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STABILIZATION OF LINEAR SYSTEMS BY TIME-DELAY FEEDBACK CONTROLS*

By

JIONGMIN YONG

Purdue University

Abstract. The notion of r-stabilizability of linear autonomous systems is introduced. If the system $[A, B]$ is completely controllable and the spectrum of A is contained in the left half of the complex plane then the system is r-stabilizable for any $r > 0$. Finally, a one-dimensional case is discussed.

1. Introduction. We consider the following system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1.1}$$

where A is an $(n \times n)$-matrix, B is an $(n \times m)$-matrix, $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, and $t \in \mathbf{R}$. (Throughout this paper, all the matrices and vectors are real.) It is well known that (see [11], for example) if $[A, B]$ is completely controllable, then for any set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \mathbf{C}$, with the property that

$$\Lambda^* \stackrel{\Delta}{=} \{\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n\} = \Lambda, \tag{1.2}$$

there exists an $(m \times n)$-matrix K such that

$$\sigma(A + BK) = \Lambda. \tag{1.3}$$

Here, $\sigma(A + BK)$ denotes the spectrum of $A + BK$. In particular, if we let $\Lambda \subset \mathbf{C}^- \stackrel{\Delta}{=} \{\lambda \in \mathbf{C} \mid \operatorname{Re} \lambda < 0\}$, then, under the following feedback control,

$$u(t) = Kx(t), \tag{1.4}$$

the system (1.1) is asymptotically stable.

A feedback control of the form (1.4) is not quite realistic because we usually have time delays. Thus, instead of (1.4) we should consider a feedback control of the form

$$u(t) = Kx(t - r), \tag{1.5}$$

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where $r > 0$ is a time lag. By substituting (1.5) into (1.1) we get

$$\dot{x}(t) = Ax(t) + BKx(t - r). \tag{1.6}$$

By [10], we know that, if we choose K such that the system

$$\dot{x}(t) = (A + BK)x(t) \tag{1.7}$$

is asymptotically stable, then for sufficiently small $r > 0$, (1.6) is also asymptotically stable. On the other hand, in [5] it was proved that if the system

$$\dot{x}(t) = Ax(t) + A_0x(t - r) \tag{1.8}$$

is asymptotically stable for all $r > 0$, then

$$\sigma(A) \subseteq \mathbf{C}^-. \tag{1.9}$$

Thus, if (1.9) is not satisfied, or equivalently, if the system

$$\dot{x}(t) = Ax(t) \tag{1.10}$$

is not asymptotically stable, then for any given K, the system (1.6) can not be asymptotically stable for all $r > 0$. Hence, a natural question can be asked. For any given $r > 0$, does there exist a K such that (1.6) is asymptotically stable? This is the question we will study in this paper.

2. Preliminaries. For any entire function $H(z)$, we let

$$\mathcal{N}(H) \equiv \mathcal{N}(H(\cdot)) \stackrel{\Delta}{=} \{z \in \mathbf{C} \mid H(z) = 0\}. \tag{2.1}$$

We denote a system of form (1.8) by $[A, A_0, r]$. We define

$$p(\lambda; A, A_0; r) = \det(\lambda I - A - A_0e^{-r\lambda}). \tag{2.2}$$

DEFINITION 2.1. Let $r > 0$ and $[A, B]$ be given. System $[A, B]$ is said to be r-stabilizable if there exists a K such that $[A, BK; r]$ is asymptotically stable.

It is well known that (see [3]) system $[A, A_0; r]$ is asymptotically stable if and only if

$$\mathcal{N}(p(\cdot; A, A_0; r)) \subseteq \mathbf{C}^-. \tag{2.3}$$

Let us observe the following:

$$\begin{aligned} p(\lambda; A, A_0; r) &\equiv \det(\lambda I - A - A_0e^{-r\lambda}) \\ &= \frac{1}{r^n}\det((r\lambda)I - rA - rA_0e^{-(r\lambda)}) \\ &= \frac{1}{r^n}p(r\lambda; rA, rA_0; 1). \end{aligned} \tag{2.4}$$

Thus, (2.3) is equivalent to the following:

$$\mathcal{N}(p(\cdot, rA, rA_0; 1)) \subseteq \mathbf{C}^-. \tag{2.5}$$

Hence, we get

PROPOSITION 2.2. Let $r > 0$, $[A, B]$ be given. Then, $[A, B]$ is r-stabilizable if and only if $[rA, B]$ is 1-stabilizable.

This proposition shows that we only need to consider the case that $r = 1$. Now, for $r = 1$, we have

$$\begin{aligned} p(\lambda; A, A_0; 1) &= P(\lambda) + Q(\lambda)e^{-\lambda} \\ &= e^{-\lambda}\left[e^{\lambda}P(\lambda) + Q(\lambda)\right] \equiv e^{-\lambda}H(\lambda), \end{aligned} \tag{2.6}$$

where $\deg P = n$, $\deg Q \leqslant n - 1$, and $P(\lambda)$ has the form

$$P(\lambda) = \lambda^n + \cdots. \tag{2.7}$$

It is clear that

$$\mathcal{N}(H) = \mathcal{N}(p(\cdot; A, A_0; 1)). \tag{2.8}$$

DEFINITION 2.3. An entire function $H(\lambda)$ is said to be stable if

$$\mathcal{N}(H) \subseteq \mathbf{C}^-. \tag{2.9}$$

Thus, from (2.8), we know that system $[A, A_0; 1]$ is asymptotically stable if and only if $H(\lambda)$ defined by (2.6) is stable.

We now quote some basic results, which we will use in the sequel, about the stability of some elementary transcendental functions.

Let $h(z, t)$ be a polynomial of the form

$$h(z,t) = P(z)t + Q(z), \tag{2.10}$$

where $P(z)$, $Q(z)$ are polynomials, $\deg P = n$, $\deg Q \leqslant n - 1$, and P has the form (2.7). Let

$$H(z) = h(z, e^z). \tag{2.11}$$

Then, for $z = iy$, $y \in \mathbf{R}$, we have real-valued functions $F(y)$ and $G(y)$, such that

$$H(iy) = F(y) + iG(y). \tag{2.12}$$

Then, we have the following theorem due to Pontryagin [9]. (See also [1] and [4].)

THEOREM 2.4. If $H(z)$ is stable, then all the zeros of $F(y)$ and $G(y)$ are real and alternating. Moreover,

$$G'(y)F(y) - G(y)F'(y) > 0 \tag{2.13}$$

for all $y \in \mathbf{R}$. Conversely, $H(z)$ is stable if one of the following holds.

(a) All the zeros of $F(y)$ and $G(y)$ are real and alternating and (2.13) holds for at least one point y.

(b) All the zeros of $F(y)$ are real, and for each zero $y = y_0$,

$$F'(y_0)G(y_0) < 0. \tag{2.14}$$

(c) All the zeros of $G(y)$ are real, and for each zero $y = y_0$,

$$G'(y_0)F(y_0) > 0. \tag{2.15}$$

To close this section, we make the following simple remark.

REMARK 2.5. Since $H(z)$ is entire and takes real values on the real-axis, by the Schwarz Reflection Principle, we know that $F(y)$ is an even function and $G(y)$ is an odd function.

3. Fundamental lemmas. We are going to prove two fundamental results in this section. These results will play a crucial role in proving our main result.

In this section, we always assume that $H(z)$ is of the form (2.11) and that $F(y)$ and $G(y)$ are defined by (2.12).

LEMMA 3.1. Suppose $H(z)$ is stable. Then, there exists a $c \in \mathbf{R}$ such that

$$H_1(z) \overset{\Delta}{=} zH(z) + c \tag{3.1}$$

is stable.

Proof. Since $H(z)$ is stable, by Remark 2.5, we have

$$0 \neq H(0) = F(0) + iG(0) = F(0). \tag{3.2}$$

Thus, without loss of generality, we assume $F(0) > 0$. By (3.1), we have

$$\begin{aligned} H_1(iy) &= iy(F(y) + iG(y)) + c \\ &= [-yG(y) + c] + iyF(y) \\ &= F_1(y) + iG_1(y), \end{aligned} \tag{3.3}$$

where

$$\begin{cases} F_1(y) = c - yG(y) \\ G_1(y) = yF(y). \end{cases} \tag{3.4}$$

Since $H(z)$ is stable, by Theorem 2.4, we know that all the zeros of $F(y)$ are real. Thus, all the zeros of $G_1(y)$ are real.

Now, we choose $c \in \mathbf{R}$ such that

$$0 < c < \tfrac{1}{2} \inf\{|y_0 G(y_0)| \,|\, F(y_0) = 0\}. \tag{3.5}$$

Note that the right-hand side of (3.5) is positive because it is easy to see that

$$\inf_{y \in \mathbf{R}} |H(iy)| > 0. \tag{3.6}$$

By choosing such a c, we have

$$G_1'(0)F_1(0) = F(0)c > 0. \tag{3.7}$$

Also, for any $y_0 \in \mathcal{N}(F)$, we have

$$\begin{aligned} G_1'(y_0)F_1(y_0) &= (F(y_0) + y_0 F'(y_0))(c - G(y_0)y_0) \\ &= y_0 F'(y_0)(c - y_0 G(y_0)) \\ &\geqslant -y_0^2 F'(y_0)G(y_0) - |cy_0 F'(y_0)| \\ &\geqslant -y_0^2 F'(y_0)G(y_0) - \tfrac{1}{2}|y_0 G(y_0) \cdot y_0 F'(y_0)| \\ &= -\tfrac{1}{2} y_0^2 F'(y_0) G(y_0) > 0. \end{aligned} \tag{3.8}$$

Here, we have used (2.13). Since it is clear that

$$\mathcal{N}(G_1) = \mathcal{N}(F) \cup \{0\}, \tag{3.9}$$

it follows from (c) of Theorem 2.4 that $H_1(z)$ is stable. □

From the proof, we see that c is explicitly determined by the following:

$$\begin{cases} F(0)c > 0, \\ |c| < \inf\{|y_0 G(y_0)| \mid y_0 \in \mathcal{N}(F)\}. \end{cases} \tag{3.10}$$

LEMMA 3.2. Suppose $H(z)$ is stable. Then, for any $a > 0$, there exist $b, c \in \mathbf{R}$, such that

$$H_2(z) = (z^2 + a^2)H(z) + bz + c \tag{3.11}$$

is stable.

Proof. From (3.11), we have

$$\begin{aligned} H_2(iy) &= (a^2 - y^2)H(iy) + biy + c \\ &= (a^2 - y^2)(F(y) + iG(y)) + byi + c \\ &= \left[(a^2 - y^2)F(y) + c\right] + i\left[(a^2 - y^2)G(y) + by\right] \\ &= F_2(y) + iG_2(y), \end{aligned} \tag{3.12}$$

where

$$\begin{cases} F_2(z) = (a^2 - y^2)F(y) + c \\ G_2(z) = (a^2 - y^2)G(y) + by. \end{cases} \tag{3.13}$$

Now, we have, by the stability of $H(z)$, that

$$0 \neq H(ia) = F(a) + iG(a). \tag{3.14}$$

Thus, we have two cases.

Case I. $F(a) \neq 0$. Then, we take

$$\begin{cases} c = 0 \\ bF(a) > 0 \\ |b| \leqslant \frac{1}{2} \inf\left\{ \dfrac{|a^2 - y_0^2|}{|y_0|} |G(y_0)| \mid y_0 \in \mathcal{N}(F) \right\} \stackrel{\Delta}{=} \delta_1. \end{cases} \tag{3.15}$$

By an argument similar to that used in the proof of Lemma 3.1, we have $\delta_1 > 0$. It is clear that

$$\mathcal{N}(F_2) = \mathcal{N}(F) \cup \{\pm a\}. \tag{3.16}$$

Thus, all zeros of F_2 are real. Observe that

$$F_2'(y)G_2(y) = \{(a^2 - y^2)F'(y) - 2yF(y)\}\left[(a^2 - y^2)G(y) + by\right]. \tag{3.17}$$

Hence, we have, by (3.15), that

$$F_2'(a)G_2(a) = -2aF(a) \cdot ba = -2a^2bF(a) < 0. \tag{3.18}$$

By Remark 2.5, $F(y)$ is an even function. Thus,

$$F_2'(-a)G_2(-a) = -2a^2bF(-a) = -2a^2bF(a) < 0. \tag{3.19}$$

Now, for any $y_0 \in \mathcal{N}(F)$, we have

$$\begin{aligned} F_2'(y_0)G_2(y_0) &= (a^2 - y_0^2)F'(y_0)\left[(a^2 - y_0^2)G(y_0) + by_0\right] \\ &\leqslant -(a^2 - y_0^2)^2|F'(y_0)G(y_0)| + |by_0||F'(y_0)|\,|(a^2 - y_0^2)| \\ &\leqslant -\tfrac{1}{2}(a^2 - y_0^2)^2|F'(y_0)G(y_0)| < 0. \end{aligned} \tag{3.20}$$

Here, we have used (2.13). Thus, by Theorem 2.4, part (b), we get the stability of $H_2(z)$.

Case II. $G(a) \neq 0$. Then, we take

$$\begin{cases} b = 0, \\ cG(a) < 0, \\ |c| \leqslant \frac{1}{2}\inf\left\{|a^2 - y_0^2|\,|F(y_0)|\,|\, y_0 \in \mathcal{N}(G)\right\} \stackrel{\Delta}{=} \delta_2. \end{cases} \tag{3.21}$$

An argument similar to that used above gives $\delta_2 > 0$. In this case, we have

$$\mathcal{N}(G_2) = \mathcal{N}(G) \cup \{\pm a\}. \tag{3.22}$$

We have

$$F_2(a)G_2'(a) = c(-2aG(a)) = -2acG(a) > 0. \tag{3.23}$$

Since $G(y)$ is an odd function (by Remark 2.5), we also have

$$F_2(-a)G_2'(-a) = 2acG(-a) = -2acG(a) > 0. \tag{3.24}$$

Now, for any $y_0 \in \mathcal{N}(G)$, by (3.21) and (2.15), we have

$$\begin{aligned} F_2(y_0)G_2'(y_0) &= \left[(a^2 - y_0^2)F(y_0) + c\right](a^2 - y_0^2)G'(y_0) \\ &\geqslant (a^2 - y_0^2)^2F(y_0)G'(y_0) - |c(a^2 - y_0^2)G'(y_0)| \\ &\geqslant \tfrac{1}{2}(a^2 - y_0^2)^2F(y_0)G'(y_0) > 0. \end{aligned} \tag{3.25}$$

Thus, by Theorem 2.4, part (c), $H_2(z)$ is stable. □

4. Stabilization.

4.1. *Single-input case*. Suppose we are given a single-input system $[A, b]$ which is completely controllable. Then, it is well known that (cf. [11]) we have the following canonical representation:

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{4.1}$$

THEOREM 4.1. Suppose $[A, b]$ is completely controllable, and

$$\sigma(A) \subseteq \mathbf{C}^- \cup \mathbf{C}^0 \equiv \{\lambda \in \mathbf{C} \,|\, \operatorname{Re}\lambda \leqslant 0\}. \tag{4.2}$$

Then, for any $r > 0$, $[A, b]$ is r-stabilizable.

Proof. We first prove that $[A, b]$ is 1-stabilizable. We let

$$k = (-k_0, -k_1, \dots, -k_{n-1})^T \in \mathbf{R}^n. \tag{4.3}$$

Then we have, by assuming (4.1), that

$$\begin{aligned} p(\lambda; A, bk^T; 1) &\equiv \det(\lambda I - A - bk^T e^{-\lambda}) \\ &= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 + e^{-\lambda}(k_{n-1}\lambda^{n-1} + \cdots + k_0) \\ &= e^{-\lambda}\{ e^{\lambda}(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0) + (k_{n-1}\lambda^{n-1} + \cdots + k_0)\}. \end{aligned} \tag{4.4}$$

By (4.2), we can assume that

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = \lambda^l P_1(\lambda) P_2(\lambda), \tag{4.5}$$

where $l \geqslant 0$ and

$$\begin{cases} \mathcal{N}(P_1) \subseteq \mathbf{C}^- \equiv \{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda < 0\} \\ \mathcal{N}(P_2) \subseteq \mathbf{C}^0 \setminus \{0\} \equiv \{\lambda \in \mathbf{C}\setminus\{0\} \mid \operatorname{Re}\lambda = 0\}. \end{cases} \tag{4.6}$$

We claim that there exists a polynomial, Q_1, with real coefficients and $\deg Q_1 < l$, such that

$$H(\lambda) \overset{\Delta}{=} \lambda^l e^{\lambda} + Q_1(\lambda) \tag{4.7}$$

is stable. If $l = 0$, we take $Q_1(\lambda) \equiv 0$. If $l > 0$, consider e^{λ} which is of the form (2.11) and is stable. By Lemma 3.1, there exists a $c_1 \in \mathbf{R}$, such that $\lambda e^{\lambda} + c_1$ is stable. Again there exists a $c_2 \in \mathbf{R}$ such that

$$\lambda(\lambda e^{\lambda} + c_1) + c_2 = \lambda^2 e^{\lambda} + c_1\lambda + c_2 \tag{4.8}$$

is stable. By induction, our claim follows. Now, by applying Lemma 3.2, and using a similar argument as above, we can find a polynomial $Q_2(\lambda)$ with real coefficients and with $\deg Q_2 < \deg P_2$, such that $P_2(\lambda)H(\lambda) + Q_2(\lambda)$ is stable. The polynomial Q defined by

$$Q(\lambda) = Q_1(\lambda)P_2(\lambda) + Q_2(\lambda) \tag{4.9}$$

then has real coefficients, and satisfies

$$\begin{aligned} \deg Q &\leqslant \max\{\deg Q_1 + \deg P_2, \deg Q_2\} \\ &= \deg Q_1 + \deg P_2 < l + \deg P_2. \end{aligned} \tag{4.10}$$

If we let the vector $k^T = (k_0, \dots, k_{n-1})$ in R^n be defined by

$$k_{n-1}\lambda^{n-1} + \cdots + k_0 = P_1(\lambda)Q(\lambda), \tag{4.11}$$

we have

$$\begin{aligned} p(\lambda; A, bk^T; 1) &= e^{-\lambda}\{ e^{\lambda}\lambda^l P_1(\lambda)P_2(\lambda) + P_1(\lambda)Q(\lambda)\} \\ &= e^{-\lambda}P_1(\lambda)\{ e^{\lambda}\lambda^l P_2(\lambda) + Q_1(\lambda)P_2(\lambda) + Q_2(\lambda)\} \\ &= e^{-\lambda}P_1(\lambda)\{[\lambda^l e^{\lambda} + Q_1(\lambda)]P_2(\lambda) + Q_2(\lambda)\} \\ &= e^{-\lambda}P_1(\lambda)\{ P_2(\lambda)H(\lambda) + Q_2(\lambda)\}, \end{aligned} \tag{4.12}$$

which is stable. Thus, we have proved that $[A, b]$ is 1-stabilizable. Now, for any $r > 0$, we know that $[rA, b]$ is still completely controllable and also we have

$$\sigma(rA) \subseteq \mathbf{C}^- \cup \mathbf{C}^0. \tag{4.13}$$

Thus, by the above proof, we know that $[rA, b]$ is 1-stabilizable. Thus, by Proposition 2.2, $[A, b]$ is r-stabilizable. □

4.2. *Multi-input case*. Now, we consider a multi-input system $[A, B]$ which is completely controllable. Then, by p. 44 of [11], we have the following representation.

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_l \end{pmatrix}_{n\times n}, \qquad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1l} & * \\ & b_{22} & \cdots & b_{2l} & * \\ & & \ddots & \vdots & \vdots \\ & & & b_{ll} & * \end{pmatrix}_{n\times m}, \tag{4.14}$$

where, for $1 \leqslant j \leqslant l$,

$$A_j = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -a_0^j & -a_1^j & \cdots & -a_{n_j-2}^j & -a_{n_j-1}^j \end{pmatrix}_{n_j\times n_j}, \qquad b_{jj} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}_{n_j\times 1}, \tag{4.15}$$

and $*$ represents the entries in which we are not interested. It is clear that in this case, we must have $m \geqslant l$. Now, we let

$$K = \begin{pmatrix} \hat{K} \\ 0 \end{pmatrix}_{m\times n}, \tag{4.16}$$

where

$$\hat{K} = \begin{pmatrix} k_{11}^T & & & \\ & k_{22}^T & & \\ & & \ddots & \\ & & & k_{ll}^T \end{pmatrix}_{l\times n} \qquad k_{jj} \in \mathbf{R}^{n_j},\ 1 \leqslant j \leqslant l. \tag{4.17}$$

Then, we have

$$A + BK = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_l \end{pmatrix} + \begin{pmatrix} b_{11}k_{11}^T & b_{12}k_{22}^T & \cdots & b_{1l}k_{ll}^T \\ & b_{22}k_{22}^T & \cdots & b_{2l}k_{ll}^T \\ & & \ddots & \vdots \\ & & & b_{ll}k_{ll}^T \end{pmatrix}. \tag{4.18}$$

Thus, we get

$$p(\lambda; A, BK; r) = \prod_{j=1}^{l} p\left(\lambda; A_j, b_{jj}k_{jj}^T; r\right), \tag{4.19}$$

and we have the following result.

THEOREM 4.2. Suppose $[A, B]$ is completely controllable and

$$\sigma(A) \subseteq \mathbf{C}^{-} \cup \mathbf{C}^{0}. \tag{4.20}$$

Then, for any $r > 0$, $[A, B]$ is r-stabilizable.

Proof. By (4.15), we see that $[A_j, b_{jj}]$ is completely controllable for each $1 \leqslant j \leqslant l$. Also, we have that

$$\sigma(A_j) \subseteq \sigma(A) \subseteq \mathbf{C}^{-} \cup \mathbf{C}^{0}, \quad 1 \leqslant j \leqslant l. \tag{4.21}$$

Thus, by Theorem 4.1, for any $r > 0$, we can find k_{jj}, such that $p(\lambda; A_j, b_{jj}k_{jj}^T; r)$ is stable for $1 \leqslant j \leqslant l$. Hence, by (4.19), we get the r-stabilizability of $[A, B]$. □

5. One-dimensional case. In this section, we consider a one-dimensional system

$$\dot{x}(t) = ax(t) + u(t), \tag{5.1}$$

where $a \in \mathbf{R}$. By Proposition 2.2, we know that (5.1) is r-stabilizable for $r > 0$, if and only if for some $k \in \mathbf{R}$

$$\dot{x}(t) = rax(t) + kx(t-1) \tag{5.2}$$

is asymptotically stable. It is easy to see that

$$\begin{aligned} p(\lambda; ra, k; 1) &= \lambda - ra - e^{-\lambda}k \\ &= e^{-\lambda}\left[e^{\lambda}(\lambda - ra) - k\right]. \end{aligned} \tag{5.3}$$

Let us cite the following theorem due to Hayes.

THEOREM 5.1 ([6]). All the roots of

$$ze^{z} - pe^{z} - q = 0 \tag{5.4}$$

have negative real parts (where, $p, q \in \mathbf{R}$) iff

(a) $p < 1$,

(b) $p < -q < \sqrt{a_1^2 + p^2}$

where, for $p = 0$, $a_1 = \pi/2$; and for $p \neq 0$, $0 < a_1 < \pi$, $a_1 = p \tan a_1$.

As a consequence of Hayes' Theorem, we have

THEOREM 5.2. System (5.1) is r-stabilizable iff

$$ra < 1. \tag{5.5}$$

Proof. We know that system (5.1) is r-stabilizable if and only if for some $k \in \mathbf{R}$ (see (5.3)),

$$e^{\lambda}(\lambda - ra) - k = 0 \tag{5.6}$$

has no roots in $\mathbf{C}^{+} \cup \mathbf{C}^{0}$. Then, we take p to be ra and we can find $k \equiv q$ satisfying (b) of Theorem 5.1. Hence our theorem follows. □

Moreover, for system (5.1), we have the following:

THEOREM 5.3. Suppose $k \in \mathbf{R}$, $r > 0$, such that $[a, k; r]$ is asymptotically stable. Then, for any $\hat{r} \in [0, r]$, $[a, k; \hat{r}]$ is also asymptotically stable.

REMARK 5.4. This theorem says that if k can be used to r-stabilize system (5.1), then it also can be used to $\hat{r}$-stabilize (5.1), for any $\hat{r} \in [0, r]$.

Proof of Theorem 5.3. In our case, by Theorem 5.1, we have

$$\begin{cases} ra < 1 \\ ra < -rk < \sqrt{a_1^2 + r^2 a^2}, \end{cases} \tag{5.7}$$

where, for $a = 0$, $a_1 = \pi/2$; for $a \neq 0$, $0 < a_1 < \pi$, $a_1 = ra \tan a_1$. Then, we have

$$a + k < 0. \tag{5.8}$$

This implies that $[a, k; 0]$ is asymptotically stable. Now, we consider $\hat{r} \in (0, r]$. It is clear that (see (5.7))

$$\begin{cases} \hat{r}a < 1 \\ \hat{r}a < -\hat{r}k \end{cases} \tag{5.9}$$

are always true. Thus, we only need to show that

$$-\hat{r}k < \sqrt{\hat{a}_1^2 + \hat{r}^2 a^2}, \tag{5.10}$$

where, for $a = 0$, $\hat{a}_1 = \pi/2$; for $a \neq 0$, $0 < \hat{a}_1 < \pi$, $\hat{a}_1 = \hat{r}a \tan \hat{a}_1$.

Case 1. $a = 0$.

Then, we have $a_1 = \hat{a}_1 = \pi/2$, and thus (5.10) is equivalent to

$$-rk < \frac{\pi}{2}. \tag{5.11}$$

This is true because by (5.7) and $\hat{r} \in (0, r]$, we have

$$-k < \frac{\pi}{2r} \leqslant \frac{\pi}{2\hat{r}}. \tag{5.12}$$

Thus, our theorem is true for this case.

Case 2. $a \neq 0$.

In order to have (5.10), it suffices to have

$$\sqrt{\left(\frac{a_1}{r}\right)^2 + a^2} \leqslant \sqrt{\left(\frac{\hat{a}_1}{\hat{r}}\right) + a^2}, \tag{5.13}$$

i.e.,

$$\frac{a_1^2}{r^2} \leqslant \frac{\hat{a}_1^2}{\hat{r}^2}, \tag{5.14}$$

or, equivalently, by the definitions of a_1 and $\hat{a}_1$,

$$\tan^2 a_1 \leqslant \tan^2 \hat{a}_1. \tag{5.15}$$

(1) $a > 0$. We consider the function $h(s)$ defined implicitly by

$$h(s) = s \tan h(s), \quad 0 < s < 1, 0 < h(s) < \frac{\pi}{2}. \tag{5.16}$$

Since

$$\begin{aligned} h'(s) &= \tan h(s) + s \sec^2 h(s) \cdot h'(s) \\ &= \tan h(s) + \frac{h(s)}{\tan h(s)} \sec^2 h(s) \cdot h'(s) \\ &= \tan h(s) + \frac{2h(s)}{\sin 2h(s)} \cdot h'(s), \end{aligned} \tag{5.17}$$

thus

$$h'(s) = \tan h(s) \cdot \left[1 - \frac{2h(s)}{\sin 2h(s)}\right]^{-1} < 0, \quad 0 < s < 1, \quad 0 < h(s) < \frac{\pi}{2}, \tag{5.18}$$

i.e., $h(s)$ is decreasing. Thus

$$0 < a_1 = h(ra) \leqslant h(\hat{r}a) = \hat{a}_1 < \frac{\pi}{2}. \tag{5.19}$$

Hence, (5.15) follows.

(2) $a < 0$. We consider the function $h(s)$ defined implicitly by

$$h(s) = s \tan h(s), \quad -\infty < s < 0, \frac{\pi}{2} < h(s) < \pi. \tag{5.20}$$

We have

$$h'(s) = \tan h(s) \cdot \left[1 - \frac{2h(s)}{\sin 2h(s)}\right]^{-1} < 0, \quad -\infty < s < 0, \frac{\pi}{2} < h(s) < \pi, \tag{5.21}$$

since $\tan h(s) < 0$, $\sin 2h(s) < 0$ for $\pi/2 < h(s) < \pi$. Thus, $h(s)$ is decreasing. Noting $ra \leqslant \hat{r}a$, we have

$$\pi > a_1 = h(ra) \geqslant h(\hat{r}a) = \hat{a}_1 > \frac{\pi}{2}. \tag{5.22}$$

Since $\tan x$ is increasing and negative for $x \in (\pi/2, \pi)$, we get (5.15). Hence our theorem is proved. □

6. Some remarks. First of all, from the previous sections, we can easily prove the following result.

THEOREM 6.1. Suppose $[A, B]$ is completely controllable, and

$$\sigma(A) \subseteq \mathbf{C}^- \cup \mathbf{C}^0 \cup \{a\}, \tag{6.1}$$

where $a > 0$, and the Jordan blocks of A corresponding to a are of order 1. Then, for any $r > 0$, with $ra < 1$, the system $[A, B]$ is r-stabilizable.

Secondly, we should note that Theorem 5.3 is not true in general for the higher-dimensional case. To see this, we have the following.

THEOREM 6.2. Let $a > 0$. Then, there exists a $c \in \mathbf{R}$ such that

$$e^{\lambda}(\lambda^2 + a^2) + c \tag{6.2}$$

is stable iff

$$\sin a \neq 0. \tag{6.3}$$

The proof is just a modification of that of Lemma 3.2.

Now, we consider a two-dimensional system with

$$A = \begin{pmatrix} 0 & 1 \\ -a^2 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.4}$$

and $\sin a \neq 0$, $a > \pi$. Then, we have $c \in \mathbf{R}$ such that (6.2) is stable. Thus, if we take

$$k = \begin{pmatrix} -c \\ 0 \end{pmatrix}, \tag{6.5}$$

then we have

$$p(\lambda; A, bk^T; 1) = \left[e^{\lambda}(\lambda^2 + a^2) + c\right]e^{-\lambda}, \tag{6.6}$$

which is stable. Thus, $[A, bk^T; 1]$ is asymptotically stable. However, if we take

$$0 < r = \frac{\pi}{a} < 1, \tag{6.7}$$

then $\sin(ra) = 0$. Thus, by Theorem 6.2,

$$\begin{aligned} p(\lambda; A, bk^T; r) &= e^{-\lambda r}(\lambda^2 + a^2) + c \\ &= \frac{e^{-\lambda r}}{r^2}\left\{e^{(\lambda r)}\left[(r\lambda)^2 + (ra)^2\right] + r^2 c\right\} \end{aligned} \tag{6.8}$$

can not be stable; i.e., $[A, bk^T; r]$ is not asymptotically stable.

Finally, let us suggest some natural questions to be solved.

(1) From the discussion of the one-dimensional case (Theorem 5.2), it seems true that a completely controllable system $[A, B]$ is r-stabilizable iff

$$r \cdot \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\} < 1. \tag{6.9}$$

(2) Under what conditions on A does Theorem 5.3 remain true for the system $[A, B]$?

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