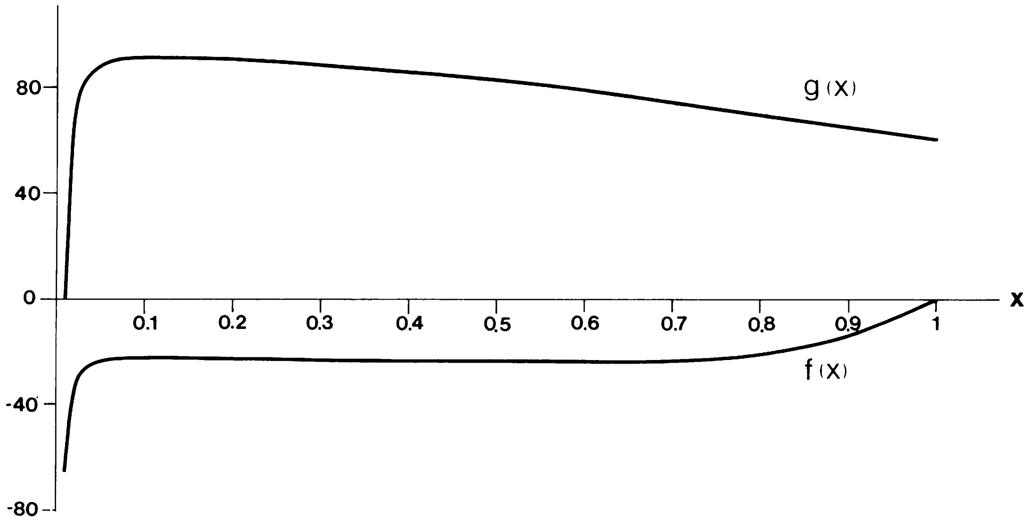

FIG. 4. Numerical solution with $\gamma = 1000$, $\varepsilon = 0.01$.

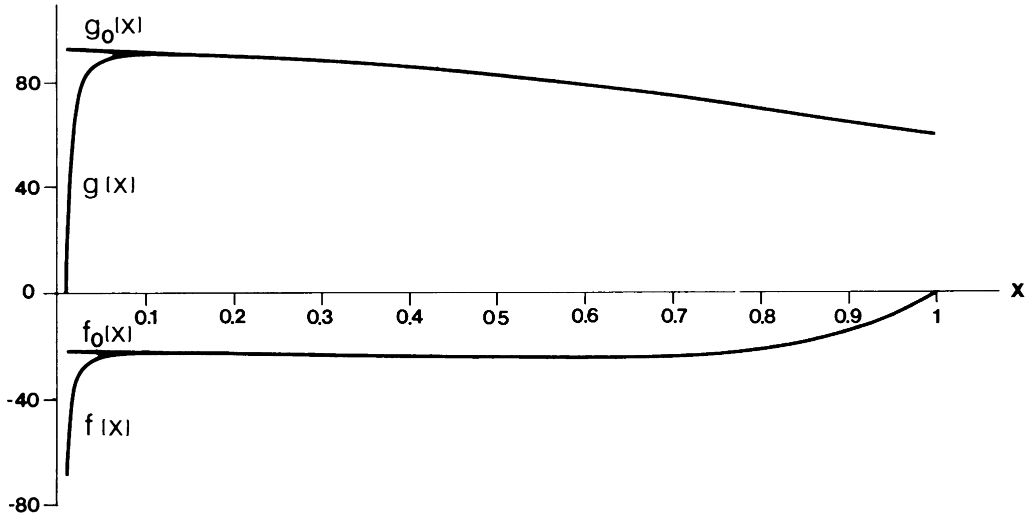

FIG. 5. Outer solution with $\gamma = 1000$, $\varepsilon = 0.01$.

(6.3)–(6.13) resulting from the matching process. The limits of the stress concentration factors as $\varepsilon \to 0$ are the same, but inclusion of the shell curvature will modify the correction terms of order $\varepsilon^2 \log \varepsilon$ in (7.2) and (7.3). Therefore, the $O(\gamma^{2/3})$ coefficient in (7.4) will also contain μ^2. But for any fixed γ and μ^2, the results will be valid for sufficiently small values of ε provided a condition like (7.4) holds.

It is clear from [1] that extension of the results to nonuniform loads is straightforward. The extension to finite rotation [2] will result in a different set of equations for the outer solution f_0, g_0. As to the inner solution, it was shown in [1] that for the annular membrane the finite rotation does not affect the solution up to and including $O(\varepsilon^2)$ terms. This situation carries over to problem A for finite rotations. Thus the layer solution obtained in Sec. 3 can be matched with an appropriate outer solution that accounts for the nonquadratic nonlinear terms of the basic equations. Similarly, it should be evident by now that an extension to arbitrary nonshallow shells of revolution will not introduce any essential novel features in the aymptotic analysis. On the other hand, the extension to nonsymmetric deformations appears to be nontrivial, as it involves a set of partial differential equations. Consequently, there will in general be a nonsymmetric stress distribution near the hole. Furthermore, we have excluded buckling from the analysis for $\mu > 0$ (symmetric snap buckling or asymmetric bifurcation buckling). This leads to an interesting new problem: given the buckling load for a shell without a hole, can the buckling load for the same structure with a small hole be calculated by a simple asymptotic analysis? We hope to return to this problem in a future paper.

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