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UNIQUENESS OF FLOW OF A SECOND-ORDER FLUID PAST A STRETCHING SHEET*

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1. Introduction. Motivated by the recent work of Rajagopal et al. [4], we investigate the flow of an incompressible second-order fluid past a stretching sheet. As described in [4] and [2], it is postulated that the constitutive equation for such a fluid is given by

$$\mathbf{T} = -p\mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2. \tag{1}$$

Here $\mathbf{T}$ is the stress tensor; p the pressure, μ, α_1, α_2 material constants, $\alpha_1 < 0$; and $\mathbf{A}_1$, $\mathbf{A}_2$ are defined by

$$\begin{aligned} \mathbf{A}_1 &= \operatorname{grad} \mathbf{v} + (\operatorname{grad} \mathbf{v})^T, \\ \mathbf{A}_2 &= \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1 \cdot \operatorname{grad} \mathbf{v} + (\operatorname{grad} \mathbf{v})^T \cdot \mathbf{A}_1. \end{aligned} \tag{2}$$

In [4], Rajagopal et al. use a perturbation analysis suggested by Beard and Walters [1] for boundary layer flows of non-Newtonian fluids. They consider a wall coinciding with the plane $y = 0$ and with the flow confined to $y > 0$. Two equal and opposite forces are applied along the x-axis so that the wall is stretched while the origin remains fixed. The steady two-dimensional boundary layer model which they develop consists of the system

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

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$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-k\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+\nu\frac{\partial^3 u}{\partial y^3}\right], \tag{4}$$

with boundary conditions

$$u=cx,\quad c>0;\ v=0\quad \text{at } y=0,$$
$$u\to 0\quad \text{as } y\to\infty. \tag{5}$$

Here $\nu=\mu/\rho$ is the kinematic viscosity and $k=-\alpha_1/\rho$. Rajagopal et al. [4] transform Eqs. (3)–(5) via the similarity transformation $\eta=(c/\nu)^{1/2}y$, $k_1=kc/\nu$, $u=cxf'$, $v=-(\nu c)^{1/2}f$ into the problem

$$f'^2-ff''=f'''-k_1\left[2f'f'''-(f'')^2-ff''''\right], \tag{6}$$

$$f'(0)=1,\qquad f(0)=0,\qquad f'(\infty)=0. \tag{7}$$

Most authors assume that k_1 is small. Rajagopal et al. [4] follow this assumption and expand the solutions in powers of k_1 to obtain numerical estimates on the behavior of the solutions of Eqs. (6) and (7). We have found that (6) and (7) have at least one solution, which is given by

$$f=(1-k_1)^{1/2}(1-e^{-\eta/(1-k_1)^{1/2}}). \tag{8}$$

This function is in exact agreement with the approximations found in [4].

We observe that if $k_1=0$ then the function given above reduces to $g(\eta)=1-e^{-\eta}$, which is a well-known solution of the problem

$$f'''+ff''-f'^2=0 \tag{9}$$

with boundary conditions (7).

The main goal in this paper is to prove that $g(\eta)$ is the only solution of (9) which satisfies (7). This same result has simultaneously been proved by McLeod and Rajagopal [3]. Their technique consists of a plane analysis. Our method of proof reduces (9) to a first-order Ricatti equation followed by an appropriate analysis.

It remains an open problem to determine the uniqueness of the solution of the full fourth-order problem of Eqs. (6) and (7) given in Eq. (8).

Uniqueness. Our goal is to prove the following:

THEOREM. The function $g(\eta)=1-e^{-\eta}$ is the only solution of (9) which satisfies (7).

Proof. We assume throughout that a second solution exists, which we denote by $f(\eta)$. A contradiction is obtained with the use of the following three technical lemmas:

LEMMA 1. (f', f'') cannot enter the set $f'\geqslant 0$, $f''>0$ on $[0,\infty)$.

Proof. If there is an $\hat\eta\geqslant 0$ for which $f'(\hat\eta)\geqslant 0$, $f''(\hat\eta)>0$, then (7) implies there exists a first $\eta_1>\hat\eta$ where $f''(\eta_1)=0$, hence $f'''(\eta_1)\leqslant 0$. However, from (9) we obtain $f'''(\eta_1)=(f'(\eta_1))^2>0$, a contradiction.

LEMMA 2. There are values $\hat\eta>\tilde\eta>0$ for which $f'(\tilde\eta)=0$, $f''(\tilde\eta)<0$, $f''(\hat\eta)=0$, and $f''<0$ on $(\tilde\eta,\hat\eta)$.

Proof. Suppose, first of all, that $f'\geqslant 0\ \forall\,\eta\geqslant 0$. From (9) and uniqueness of solutions it follows that $f''<0\ \forall\,\eta>0$. Let $h=g-f$. Then h satisfies

$$h'''+gh''-(f'+g')h'+f''h=0, \tag{10}$$

$$h(0)=h'(0)=0. \tag{11}$$

Lemma 1 implies that $f''(0) < 0$. If we assume that $f''(0) > g''(0)$ then $h''(0) < 0$, hence $h < 0$ and $h' < 0$ to the immediate right of $\eta = 0$. Since $h \to 0$ as $\eta \to \infty$ there must be a first $\eta_1 > 0$ for which $h''(\eta_1) = 0$ and $h'''(\eta_1) \geqslant 0$. However, from (10) we find that $h'''(\eta_1) = (f'(\eta_1) + g'(\eta_1))h'(\eta_1) - f''(\eta_1)h(\eta_1) < 0$, a contradiction. A similar contradiction arises if we assume that $f''(0) < g''(0)$. We conclude therefore that there is a first $\tilde{\eta} > 0$ with $f'(\tilde{\eta}) = 0$ and $f''(\tilde{\eta}) < 0$. Since f satisfies (7) there must be a first $\hat{\eta} > \tilde{\eta}$ for which $f''(\hat{\eta}) = 0$ and our lemma is proved.

Lemma 3. There is an $\bar{\eta} > \hat{\eta}$ with $f'(\bar{\eta}) = 0$ and $f''(\bar{\eta}) > 0$.

Proof. Suppose that $f' < 0 \,\forall \eta \geqslant \hat{\eta}$. From (9) it follows that $f'' > 0 \,\forall \eta > \hat{\eta}$. Further, if $f = 0$ at some first $\eta_1 > \hat{\eta}$, then $f < 0 \,\forall \eta \geqslant \eta_1$ and (9) implies that $f''' > 0 \,\forall \eta \geqslant \eta_1$. Then $f''(\eta) > f''(\eta_1) > 0$, hence $f' = 0$ for some first $\bar{\eta} > \hat{\eta}$. We assume, therefore, that $0 < f < f(\hat{\eta}) \,\forall \eta \geqslant \hat{\eta}$. Next, let $r = f''/f'$. Then r satisfies

$$r' + r^2 + fr - f' = 0, \tag{12}$$

$$r(\hat{\eta}) = 0, \qquad r'(\hat{\eta}) < 0. \tag{13}$$

A differentiation of (12) leads to

$$r'' + (2r + f)r' = 0. \tag{14}$$

Integrating (14), we obtain

$$r'(\eta) = r'(\hat{\eta}) \exp\left(-\int_{\hat{\eta}}^{\eta} (2r(s) + f(s))\, ds \right). \tag{15}$$

We let $A = \lim_{\eta \to \infty} r(\eta)$ and $f_\infty = \lim_{\eta \to \infty} f(\eta)$. Suppose, first of all, that $A < 0$ is finite. Since $r' < 0$ and $r \to A$ as $\eta \to \infty$ we conclude that $\lim_{\eta \to \infty} r'(\eta) = 0$. Thus, from (12) we obtain $A + f_\infty = 0$, hence $\lim_{\eta \to \infty}(2r(\eta) + f(\eta)) = A$. From this and (15) it follows that $\lim_{\eta \to \infty} r'(\eta) = -\infty$, a contradiction. Therefore, we conclude that $\lim_{\eta \to \infty} r(\eta) = -\infty$. Since $0 < f < f(\hat{\eta}) \,\forall \eta > 0$, it follows from (12) that there is an $\eta_1 > \hat{\eta}$ such that $r' < -r^2/2 \,\forall \eta > \eta_1$. Integrating this inequality leads to $r(\eta) < -2/(\eta - \eta_2)$, where $\eta_1 < \eta < \eta_2$ and $\eta_2 \equiv \eta_1 - 2/r(\eta_1)$. Thus $\lim_{\eta \to \eta_2} r(\eta) = -\infty$, a contradiction since $r = r''/f'$ is bounded on each subinterval of $(\hat{\eta}, \infty)$. Thus (f', f'') cannot remain in the region $f' < 0$, $f'' > 0 \,\forall \eta > \hat{\eta}$ and there must be a first $\bar{\eta} > 0$ with $f'(\bar{\eta}) = 0$, $f''(\bar{\eta}) > 0$.

To complete the proof of our theorem we conclude from Lemmas 3 and 1 that $f'' > 0$, $f' > 0$ for all $\eta > \bar{\eta}$, hence f' cannot satisfy (7), which implies that no second solution of Eqs. (7)–(9) can exist.

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