

ON UNIQUENESS AND STABILITY IN GENERALIZED THERMOELASTICITY*

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Abstract. A uniqueness theorem for the equations of generalized thermoelasticity with one relaxation time, derived by Dhaliwal and Sherief [1], is proved. The stability of the null solution in the sense of Liapounov, measured by a suitable norm, is shown. The corresponding equations for a homogeneous isotropic material, derived by Lord and Shulman [2], are considered as a special case.

1. Introduction. The equations of generalized thermoelasticity with one relaxation time for a homogeneous isotropic medium were derived by Lord and Shulman [2]. Dhaliwal and Sherief [1] obtained the corresponding equations for a general anisotropic medium, following a method established by Biot [3] for dealing with the coupled thermoelastic problem. These equations admit the so-called second sound effect in solids; i.e., they predict finite speeds of propagation for heat and displacement disturbances. This is not the situation encountered in coupled thermoelasticity, where an infinite speed of propagation is inherent.

Uniqueness theorems for the equations of generalized thermoelasticity were proved by a number of authors. For the isotropic case, Ignaczak [4] proved a uniqueness theorem for a stress-flex problem, Sherief and Dhaliwal [5] proved a general uniqueness theorem, but their proof relied heavily on the field inequality

$$-q_i \theta_{,i} \geq 0 \quad \text{on } B \times [0, \infty), \quad (1.1)$$

which was not used directly in the proof of the corresponding classical theorem by Weiner [6]. More recently, Ignaczak [7] established uniqueness by means of an associated conservation law involving higher-order time derivatives. For the anisotropic case, Dhaliwal and Sherief [1] have proved a uniqueness theorem similar to that in [5].

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2. Statement of the problem. We assume that a linear anisotropic thermoelastic material at time t occupies a closed bounded region $B(t)$ of three-dimensional space with sufficiently smooth boundary ∂B . The governing equations have the following form [1] in $B \times [0, \infty)$:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial}{\partial x_j} (c_{ijkl} u_{k,l}) - \frac{\partial}{\partial x_j} (\beta_{ij} \theta) + \rho F_i \quad (2.1)$$

and

$$\rho c_E L \left(\frac{\partial \theta}{\partial t} \right) + T_0 \beta_{ij} L \left(\frac{\partial^2 u_i}{\partial x_j \partial t} \right) = \frac{\partial}{\partial x_i} (k_{ij} \theta_{,j}). \quad (2.2)$$

In the above equations $\rho(\mathbf{x})$ denotes the density, $u_i(\mathbf{x}, t)$ are the components of the displacement vector, $c_{ijkl}(\mathbf{x})$ are the elastic moduli, $\theta(\mathbf{x}, t)$ is the temperature deviation above a reference temperature T_0 chosen such that $|\theta/T_0| \ll 1$, F_i are the components of the external applied force per unit mass, c_E is the specific heat for zero deformation, $k_{ij}(\mathbf{x})$ are the components of the conductivity tensor, and $\mathbf{x}(x_1, x_2, x_3)$ represents a point inside $B(t)$. In Eq. (2.2) we have denoted by L the linear operator

$$L = 1 + \tau_0 \frac{\partial}{\partial t}, \quad (2.3)$$

where τ_0 is the relaxation time.

Equations (2.1) and (2.2) are subject to the initial conditions that at $t = 0$ on B , u_i , \dot{u}_i , θ , and $\dot{\theta}$ are prescribed functions of \mathbf{x} and to the boundary conditions that for $t \geq 0$, θ is prescribed on ∂B , u_i is prescribed on a portion $\overline{\partial B_1}$ of ∂B , while the traction $c_{ijkl} u_{k,l} n_j$ is given on $\partial B - \overline{\partial B_1}$. We assume also that u_i and θ are continuous in the closure of B .

Throughout this paper, the convention is adopted of summing over repeated indices whose range is 1, 2, 3.

3. Stability under perturbation of the initial data. Let (u_i^1, θ^1) and (u_i^2, θ^2) denote two solutions of equations (2.1) and (2.2) with the same force F_i and the same boundary conditions but with (possibly) different initial conditions. We shall consider the difference functions

$$u_i = u_i^1 - u_i^2, \quad \theta = \theta^1 - \theta^2,$$

and take

$$U_i = L u_i, \quad \phi = L \theta. \quad (3.1)$$

Applying the operator L to (2.1) (with zero F_i) and to the initial and boundary conditions, we see that (u_i, θ) satisfies

$$\rho \frac{\partial^2 U_i}{\partial t^2} = \frac{\partial}{\partial x_j} (c_{ijkl} U_{k,l}) - \frac{\partial}{\partial x_j} (\beta_{ij} \phi) \quad \text{in } B \times [0, \infty), \quad (3.2)$$

$$\rho c_E \frac{\partial \phi}{\partial t} + T_0 \beta_{ij} \frac{\partial^2 U_i}{\partial x_j \partial t} = \frac{\partial}{\partial x_i} (k_{ij} \theta_{,j}) \quad \text{in } B \times [0, \infty), \quad (3.3)$$

$$U_i(\mathbf{x}, t) = 0 \quad \text{on } \overline{\partial B_1} \times [0, \infty), \quad (3.4a)$$

$$c_{ijkl}U_{k,l}n_j = 0 \quad \text{on } (\partial B - \overline{\partial B_1}) \times [0, \infty), \tag{3.4b}$$

$$\phi(\mathbf{x}, t) = 0 \quad \text{on } \partial B \times [0, \infty), \tag{3.5}$$

$$U_i(\mathbf{x}, 0) = U_{i0}(\mathbf{x}), \quad \dot{U}_i(x, 0) = U_{i1}(\mathbf{x}) \quad \text{on } B(0), \tag{3.6}$$

$$\theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \quad \phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}), \quad \dot{\phi}(\mathbf{x}, 0) = \phi_1(\mathbf{x}) \quad \text{on } B(0). \tag{3.7}$$

THEOREM. If (U_i, θ) is a solution of (3.1)–(3.7), then the function

$$J(t) = T_0 E(t) + \int_0^t \int_{B(\eta)} k_{ij} \theta_{,i} \theta_{,j} d\mathbf{x} d\eta, \tag{3.8}$$

is a constant independent of time, where

$$E(t) = \frac{1}{2} \int_{B(t)} \left\{ \rho \frac{\partial U_i}{\partial t} \frac{\partial U_i}{\partial t} + c_{ijkl} U_{i,j} U_{k,l} + \frac{\rho c_E}{T_0} \phi^2 + \frac{\tau_0}{T_0} k_{ij} \theta_{,i} \theta_{,j} \right\} d\mathbf{x}, \tag{3.9}$$

provided that k_{ij} and c_{ijkl} satisfy the standard symmetry conditions

$$k_{ij} = k_{ji}, \quad c_{ijkl} = c_{klij}. \tag{3.10}$$

Proof. From Eq. (3.2), it follows that

$$\int_0^t \int_{B(\eta)} \frac{\partial U_i}{\partial \eta} \left\{ \rho \frac{\partial^2 U_i}{\partial \eta^2} - \frac{\partial}{\partial x_j} (c_{ijkl} U_{k,l}) + \frac{\partial}{\partial x_j} (\beta_{ij} \phi) \right\} d\mathbf{x} d\eta = 0.$$

Integrating by parts, we get

$$\begin{aligned} & \int_0^t \int_{B(\eta)} \rho \frac{\partial U_i}{\partial \eta} \frac{\partial^2 U_i}{\partial \eta^2} d\mathbf{x} d\eta - \int_0^t \int_{\partial B} c_{ijkl} U_{k,l} \frac{\partial U_i}{\partial \eta} n_j dS d\eta \\ & + \int_0^t \int_{B(\eta)} c_{ijkl} \frac{\partial^2 U_i}{\partial x_j \partial \eta} U_{k,l} d\mathbf{x} d\eta + \int_0^t \int_{\partial B} \beta_{ij} \frac{\partial U_i}{\partial \eta} n_j \phi dS d\eta \\ & - \int_0^t \int_{B(\eta)} \phi \beta_{ij} \frac{\partial^2 U_i}{\partial x_j \partial \eta} d\mathbf{x} d\eta = 0. \end{aligned}$$

The two surface integrals in the last equation vanish by virtue of equations (3.4) and (3.5), respectively. We thus get

$$\int_0^t \int_{B(\eta)} \left\{ \rho \frac{\partial U_i}{\partial \eta} \frac{\partial^2 U_i}{\partial \eta^2} + c_{ijkl} \frac{\partial^2 U_i}{\partial x_j \partial \eta} \frac{\partial U_k}{\partial x_l} - \phi \beta_{ij} \frac{\partial^2 U_i}{\partial x_j \partial \eta} \right\} d\mathbf{x} d\eta = 0. \tag{3.11}$$

Eliminating $\beta_{ij} \partial^2 U_i / \partial x_j \partial \eta$ between equations (3.3) and (3.11) yields

$$\begin{aligned} & \int_0^t \int_{B(\eta)} \left\{ \rho \frac{\partial U_i}{\partial \eta} \frac{\partial^2 U_i}{\partial \eta^2} + c_{ijkl} \frac{\partial^2 U_i}{\partial x_j \partial \eta} \frac{\partial U_k}{\partial x_l} + \frac{\rho c_E}{T_0} \phi \frac{\partial \phi}{\partial \eta} \right\} d\mathbf{x} d\eta \\ & - \frac{1}{T_0} \int_0^t \int_{B(\eta)} \phi \frac{\partial}{\partial x_i} (k_{ij} \theta_{,j}) d\mathbf{x} d\eta = 0. \end{aligned}$$

Integrating the last integral by parts and using equation (3.5), gives

$$\int_0^t \int_{B(\eta)} \left\{ \rho \frac{\partial U_i}{\partial \eta} \frac{\partial^2 U_i}{\partial \eta^2} + c_{ijkl} \frac{\partial^2 U_i}{\partial x_j \partial \eta} \frac{\partial U_k}{\partial x_l} + \frac{\rho c_E}{T_0} \phi \frac{\partial \phi}{\partial \eta} \right\} d\mathbf{x} d\eta + \frac{1}{T_0} \int_0^t \int_{B(\eta)} k_{ij} \phi_{,i} \theta_{,j} d\mathbf{x} d\eta = 0.$$

Substituting for ϕ from the second part of equation (3.1) and using (2.3), we arrive at

$$\int_0^t \int_{B(\eta)} \left\{ \rho \frac{\partial U_i}{\partial \eta} \frac{\partial^2 U_i}{\partial \eta^2} + c_{ijkl} \frac{\partial^2 U_i}{\partial x_j \partial \eta} \frac{\partial U_k}{\partial x_l} + \frac{\rho c_E}{T_0} \phi \frac{\partial \phi}{\partial \eta} + \frac{\tau_0}{T_0} k_{ij} \frac{\partial^2 \theta}{\partial x_i \partial \eta} \frac{\partial \theta}{\partial x_j} \right\} d\mathbf{x} d\eta + \frac{1}{T_0} \int_0^t \int_{B(\eta)} k_{ij} \theta_{,i} \theta_{,j} d\mathbf{x} d\eta = 0. \tag{3.12}$$

Using the symmetry conditions (3.10) and equation (3.9), equation (3.12) yields

$$\int_0^t \frac{\partial}{\partial \eta} E(\eta) d\eta + \frac{1}{T_0} \int_0^t \int_{B(\eta)} k_{ij} \theta_{,i} \theta_{,j} d\mathbf{x} d\eta = 0,$$

which gives

$$T_0 E(t) + \int_0^t \int_{B(\eta)} k_{ij} \theta_{,i} \theta_{,j} d\mathbf{x} d\eta = T_0 E(0). \tag{3.13}$$

Recalling equation (3.8), equation (3.13) can be written in the final form

$$J(t) \equiv J(0), \tag{3.14}$$

which completes the proof.

COROLLARY 1. There is at most one solution of equations (2.1) and (2.2) satisfying the prescribed initial and boundary conditions, provided that

- (i) $\rho > 0$, $c_E > 0$, $T_0 > 0$, and $\tau_0 > 0$, (3.15)
- (ii) k_{ij} and c_{ijkl} satisfy the symmetry conditions (3.10),
- (iii) k_{ij} and c_{ijkl} satisfy the positive definiteness conditions

$$k_{ij} \xi_i \xi_j \geq c_1 \xi_i \xi_j, \tag{3.16a}$$

$$c_{ijkl} \xi_i \xi_j \xi_k \xi_l \geq c_2 \xi_i \xi_j \xi_k \xi_l \tag{3.16b}$$

for some positive constants c_1 and c_2 and for all nonzero tensors ξ_i and ξ_{ij} .

Proof. As usual, to prove uniqueness the difference functions must satisfy equations (3.1)–(3.5) and the homogeneous counterparts of equations (3.6) and (3.7), namely

$$U_i(\mathbf{x}, 0) = \dot{U}_i(\mathbf{x}, 0) = 0 \quad \text{on } B(0), \tag{3.17}$$

$$\theta(\mathbf{x}, 0) = \phi(\mathbf{x}, 0) = \dot{\phi}(\mathbf{x}, 0) = 0 \quad \text{on } B(0). \tag{3.18}$$

Using the initial conditions (3.17) and (3.18), it follows from (3.9) that

$$E(0) = 0. \tag{3.19}$$

Equations (3.9), (3.13), and (3.19) give

$$T_0 \int_{B(t)} \left\{ \rho \frac{\partial U_i}{\partial t} \frac{\partial U_i}{\partial t} + c_{ijkl} U_{k,l} + \frac{\rho c_E}{T_0} \phi^2 + \frac{\tau_0}{T_0} k_{ij} \theta_{,i} \theta_{,j} \right\} dx + 2 \int_0^t \int_{B(\eta)} k_{ij} \theta_{,i} \theta_{,j} dx d\eta = 0. \quad (3.20)$$

Using conditions (3.16) in (3.20) and taking account of (3.15), we see that all terms in the resulting inequality are nonnegative; this gives

$$T_0 \int_{B(t)} \left\{ \rho \frac{\partial U_i}{\partial t} \frac{\partial U_i}{\partial t} + c_2 U_{i,j} U_{i,j} + \frac{\rho c_E}{T_0} \phi^2 + \frac{\tau_0}{T_0} c_1 \theta_{,i} \theta_{,i} \right\} dx + 2 \int_0^t \int_{B(\eta)} c_1 \theta_{,i} \theta_{,i} dx d\eta = 0. \quad (3.21)$$

Due to (3.15), all terms on the left-hand side of equation (3.21) are nonnegative; therefore

$$\frac{\partial U_i}{\partial t} = \phi = 0 \quad \text{on } \bar{B} \times [0, \infty). \quad (3.22)$$

The first part of equation (3.17) together with (3.22) yields

$$U_i = \phi = 0 \quad \text{on } \bar{B} \times [0, \infty). \quad (3.23)$$

Using (3.1), (3.23) may be written as

$$u_i + \tau_0 \frac{\partial u_i}{\partial t} = 0, \quad \theta + \tau_0 \frac{\partial \theta}{\partial t} = 0 \quad \text{on } \bar{B} \times [0, \infty). \quad (3.24)$$

As was noted by Ignaczak [7] that equations (3.24) may be written in the equivalent form

$$u_i(\mathbf{x}, t) = u_i(\mathbf{x}, 0) \exp\left(\frac{-t}{\tau_0}\right) \quad \text{on } \bar{B} \times [0, \infty), \quad (3.25)$$

$$\theta(\mathbf{x}, t) = \theta(\mathbf{x}, 0) \exp\left(\frac{-t}{\tau_0}\right) \quad \text{on } \bar{B} \times [0, \infty). \quad (3.26)$$

Now, using (3.17) and (3.18), the last two equations yield

$$u_i(\mathbf{x}, t) = \theta(\mathbf{x}, t) = 0 \quad \text{on } \bar{B} \times [0, \infty),$$

which completes the proof.

COROLLARY 2. The null solution of equations (2.1) and (2.2) is Liapounov stable in the J -norm provided that conditions (3.10), (3.15), and (3.16) are satisfied.

The proof of this corollary follows directly from equation (3.14) and resembles somewhat the linear version of one obtained by Ericksen [8, 9] in discussing thermoelastic stability. It also resembles a result of Knops and Payne [10] for the coupled thermoelastic equations.

4. The isotropic case. For this case the elastic moduli c_{ijkl} are given by the relation

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (4.1)$$

while the components of the conductivity tensor are given by

$$k_{ij} = k\delta_{ij}, \quad (4.2)$$

where λ and μ are Lamé's constants and k is the coefficient of thermal conductivity.

After some manipulations and using the fact that the tensor e_{ij} defined by

$$e_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}) \quad (4.3)$$

is symmetric, the function $J(t)$ in (3.8) can be written in the form

$$J(t) = \frac{T_0}{2} \int_{B(t)} \left\{ \rho \frac{\partial U_i}{\partial t} \frac{\partial U_i}{\partial t} + \lambda e_{kk}^2 + 2\mu e_{ij}e_{ij} + \frac{\rho c_E}{T_0} \phi^2 + \frac{\tau_0 k}{T_0} \theta_{,i} \theta_{,i} \right\} d\mathbf{x} \\ + k \int_0^t \int_{B(\eta)} \theta_{,i} \theta_{,i} d\mathbf{x} d\eta = J(0). \quad (4.4)$$

Uniqueness and stability in this case follow from equation (4.4) as before, provided that

$\rho, \lambda, \mu, c_E, T_0, k,$ and τ_0 are all nonnegative.

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