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NONLINEAR FOCUSSING IN MAGNETIC FLUIDS*

By

S. K. MALIK and M. SINGH

Simon Fraser University, Burnaby, B.C.

Abstract. The phenomenon of nonlinear focussing or collapse is presented for two superposed magnetic fluids subjected to a normal magnetic field. We show that the focussing is direction-dependent and is more pronounced at shorter wavelengths as well as at higher values of density ratio. Nonlinear focussing occurs if the dimensions of the system are higher than one and the magnetic field is in the subcritical regime. Because of this nonlinear effect, the regular pattern formation may develop local spots of highly irregular behaviour.

1. Introduction. Cowley and Rosensweig [1] (see also Rosensweig [2]) have investigated the linear stability of two superposed magnetic fluids in the presence of an externally applied magnetic field. They show that the magnetic field acting normal to the surface of separation has a destabilizing effect on the interface. When the applied field H is slightly larger than the critical magnetic field H_c, the interface deforms to form a regular hexagonal pattern. The nonlinear stability of such a configuration for $H \geqslant H_c$ has been examined by various authors ([2] and the references therein). A generalized formulation of the evolution equation governing the wave amplitude was developed by Malik and Singh [3], who obtained the nonlinear Klein-Gordon equation leading to the derivation of bell-shaped solitons and kink solutions as special cases.

On the other hand, for a tangential applied magnetic field, Zelazo and Melcher [4] studied the linear stability in superposed ferro-fluids and showed thereby that the magnetic field exerts a stabilizing influence on the fluid interface. In investigating the nonlinear evolution of two-dimensional wave packets on the surface of the superposed magnetic fields in the presence of a tangential magnetic field, Malik and Singh [5] demonstrated the existence of modulational instability. The magnetic field, however, suppresses such an instability.

The phenomenon of self-focussing or collapse has been the subject of extensive study in nonlinear optics by Zakharov and Synakh [6], in plasma physics by Zakharov [7], and in

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two-layer Kelvin–Helmholtz instability by Gibbon and McGuinness [8]. When self-focussing occurs in a medium, the wave field becomes singular at a point, called the focus, where the amplitude of the wave packet becomes infinite, leading to the appearance of turbulent bursts. Such an effect has been reported in various numerical and laboratory experiments ([6] and the references therein). Newell [9] has shown that this effect is possible only if the system is multi-dimensional and that it takes place in the subcritical region. In a different context, Zakharov and Shabat [10] have demonstrated that in the one-dimensional case, the equation for the evolution of the amplitude is a nonlinear Schrödinger equation giving soliton solutions.

The aim of this presentation is to examine the stability of two superposed magnetic fluids subjected to a normal magnetic field in the subcritical region characterized by $H < H_c$. The system is linearly stable, and we wish to investigate the effect of cubic nonlinearity in this regime. In Sec. 2 we formulate the nonlinear boundary value problem, giving the basic equations and the mathematical procedure for solving them. In Sec. 3 we obtain the equation governing the evolution of the amplitude, which, on using canonical transformation, results in the two-dimensional nonlinear Schrödinger equation. From this equation, we derive the various criteria of stability and exhibit the existence of a self-focussing mechanism in magnetic fluids. It may be noted that a similar phenomenon also occurs in superposed dielectric fluids under the influence of a normal uniform electric field.

2. Basic equations. We consider the three-dimensional flow of two superposed semi-infinite magnetic fluids. The half space $z < 0$ is occupied by the magnetic fluid of density ρ_1 and permeability μ_1, whereas the region $z > 0$ contains the magnetic fluid of density ρ_2 and permeability μ_2. The fluids are subjected to an external magnetic field $\mathbf{H}(0, 0, H)$ acting normal to the interface $z = 0$. Both the fluids are homogeneous, incompressible, and irrotational. The equations govening the flow are

$$\nabla^2\phi^{(i)} = 0, \quad (i = 1, 2), \tag{1}$$

$$\nabla^2\psi^{(i)} = 0, \quad (i = 1, 2), \tag{2}$$

where $i = 1$ denotes the region $z < 0$ and $i = 2$ the region $z > 0$. Here ϕ and ψ are the velocity potential and the magnetic potential, respectively. Since away from the surface the motion vanishes,

$$|\nabla\phi^{(1)}| \to 0 \quad \text{as } z \to -\infty \tag{3a}$$

and

$$|\nabla\phi^{(2)}| \to 0 \quad \text{as } z \to \infty. \tag{3b}$$

The kinematic condition at the interface is

$$\frac{\partial\eta}{\partial t} - \frac{\partial\phi^{(i)}}{\partial z} + \frac{\partial\phi^{(i)}}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi^{(i)}}{\partial y}\frac{\partial\eta}{\partial y} = 0 \quad \text{at } z = \eta(x, y, t), \tag{4}$$

where $\eta(x, y, t)$ stands for the elevation of the interface. The continuity of the normal and tangential components of the magnetic field across the interface requires

$$\mu_1 H_{1n} = \mu_2 H_{2n} \quad \text{at } z = \eta(x, y, t) \tag{5}$$

and

$$H_{1T} = H_{2T} \quad \text{at } z = \eta(x, y, t). \tag{6}$$

Since the normal stress across the interface must be continuous, we obtain

$$\frac{\partial \phi^{(1)}}{\partial t} - \rho \frac{\partial \phi^{(2)}}{\partial t} + (1 - \rho) g \eta + \frac{1}{2} (\nabla \phi^{(1)})^2 - \frac{1}{2} \rho (\nabla \phi^{(2)})^2$$
$$- \frac{T}{\rho_1} \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) \left[1 + \left(\frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial y} \right)^2 \right]^{-3/2} = \frac{\mu_1 - \mu_2}{8 \pi \rho_1} \left[H_1^2 + (\mu - 1) H_{1n}^2 \right], \tag{7}$$

where T denotes the surface tension, $\mathbf{g}(0, 0, 1)$ the gravitational force per unit mass, $\rho = \rho_2/\rho_1$, and $\mu = \mu_1/\mu_2$. As the boundary conditions (4) to (7) are given at the free surface $z = \eta(x, y, t)$, one needs *a priori* information about $\eta(x, y, t)$. To surmount this difficulty, we use Maclaurin's expansions about $z = 0$ of the physical quantities appearing in equations (4)–(7), thereby reducing the conditions at the unperturbed level $z = 0$.

To obtain the asymptotic solution to the system of equations (1)–(7), we introduce the three sets of slow variables $t_n = \varepsilon^n t$, $n = 0, 1, 2$, $x_n = \varepsilon^n x$, and $y_n = \varepsilon^n y$, where ε is a small dimensionless parameter representing the size of the perturbations. The method we use is that of multiple scales, which relies on the fact that the wave amplitude is being modulated slowly in space and time. We assume that

$$\eta(x, y, t) = \sum_{n=1}^{3} \varepsilon^n \eta_n(x_0, x_1, x_2, y_0, y_1, y_2; t_0, t_1, t_2) + O(\varepsilon^4), \tag{8}$$

$$\phi(x, y, t) = \sum_{n=1}^{3} \varepsilon^n \phi_n(x_0, x_1, x_2, y_0, y_1, y_2; t_0, t_1, t_2) + O(\varepsilon^4), \tag{9}$$

and

$$\psi(x, y, t) = \sum_{n=1}^{3} \varepsilon^n \psi_n(x_0, x_1, x_2, y_0, y_1, y_2; t_0, t_1, t_2) + O(\varepsilon^4), \tag{10}$$

$$\frac{\partial}{\partial t} = \sum_{n=0}^{2} \varepsilon^n \frac{\partial}{\partial t_n} + O(\varepsilon^3), \tag{11}$$

$$\frac{\partial}{\partial x} = \sum_{n=0}^{2} \varepsilon^n \frac{\partial}{\partial x_n} + O(\varepsilon^3), \tag{12}$$

and

$$\frac{\partial}{\partial y} = \sum_{n=0}^{2} \varepsilon^n \frac{\partial}{\partial y_n} + O(\varepsilon^3). \tag{13}$$

The expansions (8) to (13) are uniformly valid for $-\infty < x < \infty$, $-\infty < y < \infty$, and $0 < t < \infty$. On substitution in equations (1)–(7), we get the linear and the successive nonlinear partial differential equations of the various orders. The solution of the problem in any order can be deduced with knowledge of the solutions of all the previous orders.

3. Linear theory, critical magnetic field. The travelling wave solutions of the first-order problem with respect to scales x_0, y_0, and t_0 are given by

$$\eta_1 = A(x_0, y_0, t_0)\exp(i\theta) + \text{c.c.}, \tag{14}$$

$$\phi^{(j)} = (-1)^j i\frac{\omega}{k}A(x_0, y_0, t_0)\exp\left(i\theta + (-1)^{j-1}kz\right) + \text{c.c.}, \quad j = 1, 2, \tag{15,16}$$

$$\psi_1^{(1)} = \frac{H}{\mu}\left(\frac{1-\mu}{1+\mu}\right)A\exp(i\theta + kz) + \text{c.c.}, \tag{17}$$

$$\psi_1^{(2)} = -H\left(\frac{1-\mu}{1+\mu}\right)A\exp(i\theta - kz) + \text{c.c.}, \tag{18}$$

where

$$\theta = lx_0 + my_0 - \omega t_0, \qquad k = (l^2 + m^2)^{1/2}. \tag{19}$$

Here H is the uniform part of the magnetic field in the lower fluid. All the physical quantities have been normalized with respect to the characteristic length $l_c = (T/\rho_1 g)^{1/2}$ and time $t_c = (l_c/g)^{1/2}$. The solution to equations (14)–(19) furnishes the dispersion relation

$$D(\omega, k) = -\omega^2(1+\rho) + (1-\rho)k + k^3 - \frac{(\mu-1)^2}{\mu(\mu+1)}A_0^2 k^2, \tag{20}$$

where A_0 is the normalized Alfvén velocity. Equation (20) yields the critical value of the magnetic field at $k = k_c = (1-\rho)^{1/2}$:

$$H_c = \left[\frac{8\pi\mu(\mu+1)}{(\mu-1)^2}\right]^{1/2}((1-\rho)Tg)^{1/4}. \tag{21}$$

This result was obtained earlier by Cowley and Rosensweig [1]. It is clear from equation (20) that $H = H_c$ is a point of bifurcation. For $H \geqslant H_c$, the flow is supercritical, as examined earlier by Malik and Singh [3]. In this paper, however, we are interested in investigating the nonlinear stability of waves in the subcritical region $H < H_c$.

4. Amplitude modulation of a travelling wave packet. With a view to deriving the equation for the evolution of the amplitude, we proceed to solve the second- and third-order problems. The nonsecularity conditions for the existence of uniformly valid solutions in the second-order problem are

$$\frac{\partial A}{\partial t_1} + V_l\frac{\partial A}{\partial x_1} + V_m\frac{\partial A}{\partial y_1} = 0 \tag{22}$$

and its complex conjugate. Here, the group velocities V_l and V_m are given by

$$V_l = \frac{k}{2\omega(1+\rho)}\left[\frac{\omega^2 l(1+\rho)}{k^3} + 2l - \frac{l}{k}\frac{(\mu-1)^2}{\mu(\mu+1)}A_0^2\right], \tag{23}$$

$$V_m = \frac{k}{2\omega(1+\rho)}\left[\frac{\omega^2 m(1+\rho)}{k^3} + 2m - \frac{m}{k}\frac{(\mu-1)^2}{\mu(\mu+1)}A_0^2\right]. \tag{24}$$

The quantities V_l and V_m appearing in (22) could be eliminated by transforming to the group velocity frame of reference. On substituting the second-order solutions into the third-order problem, we obtain

$$2i\left(\frac{\partial A}{\partial t_2} + V_l\frac{\partial A}{\partial x_2} + V_m\frac{\partial A}{\partial y_2}\right) + P_1\frac{\partial^2 A}{\partial x_1^2} + 2P_2\frac{\partial^2 A}{\partial x_1\partial y_1} + P_3\frac{\partial^2 A}{\partial y_1^2} = Q|A|^2 A, \tag{25}$$

where

$$P_1 = \frac{k}{\omega(1+\rho)}\left[\frac{\omega^2(1+\rho)}{k^5}(m^2 - 2l^2) + 2 - \frac{(\mu-1)^2 A_0^2 m^2}{\mu(\mu+1)k^3} + \frac{4V_l\omega l(1+\rho)}{k^3} - \frac{2(1+\rho)}{k^3}V_l^2\right], \tag{26}$$

$$P_2 = \frac{k}{\omega(1+\rho)}\left[\frac{lm}{k^5}\left(-3\omega^2(1+\rho) + \frac{(\mu-1)^2}{\mu(\mu+1)}k^2A_0^2\right) + \frac{2\omega(1+\rho)}{k^3}(mV_m + lV_l) - \frac{2(1+\rho)}{k}V_lV_m\right], \tag{27}$$

$$P_3 = \frac{k}{\omega(1+\rho)}\left[\frac{\omega^2(1+\rho)}{k^5}(l^2 - 2m^2) + 2 - \frac{(\mu-1)^2 l^2 A_0^2}{\mu(\mu+1)k^3} + \frac{4V_m\omega m(1+\rho)}{k^3} - \frac{2(1+\rho)}{k^3}V_m^2\right], \tag{28}$$

$$Q = \frac{k}{2\omega(1+\rho)}\left[4(1-\rho)\omega^2\Lambda + 4(1+\rho)\omega^2 k - 3k^4 - \frac{4k^3(\mu-1)^3A_0^2}{\mu(\mu+1)^2}(\mu^2 - 6\mu + 1)\right], \tag{29}$$

$$\Lambda = \left[\omega^2(1-\rho) - \frac{k^2A_0^2(\mu-1)^3}{\mu(\mu+1)^2}\right]\left[1 - \rho - 2k^2\right]^{-1}. \tag{30}$$

It is interesting to observe that equation (25) has a singularity in Λ for $k^2 = \frac{1}{2}(1-\rho)$. This corresponds to the case of second harmonic resonance, where the fundamental cannot exist without the presence of its first harmonic. We should remark that the analysis given in this paper is not valid in the neighbourhood of such a resonance. To investigate the behaviour near this resonance, the method used earlier by Malik and Singh [11] can be employed.

Equation (25), which is elliptic or hyperbolic according to the sign of $(P_2^2 - P_1P_3)$, initially arose in the study of Kelvin–Helmholtz instability [8, 12]. When $P_2^2 - P_1P_3 < 0$, it can be reduced to the two-dimensional nonlinear Schrödinger equation with the aid of the transformations

$$\alpha = y - \frac{P_2}{P_1}x, \qquad \beta = -\frac{1}{P_1}\left(P^2 - P_1P_3\right)^{1/2}x. \tag{31}$$

Rewriting equation (25) in the group velocity frame of reference, we obtain

$$2i\frac{\partial A}{\partial t} + \Delta_1\frac{\partial^2 A}{\partial \xi^2} + \Delta_2\frac{\partial^2 A}{\partial \eta^2} = Q|A|^2 A, \tag{32}$$

where

$$\xi = \left(P_3 - \frac{P_2^2}{P_1}\right)^{-1/2}\alpha, \qquad \eta = \left(P_3 - \frac{P_2^2}{P_1}\right)^{-1/2}\beta,$$

$$t_2 = \tau, \qquad \Delta_1 = \Delta_2 = 1.$$

On the other hand, if we assume $P_2^2 - P_1P_3$ to be positive and proceed as before, we still recover equation (32), except that $\Delta_1 = 1$ and $\Delta_2 = -1$. Here the nonlinear interaction parameter Q and group velocity rate parameters P change sign depending on the values of the wave numbers l, m and the Alfvén velocity A_0. Following Zakharov [7], the two integrals of motion for equation (32) are

$$I_1 = \iint |A|^2\, d\xi\, d\eta, \tag{33}$$

$$I_2 = \iint \left(\left|\frac{\partial A}{\partial \xi}\right|^2 + \left|\frac{\partial A}{\partial \eta}\right|^2 + \tfrac{1}{2}Q|A|^4\right) d\xi\, d\eta. \tag{34}$$

Furthermore, one can show by direct calculation that

$$\frac{\partial^2}{\partial \tau^2}\iint (\xi^2 + \eta^2)|A|^2\, d\xi\, d\eta = 8I_2. \tag{35}$$

Since I_2 is a constant of motion, integration of (35) yields

$$\iint (\xi^2 + \eta^2)|A|^2\, d\xi\, d\eta = 4\tau^2 I_2 + c_1\tau + c_2, \tag{36}$$

where c_1 and c_2 are constants of integration. From equations (34) and (35), if $Q < 0$ then a wide class of initial data will render I_2 negative. Since the left-hand side of equation (36) is positive definite, the right-hand side will be negative for not too large a value of τ. When I_2 is negative, it leads to a singularity at a certain time $\tau = \tau_0$ where the solution will cease to exist. This singularity would mean that in finite time $|A|$ tends to zero everywhere except at the singular point, called the focus. This phenomenon is termed collapse or strong focussing. For $Q < 0$, collapse will occur if either

$$\Delta_1 = \Delta_2 = 1, \qquad P_2^2 - P_1P_3 < 0, \tag{37}$$

or

$$\Delta_1 = 1, \qquad \Delta_2 = -1, \qquad P_2^2 - P_1P_3 > 0. \tag{38}$$

Berkshire and Gibbon [13] have shown that when condition (38) holds, it is possible to have a focussing effect locally, if not an entire collapse. The nature of the singularity is a subject of great interest. Berkshire and Gibbon [13] established a close analogy to the Sundman results on collapse in the N-body problem in classical mechanics by considering the integral in equation (36) as the moment of inertia, and went on to describe the singularity as $(\tau_0 - \tau)^{-1/2}$.

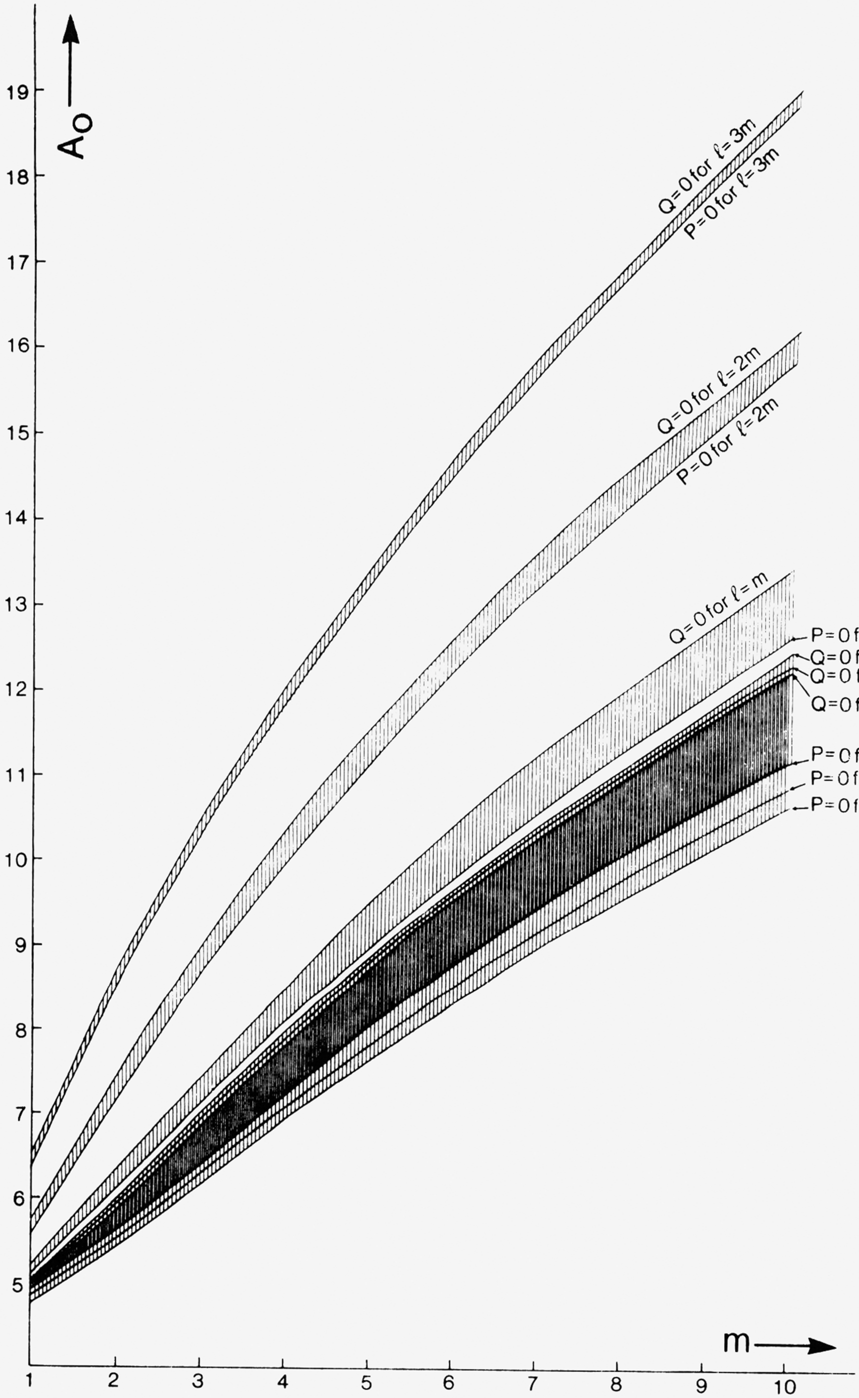

FIG. 1. Plot of normalized Alfvén velocity A_0 versus wave number m. The shaded regions between the transition curves $P = Q = 0$ exhibit strong focussing.

We have computed numerically the collapse criteria (37) and (38) for different values of l, m, μ, ρ, and A_0. The numerical results reveal the existence of collapse for wave numbers $k > k_c = [\frac{1}{2}(1-\rho)]^{1/2}$. Figure 1 exhibits the sketch of m versus A_0 for $\mu = 1.5$, $\rho = 0.4$, and various values of l. The values taken for A_0 are smaller than the critical value of the magnetic field allowed by the linear theory. The graphs outline the transition curves across which $P(= P_2^2 - P_1P_3)$ and Q change sign, indicating the regions where nonlinear focussing takes place. Figure 2 shows how variation of the density ratio ρ affects the transition regions. We conclude that the focussing takes place at the higher values of wave numbers. Furthermore, the effect is more pronounced when the wave numbers m exceed the wave numbers l, implying thereby a strong directional dependence. The focussing occurs for $0.3 < \rho < 0.9$, and the higher the value of ρ, the stronger is the effect. Moreover, part of the region which is stable in linear theory becomes unstable because of nonlinear focussing. The transition occurs from a marginal state to an excited one for the subcritical values of the applied magnetic field. There is a strong interaction between the neighbouring wave numbers, which may lead to the formation of irregular patches at the local level.

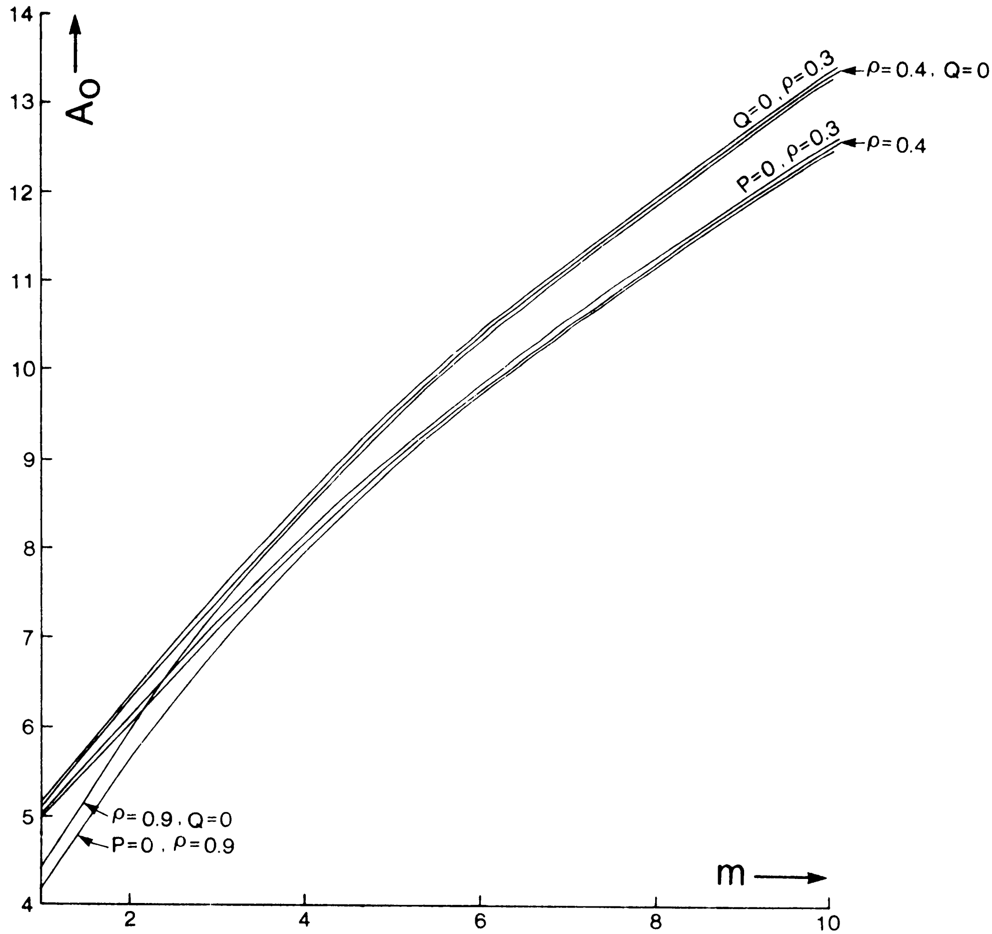

FIG. 2. Sketch of A_0 versus wave number m for different values of the density ratio.

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