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THE MOVING LOAD ON A STRING AS FREE BOUNDARY PROBLEM*

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Abstract. A free boundary problem for the nonhomogeneous wave equation is studied. Such a problem arises when the motion of a load on a string is analyzed without supposing that the load velocity is known and fixed. Preliminarily the complementary equations which characterize the free boundary are determined. The mechanical problem is then solved by proving a uniqueness and existence theorem.

1. Introduction. The problem of a moving load $\mathscr{G}$ on a beam or a string $\mathscr{S}$ usually has been approached under the hypothesis that the load velocity is known and fixed [see, e.g., 3–5].

But it is clear that, while the motion of the load influences the vibrations of the body $\mathscr{S}$ on which it is constrained, the motion of $\mathscr{G}$ is reciprocally modified by the oscillations of $\mathscr{S}$. From this follows the impossibility of determining the motion of $\mathscr{G}$ without at the same time finding the vibrations of $\mathscr{S}$. The analysis of the interferences of the motion of $\mathscr{G}$ with $\mathscr{S}$ and vice versa therefore requires the study of a free boundary problem for the equation governing the motion of $\mathscr{S}$.

Such a problem is discussed in this paper in the case where $\mathscr{S}$ is represented by a string which is supposed to be elastic, subject to external forces, and infinite.

2. Statement of the problem. Let us take the x-axis to be along $\mathscr{S}$ when it is in equilibrium under the action of the tension only. Let $y(x,t)$ be the position at time t of the point of the string labelled by x. Assume that the equation governing the free transverse motion of $\mathscr{S}$ is

$$y_{tt} - y_{xx} = f, \tag{2.1}$$

where $f(x,t)$ is the external force. Next consider a load $\mathscr{G}(x_G, y_G)$, of mass m, constrained to move on the string. Suppose that $\mathbf{F}(G, \dot{G}, t)$ is the force applied to $\mathscr{G}$ and indicate by F_1, F_2 the components of $\mathbf{F}$ along the axes x and y.

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Let s be the curvilinear abscissa that indicates the position of $\mathscr{G}$ on the string. In the linear theory $x_G = s$. In addition, we set $y_G = a$.

We have the following equations:

$$y_{tt} - y_{xx} = f, \quad t > 0, -\infty < x < s(t), \, s(t) < x < +\infty. \tag{2.2}$$

$$y(s(t), t) = a(t), \quad t > 0, \tag{2.3}$$

together with initial conditions

$$y(x,0) = y_0(x), \qquad y_t(x,0) = y_1(x), \tag{2.4}$$

$$s(0) = s_0, \qquad \dot{s}(0) = \dot{s}_0, \tag{2.5}$$

and compatibility conditions

$$a_0 = a(0) = y_0(s_0), \qquad \dot{a}_0 = \dot{a}(0) = y_0'(s_0^{\pm})\dot{s}_0 + y_1(s_0^{\pm}). \tag{2.6}$$

System (2.2)–(2.6) does not suffice for determination of $y(x,t)$, $s(t)$, and $a(t)$. We need two more equations, which, as will be shown later, are the following:

$$m(\dot{a}\ddot{a} + \dot{s}\ddot{s}) - F_2\dot{a} - F_1\dot{s} = (1 - \dot{s}^2)(y_x^+ - y_x^-)\{\dot{a} - \tfrac{1}{2}\dot{s}(y_x^+ + y_x^-)\}, \quad x = s(t), \, t > 0, \tag{2.7}$$

$$(1 - \dot{s}^2)(y_x^+ - y_x^-) + F_2 - m\ddot{a} = 0, \quad x = s(t), \, t > 0, \tag{2.8}$$

where y_x^+ and y_x^- are defined by

$$y_x^+ = y_x(s(t)^+, t), \qquad y_x^- = y_x(s(t)^-, t). \tag{2.9}$$

Let x_A and x_B be two points of $\mathscr{S}$. Let $s(t)$ be a curve for which $x_A < s(t) < x_B$, $t \in [0,T]$. Assume $|\dot{s}| < 1$ and set $\Omega_{1T} = \{(x,t)\colon\ 0 < t < T,\ x_A < x < s(t)\}$, $\Omega_{2T} = \{(x,t)\colon 0 < t < T,\ s(t) < x < x_B\}$, $\Omega_T = \Omega_{1T} \cup \Omega_{2T}$. From Eq. (2.1) one deduces

$$\int_{\partial\Omega_{jT}} y_t y_x \, dt + \frac{1}{2}\int_{\partial\Omega_{jT}} (y_t^2 + y_x^2)\, dx = -\int_{\Omega_{jT}} f y_t \, d\Omega_{jT}, \quad j = 1,2.$$

From here, by setting

$$E(t) = \frac{1}{2}\int_{x_A}^{x_B} \{y_x^2(x,t) + y_t^2(x,t)\}\, dx, \qquad W_f = \int_{\Omega_T} y_t f \, d\Omega_T,$$

$$W_A = -\int_0^T (y_t y_x)(x_A, t)\, dt,$$

$$W_B = \int_0^T (y_t y_x)(x_B, t)\, dt,$$

and indicating by $[g]$ the jump of a function $g(x,t)$ on $x = s(t)$, one obtains

$$E(T) = E(0) + W_A + W_B + W_f - \int_0^T \left\{[y_x y_t] + \frac{1}{2}\dot{s}[y_x^2 + y_t^2]\right\} dt. \tag{2.10}$$

On the other side, denoting by $K(t)$ the kinetic energy of $\mathscr{G}$, the conservation of the energy of the system yields

$$\int_0^T (\dot{E} + \dot{K})\, dt = W_A + W_B + \int_0^T (F_1\dot{s} + F_2\dot{a})\, dt + W_f. \tag{2.11}$$

By comparing (2.10) with (2.11) and using (2.3) and (2.9), we get (2.7).

Now we deduce Eq. (2.8). Let F^* be a concentrated force applied to $x = s(t)$. From the momentum balance for a one-dimensional continuum, one has

$$\frac{d}{dt}\int_{x_A}^{x_B} y_t(x,t)\,dx = -y_x(x_A,t) + y_x(x_B,t) + \int_{x_A}^{x_B} f(x,t)\,dx + F^*(s(t),t),$$

and therefore

$$-\dot{s}[y_t] + \int_{x_A}^{x_B} y_{tt}(x,t)\,dx = y_x(x_B,t) - y_x(x_A,t) + \int_{x_A}^{x_B} f(x,t)\,dx + F^*(s(t),t).$$

If we make x_A and x_B tend to s and use (2.3) we get $\dot{s}^2[y_x] = [y_x] + F^*$. From here, we obtain (2.8), since in our case $F^* = F_2 - m\ddot{a}$.

3. Solution of the free boundary problem. In this section we will indicate by $C^{(r,1)}(I)$ the space of all functions which have continuous derivatives up to order r and verify a Lipschitz condition in every subset of I.

Suppose that there exist three functions y, s, a satisfying the system (2.2)–(2.8) and show how it is possible to determine them in a unique way. Set $\Omega_{11} = \{(x,t)\colon 0 < t < T, -\infty < x < s_0 - t\}$, $\Omega_{22} = \{(x,t)\colon 0 < t < T,\ s_0 + t < x < +\infty\}$, $T > 0$. Note that the solution of (2.1) in Ω_{jj}, $j = 1,2$, depends only on the initial data. If $y_i \in C^{(1-i,1)}(\mathbf{R} - \{S_0\})$, $i = 0,1$, $f \in C^{(0,1)}(\mathbf{R} \times [0,+\infty[)$, there exists, in the sense of distributions, a unique solution $y \in C^1(\Omega_{11} \cup \Omega_{22})$ of (2.1) verifying (2.4) (see, e.g. [1]). Moreover, setting

$$\phi(x) = y(x, -x + s_0),\quad s_0 - T < x < s_0,\qquad \psi(x) = y(x, x - s_0),\quad s_0 < x < s_0 + T, \tag{3.1}$$

we can see that

$$\phi \in C^{(1,1)}(]s_0 - T, s_0[),\qquad \psi \in C^{(1,1)}(]s_0, s_0 + T[). \tag{3.2}$$

Let us now consider the set $\Omega_{12} = \{(x,t)\colon 0 < t < T,\ s_0 - t < x < s(t)\}$ and assume a, $s \in C^2([0,T])$, $|\dot{s}| < 1$. This last assumption means that the velocity of the moving load is not greater than the propagation speed along the string of the disturbance created by the motion of $\mathcal{G}$ on $\mathcal{S}$. We next observe that there exists a unique explicit solution $u(P_0) \in C^1(\Omega_{12})$ of (2.1), in the sense of distributions, which satisfies conditions (2.3) and $(3.1)_1$ (see, e.g., [1]). If we differentiate $u(P_0)$ with respect to x_0 and let P_0 tend to $(s(t),t)$ in Ω_{12}, we have

$$y_{x_0}(s(t)^-,t) = \{1 + \dot{s}(t)\}^{-1}\{\dot{a}(t) + f_1(s(t),t)\}, \tag{3.3}$$

where $f_1(s,t) = \phi'((s_0 + s - t)/2) - \int_{(s_0+s-t)/2}^{s} f(\sigma, \sigma - s + t)\,d\sigma$.

We apply the same argument to $\Omega_{21} = \{(x,t)\colon 0 < t < T,\ s(t) < x < s_0 + t\}$, obtaining

$$y_{x_0}(s(t)^+,t) = \{1 - \dot{s}(t)\}^{-1}\{-\dot{a}(t) + f_2(s(t),t)\}, \tag{3.4}$$

where $f_2(s,t) = \psi'((s + t + s_0)/2) + \int_s^{(s+t+s_0)/2} f(\sigma, -\sigma + s + t)\,d\sigma$.

We now reconsider Eqs. (2.7) and (2.8) and substitute in them the expressions for $y_{x_0}^{\pm}$ given by (3.3) and (3.4). We get

$$m\ddot{s} = F_1(s, \dot{s}, a, \dot{a}, t) - \frac{1}{2}\{m\ddot{a} - F_2(s, \dot{s}, a, \dot{a}, t)\}\{(1 - \dot{s})f_1(s, t) + (1 + \dot{s})f_2(s, t) - 2\dot{a}\dot{s}\}/(1 - \dot{s}^2), \quad (3.5)$$

$$m\ddot{a} = F_2(s, \dot{s}, a, \dot{a}, t) - 2\dot{a} - (1 - \dot{s})f_1(s, t) + (1 + \dot{s})f_2(s, t), \quad (3.6)$$

where only the unknown functions $s(t)$ and $a(t)$ appear. Initial conditions (2.5) and (2.6) are associated with Eqs. (3.5) and (3.6).

On the other side, $f_j \in C^{(0,1)}(]s_0 - T, s_0 + T[\times]0, T[)$, $j = 1, 2$, because of (3.2). Moreover, suppose that the functions $F_j(s, \dot{s}, a, \dot{a}, t)$ are continuous and satisfy locally a Lipschitz condition. Under these hypotheses the Cauchy problem (3.5), (3.6), (2.5), (2.6) has a unique solution $(a, s) \in C^2([0, T]) \times C^2([0, T])$, with $T > 0$ depending on the initial conditions. When the free boundary $s(t)$ and the function $a(t) = u(s(t), t)$ are explicitly determined, one can also find the function $y(x, t)$ satisfying Eq. (2.1) on the sets Ω_{12}, Ω_{21}. Thus, setting $\Omega_1 = \Omega_{11} \cup \Omega_{12}$, $\Omega_2 = \Omega_{21} \cup \Omega_{22}$, we can state the following

THEOREM. Suppose that the functions $F_j(s, \dot{s}, a, \dot{a}, t)$, $j = 1, 2$, are continuous and satisfy locally a Lipschitz condition for s, $\dot{s}$, a, $\dot{a}$. Assume $f \in C^{(0,1)}(\mathbf{R} \times [0, +\infty[)$, $y_i \in C^{(1-i,1)}(\mathbf{R} - \{S_0\})$, $i = 0, 1$, and $|\dot{s}| < 1$. Then there exists a unique solution $y \in C^1(\Omega_1) \cup C^1(\Omega_2)$, $s \in C^2([0, T])$, $a \in C^2([0, T])$ for system (2.2)–(2.8).

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