QUARTERLY OF APPLIED MATHEMATICS

VOLUME XLIV JANUARY · 1987 NUMBER 4

JANUARY 1987, PAGES 601-620

ON THE SOLVABILITY OF A TWO-DIMENSIONAL WATER-WAVE RADIATION PROBLEM*

By

G. A. ATHANASSOULIS

National Technical University of Athens

Abstract. The existence of a unique weak solution for the two-dimensional water-wave radiation problem arising when a floating rigid body oscillates on the free surface is established for all but a discrete set of oscillation frequencies. The body boundary is taken to be of $\mathscr{C}_*^{1,\alpha}$ class (see Sec. 2) and the body boundary condition is satisfied in the L^2-sense. The proof relies on an expansion theorem (Athanassoulis [1]) and on the property of the associated water-wave multipoles being a Riesz basis of $L^2(-\pi, 0)$, a fact which is established in the present paper. Under stronger geometrical restrictions on the body boundary it is proved, using a method due to Ursell [10], that the weak solution is actually a classical one; that is, the velocity field is continuous up to and including the body boundary.

1. Introduction. This paper is concerned with the study of the boundary value problem arising when a two-dimensional body floating on the free surface of an unbounded, infinitely deep, incompressible, and inviscid fluid performs time-harmonic oscillations of small amplitude about a fixed mean position. This problem is to be studied under the additional assumption that energy is radiated in both directions toward infinity, in which case it is usually called a radiation problem.

The solvability of the water-wave radiation problem for either two- or three-dimensional floating obstacles of general shape has been studied by John [2], Beale [3], Lenoir and Martin [4, 5], and Lenoir [6]. John [2] reduced the problem to a Fredholm singular integral equation of the second kind and treated the three-dimensional case assuming uniform finite depth; however, his results are also valid in the two-dimensional case for either finite or infinite depth. He established the existence and uniqueness of a classical solution when the wetted surface of the floating body satisfies certain geometric conditions, namely it is of class $\mathscr{C}^2$, it intersects the free surface perpendicularly, and it is intersected by every vertical line in at most one point. Beale [3] proved the unique weak

* Received July 1, 1983.

solvability of the same problem for all but a discrete set of oscillation frequencies, removing John's geometric restrictions. Lenoir and Martin [4, 5] treated the case of three-dimensional bodies floating in a fluid of infinite depth. Using the limiting absorption principle, they proved the existence of a generalized solution for all oscillation frequencies for which uniqueness holds. They also provided a general uniqueness theorem, which however, as pointed out by Martin and Ursell, was not correct (see the discussions of a paper by Euvrard et al. [27]). Lenoir [6] treated, by the same method, the two-dimensional case for either finite or infinite depth.

For the two-dimensional case with infinite depth, Athanassoulis [1] showed, using a method due to Ursell [7, 8] and the conformal mapping technique, that if a solution of the radiation problem exists it can be expanded in an infinite series of special functions, called water-wave multipoles.

In the present paper this multipole expansion is to be used to establish the weak solvability of the radiation problem for all but (possibly) a discrete set of oscillation frequencies (Theorem 5.3) provided the body boundary is sufficiently smooth, i.e., it belongs to the class $\mathscr{C}_*^{1,\alpha}$ (see Sec. 2) and, as a consequence, it intersects the free surface perpendicularly. It is worth mentioning that in our approach John's convexity condition is no longer necessary. Subsequently, under stronger geometric restrictions on the body boundary, it is proved that the unique weak solution is, in fact, a classical one; i.e., the velocity field is continuous up to and including the body boundary. Moreover, it is established that the multipole expansion coefficients of the velocity field are of order $O(1/m^2)$ (Theorem 6.1).

The idea of using relevant multipole expansions to prove the solvability of water-wave radiation problems goes back to Ursell (see Ursell [9, 10] and Yu and Ursell [11]), who has treated the case of a semicircular floating body. Ursell reduced the question of the solvability of the radiation problem to that of an infinite linear system and studied it by using Fredholm theory for compact operators. In our treatment the proof of the weak solvability theorem, presented in Sec. 5, makes use of the basis property of the water-wave multipoles, which is also established in the same paragraph, while the proof of the regularity theorem, presented in Sec. 6, is along lines due to Ursell [10].

In establishing the basis property of the water-wave multipoles we need some elements of the spectral theory of compact operators analytically dependent on the spectral parameter. The pertinent background material along with the needed elements of the theory of bases in Hilbert spaces is summarized in Sec. 4.

2. Notation and terminology. A Cartesian coordinate system Ox_2x_3 is introduced with the x_2-axis on the mean free surface, the x_3-axis directed vertically upward, i.e., in the direction opposite to that of the acceleration of gravity, and the center O inside the floating body. A point in the (x_2, x_3)-plane is denoted by $x = (x_2, x_3)$, or $w = x_2 + ix_3$ in complex notation.

The mean fluid domain D is defined by $D = \{x;\ x_3 < 0,\ x \notin D_B\}$, where D_B is a compactum in the lower half-plane, intersecting the x_2-axis and representing the floating body. The mean free surface of the fluid, i.e., the part of the x_2-axis lying outside D_B, is denoted by ∂D_F, and the mean wetted surface of the floating body, i.e., the common boundary of D and D_B, is denoted by ∂D_B. The symbols ∂D_B^+, D_B^+, D^+ denote the point sets which are symmetric to ∂D_B, D_B, D with respect to the x_2-axis; see Fig. 1. We also define

$$\partial D_B^* = \partial D_B \cup \partial D_B^+, \qquad D^* = D \cup D^+ \cup \partial D_F, \qquad D_B^* = D_B \cup D_B^+.$$

The class of quadratically Lebesgue-integrable, complex-valued functions, defined almost everywhere in (a, b), is denoted by $L^2(a, b)$. The scalar product and the norm in $L^2(a, b)$ are denoted by (f, g) and $\|f\| = (f, f)^{1/2}$, respectively.

The space $L^2(-\pi, 0)$ is decomposed into two orthogonal subspaces: the subspace L_A^2, spanned by $\{1, \cos\theta\}$, and the subspace $L_B^2 = (L_A^2)^\perp$, in which $\{\cos(m\theta)\}_2^\infty$ forms a complete orthogonal sequence.

Some smoothness requirements are needed for the body boundary ∂D_B.

We shall say that ∂D_B belongs to the class $\mathscr{C}_*^{1,\alpha}$ if ∂D_B^* is a simple closed curve, described parametrically by equations

$$x_2 = x_2(\theta), \qquad x_3 = x_3(\theta), \quad \theta \in [-\pi, \pi], \tag{2.1}$$

and the functions $x_2(\theta)$, $x_3(\theta)$ have the following properties:

(i) They are continuously differentiable and their first derivatives satisfy a Hölder condition with exponent $\alpha \in (0, 1]$.

(ii) Either $\alpha > 1/2$ or the first derivatives of $x_2(\theta)$ and $x_3(\theta)$ are of bounded variation.

(iii) $(\dfrac{dx_2(\theta)}{d\theta})^2 + (\dfrac{dx_3(\theta)}{d\theta})^2 \neq 0$ for every $\theta \in [-\pi, \pi]$.

The above conditions are needed to ensure the validity of Lemma 3.1 below.

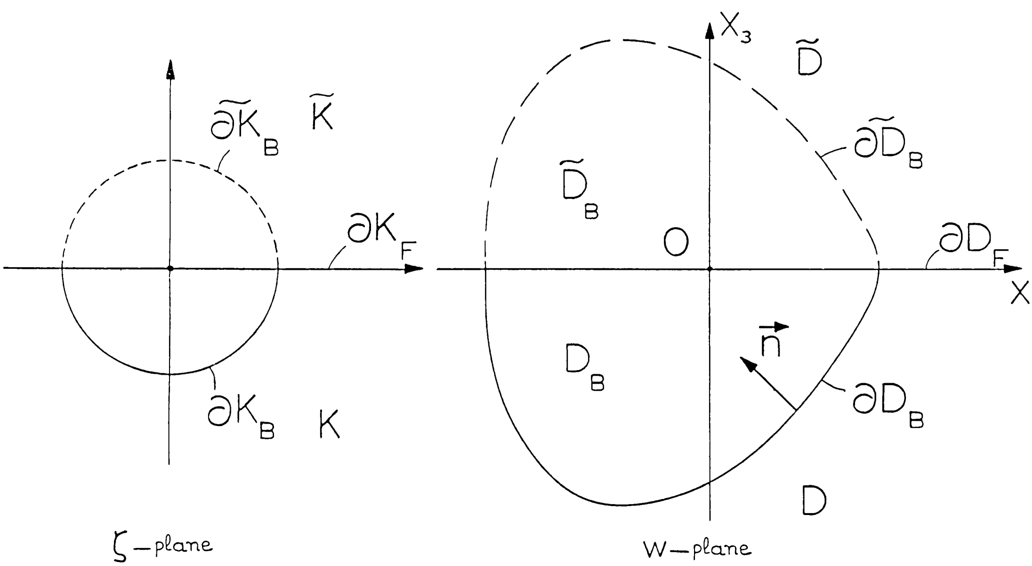

FIG. 1. Geometric description.

More restrictive assumptions on the body boundary will be introduced in Sec. 6, where the regularity of the radiation problem will be studied.

Two noninteracting imaginary units j and i are used, making necessary the distinction between the j-complex numbers $a = \alpha + j\beta(\alpha, \beta \in \mathbb{R})$, and the i-complex numbers $z = \gamma + i\delta(\gamma, \delta \in \mathbb{R})$. Products of j- and i-complex numbers also occur, leading to objects of the form

$$\alpha + j\beta + i\gamma + ij\delta, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}, (ij \equiv ji),$$

which are called ij-complex numbers. The sets of j-, i- and ij-complex numbers are denoted by $\mathbb{C}_j$, $\mathbb{C}_i$, and $\mathbb{C}_{ij}$, respectively. The basic notations and operations in $\mathbb{C}_{ij}$ are defined below.

Let A, A_s, $s = 1,2$, be ij-complex numbers. They can be represented in the form $A = z + jw$, $A_s = z_s + jw_s$, where z, w, z_s, $w_s \in \mathbb{C}_i$. Then we define:

Equality: $$A_1 = A_2 \Leftrightarrow z_1 = z_2 \quad \text{and} \quad w_1 = w_2, \tag{2.2}$$

Addition: $$A_1 + A_2 = (z_1 + z_2) + j(w_1 + w_2), \tag{2.3}$$

Zero element: $$0_{\mathbb{C}_{ij}} = O + jO, \tag{2.4}$$

Multiplication: $$A_1 \cdot A_2 = (z_1 z_2 - w_1 w_2) + j(z_1 w_2 + z_2 w_1), \tag{2.5}$$

Unit element: $$1_{\mathbb{C}_{ij}} = 1 + j0, \tag{2.6}$$

Scalar multiplication: $$\lambda A = (\lambda z) + j(\lambda w), \quad \lambda \in \mathbb{C}_i, \tag{2.7}$$

Absolute value: $$|A|_{\mathbb{C}_{ij}} = \left(|z|^2 + |w|^2\right)^{1/2}. \tag{2.8}$$

The subscript $\mathbb{C}_{ij}$ in the definition of the zero and unit elements and the absolute value will be omitted in the sequel whenever no confusion is likely to occur.

The system $(\mathbb{C}_{ij}, +, \cdot, 0, 1)$ is a commutative ring, which, however, is not a field (there exist nonzero, noninvertible elements). Furthermore, $|\cdot|_{\mathbb{C}_{ij}}$ is a norm, and the set $\mathbb{C}_{ij}$, equipped with the operations (2.3), (2.5), (2.7), and the norm (2.8), becomes a commutative Banach algebra. For a rather complete study of the algebraic and topological structure of $\mathbb{C}_{ij}$ see Athanassoulis [1, appendix I].

Finally, if $A = \alpha + j\beta + i\gamma + ij\delta \in \mathbb{C}_{ij}$, we define

$$\mathrm{Re}_i A = \alpha + j\beta, \qquad \mathrm{Im}_i A = \gamma + j\delta,$$

$$\mathrm{Re}_j A = \alpha + i\gamma, \qquad \mathrm{Im}_j A = \beta + i\delta.$$

3. Formulation of the radiation problem and some preliminary results. Under the assumptions made in Sec. 1, the fluid motion is described by a velocity potential

$$\phi(x, t) = \mathrm{Re}_j\{\phi(x) e^{j\omega t}\},$$

where $\phi(x)$, the j-complex amplitude of $\phi(x, t)$, satisfies the Laplace equation

$$\phi_{,22}(x) + \phi_{,33}(x) = 0, \quad x \in D, \tag{3.1}$$

and the boundary conditions

$$K_0\phi(x) - \phi_{,3}(x) = 0, \qquad K_0 = \omega^2/g, \quad x \in \partial D_F, \tag{3.2}$$

$$\frac{\partial\phi(x)}{\partial n} = u_n(x), \quad x \in \partial D_B, \tag{3.3}$$

$$|\phi_{,s}(x)| \to 0, \quad s = 2,3,\ x_3 \to -\infty, \tag{3.4}$$

and

$$\frac{\partial\phi(x)}{\partial|x_2|} - jK_0\phi(x) \to 0, \quad x \in D, |x_2| \to \infty. \tag{3.5}$$

Here and below, ω is the frequency of oscillation, g is the acceleration due to gravity, and $\partial/\partial n$ denotes differentiation along the normal $n = (n_2, n_3)$ of ∂D_B, directed outward with respect to the fluid.

Introducing the ij-complex potential

$$F(w) = \phi(x) + i\psi(x),$$

where $\psi(x)$ is the j-complex amplitude of the stream function $\psi(x, t)$, Eqs. (3.1)–(3.5) are transformed to

$$F(w) \text{ be } i\text{-analytic in } D, \tag{3.6}$$

$$\operatorname{Im}_i\left\{\frac{dF(w)}{dw} + iK_0F(w)\right\} = 0, \quad w \in \partial D_F, \tag{3.7}$$

$$\operatorname{Re}_i\left\{n\,\frac{dF(w)}{dw}\right\} = u_n(w), \quad w \in \partial D_B, \tag{3.8}$$

$$\left|\frac{dF(w)}{dw}\right|_{\mathbb{C}_{ij}} \to 0, \quad x_3 \to -\infty, \tag{3.9}$$

and

$$F(w) \to \Lambda^{\pm}(1 \mp ij)e^{-ik_0w}, \quad \Lambda^{\pm} \in \mathbb{C}_j,\ w \in D,\ x_2 \to \pm\infty, \tag{3.10}$$

respectively.

Relations (3.5) and (3.10) are two forms of the radiation condition. For a discussion of this condition see, e.g., Stoker [12, Secs. 4.3, 6.7] and Athanassoulis [1, Secs. 2, 4].

A conformal mapping $w = f(\zeta)$ is now introduced, transforming the exterior of the unit circle $\{\zeta\colon |\zeta| > 1\}$ in the ζ-plane onto the domain D^* in the w-plane. Such a function always exists, provided that D_B^* has at least two boundary points (Riemann's mapping theorem) and may be chosen so that the point sets $\partial K_B = \{\zeta\colon \operatorname{Im}_i\zeta \leqslant 0,\ |\zeta| = 1\}$, $\partial K_F = \{\zeta\colon \operatorname{Im}_i\zeta = 0,\ |\zeta| \geqslant 1\}$, and $K = \{\zeta\colon \operatorname{Im}_i\zeta < 0,\ |\zeta| > 1\}$ in the ζ-plane are to be transformed to the point sets ∂D_B, ∂D_F, and D in the w-plane, respectively (see Fig. 1).

Finally, we define

$$\overline{K} = \left\{\zeta\colon \operatorname{Im}_i\zeta \leqslant 0, |\zeta| \geqslant 1\right\} = K \cup \partial K_F \cup \partial K_B.$$

If ∂D_B belongs to the class $\mathbb{C}_*^{1,\alpha}$, then:

1. The function $f(\zeta)$ can be extended in the region $\{\zeta: |\zeta| \geqslant 1\}$, establishing a one-to-one and bicontinuous correspondence between the regions $\{\zeta: |\zeta| \geqslant 1\}$ and $D^* \cup \partial D_B^*$ (Osgood-Carathéodory theorem).
2. The function $f(\zeta)$ is continuously differentiable and its derivative $df(\zeta)/d\zeta$ does not vanish in $\{\zeta: |\zeta| \geqslant 1\}$ (Kellogg's theorem).
3. The function $f(\zeta)$ is expanded in a Laurent series of the form
$$f(\zeta) = \sum_{l=1}^{\infty} c_l \zeta^{2-l}, \quad c_1 > 0, |\zeta| \geqslant 1, \tag{3.11}$$
where all c_l's are real numbers, since the domain D^* is symmetric with respect to the x_2-axis.
4. The equation
$$w(\theta) = f(e^{i\theta}) = \sum_{l=1}^{\infty} c_l e^{i(2-l)\theta} \tag{3.12}$$
realizes a parametric representation of the boundary ∂D_B^* such that the functions $x_2(\theta) = \mathrm{Re}_i\{w(\theta)\}$ and $x_3(\theta) = \mathrm{Im}_i\{w(\theta)\}$ satisfy conditions (i), (ii), (iii) introduced in the definition of the $\mathscr{C}_*^{1,\alpha}$ class.

Using the change of variable $w = f(\zeta)$, we may transform Eqs. (3.6)–(3.10) into the equations

$$F_\zeta(\zeta) \text{ be } i\text{-analytic in } K, \tag{3.13}$$

$$\mathrm{Im}_i\left\{\frac{dF_\zeta(\zeta)}{d\zeta} + iK_0 \frac{df(\zeta)}{d\zeta} F_\zeta(\zeta)\right\} = 0, \quad \zeta \in \partial K_F, \tag{3.14}$$

$$\mathrm{Re}_i\left\{-\zeta \frac{dF_\zeta(\zeta)/d\zeta}{|df(\zeta)/d\zeta|}\right\} = u_n(\zeta), \quad \zeta \in \partial K_B, \tag{3.15}$$

$$\left|\frac{dF_\zeta(\zeta)}{d\zeta}\right|_{\mathbb{C}_{i,j}} \to 0, \quad \mathrm{Im}_i\zeta \to -\infty \tag{3.16}$$

$$F_\zeta(\zeta) \to \Lambda^\pm(1 \mp ij)e^{-iK_0c_1\zeta}, \quad \Lambda^\pm \in \mathbb{C}_j, \zeta \in K, \mathrm{Re}_i\zeta \to \pm\infty, \tag{3.17}$$

where $F_\zeta(\zeta)$ stands for $F(f(\zeta))$.

It should be noted that the complex potential $F_\zeta(\zeta)$ also depends on the parameter $K_0 = \omega^2/g$. To emphasize this fact we shall often write $F_\zeta(\zeta; K_0)$ instead of $F_\zeta(\zeta)$.

The three sets of equations (3.1)–(3.5), (3.6)–(3.10), and (3.13)–(3.17) are mutually equivalent under the assumption $\partial D_B \in \mathscr{C}_*^{1,\alpha}$. In the present paper we shall work with the third formulation and, for the sake of brevity, Eqs. (3.13)–(3.17) will be collectively referred to as the *radiation problem* $\mathscr{P}(K_0)$.

We shall now state an expansion theorem for the wave potential $F_\zeta(\zeta;\ K_0)$, which is of fundamental importance for the subsequent treatments. This theorem provides an extension of the multipole expansion introduced by Ursell [26] for body boundaries symmetric with respect to the vertical axis.

THEOREM 3.1. (The expansion theorem; Athanassoulis [1, Sec. 4].) A function $F_\zeta(\zeta; K_0)$ satisfies conditions (3.13), (3.14), (3.16), and (3.17) if and only if it may be represented in the form

$$F_\zeta(\zeta; K_0) = \Lambda_0 G_0(f(\zeta); K_0) + \Lambda_1 G_1(f(\zeta); K_0) + \sum_{m=2}^{\infty} d_m M_m(\zeta; K_0), \quad (3.18)$$

where

$$\Lambda_0,\ \Lambda_1,\ d_m \in \mathbb{C}_j,$$

$$M_m(\zeta; K_0) = \zeta^{-m} - iK_0 \sum_{l=1}^{\infty} \frac{(2-l)c_l \zeta^{2-l-m}}{2-l-m}, \quad m = 2, 3, \ldots, \quad (3.19)$$

$$G_0(w; K_0) = \frac{1}{2}(1-ij)K_0 e^{-iK_0 w} - \frac{jK_0}{2\pi} F_1(w; K_0), \quad (3.20a)$$

$$G_1(w; K_0) = \frac{1}{2}(1-ij)K_0 e^{-iK_0 w} - \frac{1}{2\pi} F_2(w; K_0), \quad (3.20b)$$

$$F_s(w; K_0) = e^{-iK_0 w} \int_{\infty + i0}^{w} u^{-s} e^{iK_0 u}\, du, \quad s = 1, 2, \quad (3.21a)$$

and the path of integration in the last integrals is taken to be in the lower half-plane.

REMARK 3.1.1. The functions $G_0(w; K_0)$ and $G_1(w; K_0)$ represent free surface flows which are, respectively, symmetric and antisymmetric with respect to the axis Ox_3.

REMARK 3.1.2. The function $F_1(w; K_0)$ can be written in the following alternative form:

$$F_1(w; K_0) = e^{-iK_0 w}\left\{ \gamma + \ln(iK_0 w) - \ln(-1) + \sum_{n=1}^{\infty} \frac{(iK_0 w)^n}{n n!} \right\}, \quad (3.21b)$$

where γ is Euler's constant. With the aid of the above expression we see that $F_1(w; K_0)$ is, for each $w \neq 0$, an analytic function of K_0 in $Q \subset \mathbb{C}_j$, where Q is any open, bounded, simply connected region of $\mathbb{C}_j$ containing a part of the positive real axis but not containing the origin. Moreover, the function $K_0 F_1(w; K_0)$ tends to zero as $K_0 \to 0$ for each fixed $w \neq 0$. Since

$$F_2(w; K_0) = -\frac{1}{w} + iK_0 F_1(w; K_0), \qquad \frac{dF_s(w; K_0)}{dw} = \frac{1}{w^s} - iK_0 F_s(w; K_0), \quad s = 1, 2,$$

it follows that $F_2(w; K_0)$ and $dF_s(w; K_0)/dw$ are also analytic functions of $K_0 \in Q$ and they remain bounded as $K_0 \to 0$, for each $w \neq 0$.

Now introducing (3.18) into (3.15) and differentiating term by term, we obtain

$$\operatorname{Re}_i \left\{ \Lambda_0 H_0(\zeta; K_0) + \Lambda_1 H_1(\zeta; K_0) + \sum_{m=2}^{\infty} (m d_m) \left\{ \zeta^{-m} + \frac{iK_0}{m} \sum_{l=1}^{\infty} (2-l) c_l \zeta^{2-l-m} \right\} \right\}_{\zeta = e^{i\theta}}$$
$$= V(\theta), \quad \theta \in [-\pi, 0], \quad (3.22)$$

where $H_s(\zeta; K_0) = -\zeta dG_s(f(\zeta); K_0)/d\zeta$, $s = 0, 1$, $V(\theta) = u_n(\theta)|df(e^{i\theta})/d\zeta|$, and $u_n(\theta)$ stands for $u_n(f(e^{i\theta}))$.

The termwise differentiation in (3.22) is justified, in the sense of pointwise convergence, if the series $\Sigma|2 - l|\ |c_l|$ and $\Sigma|md_m|$ converge. The convergence of the first series is a consequence of the body boundary smoothness assumption $\partial D_B \in \mathscr{C}_*^{1,\alpha}$ (see Lemma 3.1 below). The convergence of the second series will be considered in Sec. 6, where it will be established that $d_m = O(1/m^3)$, provided that $f(\zeta)$ is represented by a finite Laurent series and that ∂D_B^* is a simple, closed curve with continuous tangent. However, if the left-hand side of (3.22) is interpreted in a limiting sense, that is, as the limit when $|\zeta| \to 1+$, the weaker condition $\Sigma|md_m|^2 < \infty$ is sufficient to make (3.22) valid in the L^2-sense. This point of view will be adopted in Sec. 5, where the weak solvability of the problem $\mathscr{P}(K_0)$ will be proved without restricting $f(\zeta)$ to be a finite Laurent series.

Now setting

$$\varphi_m(\theta; K_0) = \operatorname{Re}_i\left\{ H_m\left(e^{i\theta}; K_0\right)\right\}, \quad m = 0, 1, \tag{3.23}$$

$$\varphi_m(\theta; K_0) = \cos(m\theta) + \frac{K_0}{m}\sum_{l=1}^{\infty}(2 - l)c_l \sin(l + m - 2)\theta, \quad m = 2, 3, \ldots, \tag{3.24}$$

$$\Lambda_0 = D_0, \quad \Lambda_1 = D_1, \qquad md_m = D_m, \quad m = 2, 3, \ldots, \tag{3.25}$$

we can rewrite Eq. (3.22) in the form

$$\sum_{m=0}^{\infty} D_m\varphi_m(\theta; K_0) = V(\theta), \quad \theta \in [-\pi, 0]. \tag{3.26}$$

The central question is now the following: Under what conditions and in what sense may the function $V(\theta) = u_n(\theta) \cdot |df(e^{i\theta})/d\zeta|$ be expanded with respect to the sequence $\{\varphi_m(\theta; K_0)\}_0^\infty$?

It should be noted that the forcing term $V(\theta)$ may also depend on the parameter K_0, as, for example, in the diffraction problem. But this fact does not introduce any further complication.

We now state a lemma on the conformal mapping coefficients of a $\mathscr{C}_*^{1,\alpha}$ boundary, which is often used in this work.

LEMMA 3.1. If $\partial D_B \in \mathscr{C}_*^{1,\alpha}$ and $\{c_l\}$ are the Laurent expansion coefficients of the conformal mapping function $f(\zeta)$ [see Eq. (3.11)], then

$$\sum_{l=1}^{\infty} |2 - l|\,|c_l| < \infty.$$

The proof of this lemma is based on two classical theorems on the absolute convergence of Fourier series (Zygmund [13], Theorems 3.1 and 3.6, pp. 240–241) and the Denjoy-Lusin theorem on the absolute convergence of the series of Fourier coefficients (*ibid*., p. 232).

Finally, we collect, in the form of a theorem, some properties of the functions $G_s(f(\zeta); K_0)$, $s = 0, 1$, $M_m(\zeta; K_0)$, $m = 2, 3, \ldots$, and $\varphi_m(\theta; K_0)$, $m = 0, 1, 2, \ldots$, which will be needed in Sec. 5.

THEOREM 3.2. Suppose that $\partial D_E \in \mathscr{C}_*^{1,\alpha}$. Then

(i) $G_s(f(\zeta); K_0)$, $s = 0,1$, are continuously differentiable with respect to $\zeta \in \overline{K}$ for any fixed $K_0 \in Q$. They are also analytic functions of $K_0 \in Q$, for each $\zeta \in \overline{K}$.

(ii) $M_m(\zeta; K_0)$, $m = 2, 3, \ldots$, are uniformly bounded with respect to m and continuously differentiable with respect to $\zeta \in \{\zeta\colon |\zeta| \geq 1\}$, the infinite series contained being termwise differentiable.

(iii) $\varphi_s(\theta; K_0)$, $s = 0, 1$, are continuous with respect to $\theta \in [-\pi, 0]$ for any fixed $K_0 \in Q$. They are also analytic functions of $K_0 \in Q$, for each $\theta \in [-\pi, 0]$.

(iv) $\varphi_m(\theta; K_0)$, $m = 2, 3, \ldots$, are uniformly bounded with respect to m and continuous with respect to $\theta \in [-\pi, \pi]$.

The proof of this theorem is easily carried out with the aid of Lemma 3.1.

4. Background theory. In this section we present some definitions and theorems underlying the proof of the solvability theorem, given in the next section. The materials presented concern the spectral theory of compact operators analytically dependent on the spectral parameter and the theory of bases; they are given for the Hilbert space case, although they can be properly generalized for Banach or even, in some cases, for linear topological spaces (see Harazov [22] and Singer [14]).

Let E be a separable Hilbert space and $B(E)$ be the set of all bounded linear operators in E. By $(\cdot, \cdot)$ and $\|\cdot\|$ are denoted the inner product and the induced norm in E, respectively.

DEFINITION 4.1. Let $T(\lambda)$ [resp. $g(\lambda)$] be an operator-valued (resp. vector-valued) function defined on a simply connected region $Q \subset \mathbb{C}$, with values in $B(E)$ (resp. E). We shall say that $T(\lambda)$ [resp. $g(\lambda)$] is *analytic* in Q if the scalar-valued function $(T(\lambda)x, y)$ [resp. $(g(\lambda), x)$] is analytic in Q for any x, $y \in E$.

This apparently "weak" notion of analyticity is actually equivalent to the "strong" one, based on the existence of a Fréchet derivative with respect to λ. (See, e.g., Taylor [24, p. 205] or Hille and Phillips [25, p. 93].)

THEOREM 4.1. (Gohberg and Krein [23, p. 21].) Let $T(\lambda)$ be an operator-valued function, analytic in an open, simply connected region $Q \subset \mathbb{C}$, and suppose that all values of $T(\lambda)$ are compact operators. Then, for all points $\lambda \in Q \setminus \Sigma$, where Σ is a set of isolated points of Q, the number $\tau(\lambda)$ of linearly independent solutions of the equation

$$x - T(\lambda)x = 0, \quad x \in E,$$

is constant, i.e., $\tau(\lambda) = n$. For $\lambda \in \Sigma$ we have $\tau(\lambda) > n$.

In particular, if $\tau(\lambda) = 0$ for at least one point in Q, then for all $\lambda \in Q \setminus \Sigma$ the operator $I - T(\lambda)$ has a bounded inverse.

DEFINITION 4.2. Let $\{x_n\}$, $\{y_n\}$ be two sequences with elements in E. Then:

(a) The two sequences $\{x_n\}$, $\{y_n\}$ will be said to be *quadratically near* each other if

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < \infty. \tag{4.1}$$

(b) The two sequences $\{x_n\}$, $\{y_n\}$ will be said to be *equivalent* if there exists an automorphism T such that $y_n = Tx_n$.

THEOREM 4.2. (Higgins [15, p. 75].) Suppose that $\{x_n\}$ and $\{y_n\}$ are two equivalent sequences in E. Then, $\{x_n\}$ is a (Schauder, Bessel, Hilbert, Riesz) basis of E if and only if $\{y_n\}$ is a (Schauder, Bessel, Hilbert, Riesz) basis of E.

We shall now state and prove a theorem providing a criterion for the equivalence of a sequence in a Hilbert space E to a complete orthonormal system of E. This theorem can be considered as an extension of some relative theorems of Bary (see Bary [16] or Kato [17, pp. 246–266]).

THEOREM 4.3. Suppose that $\{x_n\}$ is a complete orthonormal system of a Hilbert space E, and $\{y_n(\lambda)\}$, $\lambda \in Q \subset \mathbb{C}$, is a family of sequences in E defined by

$$y_n(\lambda) = x_n + \lambda g_n(\lambda), \tag{4.2}$$

where $g_n(\lambda)$, $n = 1, 2, \ldots$, are analytic vector-valued functions of $\lambda \in Q$, with values in E. Suppose also that

$$\sum_{n=1}^{\infty} \|g_n(\lambda)\|^2 < B < \infty, \tag{4.3}$$

where B is a positive constant, independent of λ, and that $B \cdot |\lambda| < 1$ for at least one point of Q.

Then the sequences $\{x_n\}$, $\{y_n(\lambda)\}$ are equivalent for all $\lambda \in Q \setminus \Sigma$, where Σ is a set of isolated points of Q. Moreover, $\{y_n(\lambda)\}$ is a Riesz basis of E for all $\lambda \in Q \setminus \Sigma$.

REMARK 4.3.1. The set Σ may be countable, finite, or even empty.

REMARK 4.3.2. Condition (4.3) is equivalent to the quadratic nearness of the sequences $\{x_n\}$ and $\{y_n(\lambda)\}$.

To prove Theorem 4.3 the following Lemma is needed.

LEMMA 4.1. Suppose that $\{x_n\}$ is a complete orthonormal system of E and $\{g_n\}$ is a sequence in E such that $\Sigma\|g_n\|^2 < \infty$. Then the operator T: $E \to E$ defined on $\{x_n\}$ by $Tx_n = g_n$ can be linearly extended on the whole E as a compact operator.

Proof. To start with, let us extend the operator T on the whole E. If only a finite number of α_n in the expansion $\Sigma \alpha_n x_n$ of an element $x \in E$ are different from zero, then we define $Tx = \Sigma \alpha_n T x_n = \Sigma \alpha_n g_n$. The same definition remains valid when the series $\Sigma \alpha_n x_n$ has infinite terms, provided that the series $\Sigma \alpha_n g_n$ converges in E. This is indeed the case since $(N > M)$,

$$\left\| \sum_{n=M}^{N} \alpha_n g_n \right\|^2 \leqslant \sum_{n=M}^{N} |\alpha_n|^2 \cdot \sum_{n=M}^{N} \|g_n\|^2,$$

and the series $\Sigma|\alpha_n|^2$ and $\Sigma\|g_n\|^2$ converge. The compactness of the operator T follows from the condition $\Sigma\|Tx_n\|^2 = \Sigma\|g_n\|^2 < \infty$. See, e.g., Smirnov [18, Sec. 138].

Proof of Theorem 4.3. We introduce the operator $T(\lambda)$ defined on $\{x_n\}$ by $T(\lambda)x_n = g_n(\lambda)$ and extend it on the whole E by means of the definition

$$T(\lambda)x = \sum_{n=1}^{\infty} \alpha_n g_n(\lambda), \quad \text{where } x = \sum_{n=1}^{\infty} \alpha_n x_n.$$

According to Lemma 4.1, $T(\lambda)$ is a compact operator for each $\lambda \in Q$.

Now consider the operator $A(\lambda) = I + \lambda T(\lambda)$. Obviously, $A(\lambda)x_n = y_n(\lambda)$; thus the equivalence of the sequences $\{x_n\}$ and $\{y_n(\lambda)\}$ is equivalent to the invertibility of the operator $A(\lambda)$. Since

$$\|\lambda T(\lambda)\| = |\lambda| \|T(\lambda)\| \leqslant |\lambda| \sum_{n=1}^{\infty} \|g_n(\lambda)\|^2 < |\lambda| \cdot B,$$

it follows that there exists a point $\lambda = \lambda_0$ in which $\|\lambda_0 T(\lambda_0)\| < 1$; this ensures the invertibility of $A(\lambda_0)$. Accordingly, applying Theorem 4.1, we conclude that $A(\lambda)$ is invertible for all $\lambda \in Q \setminus \Sigma$. Finally, invoking Theorem 4.2, we see that $\{y_n(\lambda)\}$ is a Riesz basis of E for all $\lambda \in Q \setminus \Sigma$. This completes the proof of the theorem.

Remark 4.3.3. Since $A(\lambda)x = \sum_{n=1}^{\infty} \alpha_n y_n(\lambda)$, where $x = \sum_{n=1}^{\infty} \alpha_n x_n$ and $\sum_{n=1}^{\infty} |\alpha_n|^2 = \|x\|^2 < \infty$, the invertibility of the operator $A(\lambda)$ is equivalent to the condition

$$\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n y_n(\lambda) = 0 \Rightarrow \alpha_n = 0 \text{ for every } n \in \mathbb{N}, \tag{4.4}$$

i.e., the l^2-linear independence of the sequence $\{y_n(\lambda)\}$. In the application to the radiation problem, condition (4.4) is, in fact, a uniqueness theorem.

5. The weak solvability of the radiation problem. The main tool in establishing the solvability of the radiation problem $\mathscr{P}(K_0)$ is the basis property of the sequence $\{\varphi_m(\theta; K_0)\}_0^{\infty}$ in the space $L^2(-\pi, 0)$. Let us note that the functions $\varphi_m(\theta; K_0)$ depend analytically on the parameter $K_0 \in \mathbb{C}_j$, which corresponds to the λ parameter in the general theory of the preceding section.

Lemma 5.1. Suppose that $\partial D_B \in \mathscr{C}_*^{1,\alpha}$, $K_0 \in \mathbb{C}_j$, and consider the multipoles

$$\varphi_m(\theta; K_0) = \cos(m\theta) + \frac{K_0}{m} \sum_{l=1}^{\infty} (2-l) c_l \sin(l+m-2)\theta, \quad m = 2, 3, \ldots. \tag{5.1}$$

Then,

$$\begin{aligned} &\text{(i)} \qquad \varphi_m(\theta; K_0) \in L^2(-\pi, 0),\ m = 2, 3, \ldots, \\ &\text{(ii)} \qquad \left\| \sum_{l=1}^{\infty} (2-l) c_l \sin(l+m-2)\theta \right\|^2 = \frac{\pi}{2} \sum_{l=1}^{\infty} (2-l)^2 c_l^2 < \infty, \end{aligned} \tag{5.2}$$

whereby it follows that

$$\sum_{m=2}^{\infty} \|\varphi_m(\theta; K_0) - \cos(m\theta)\|^2 < \infty. \tag{5.3}$$

The proof of the above results is straightforward, so it will be omitted.

According to (5.3) the sequence $\{\varphi_m(\theta; K_0)\}_2^{\infty}$ is quadratically near to the system $\{\cos(m\theta)\}_2^{\infty}$, which is an orthogonal basis of the Hilbert space L_B^2. However, we cannot deduce any completeness or basis property of $\{\varphi_m(\theta; K_0)\}_2^{\infty}$ from this fact, since in general $\varphi_m(\theta; K_0) \notin L_B^2$, $m = 2, 3, \ldots$.

To proceed, we shall modify the multipoles $\varphi_m(\theta; K_0)$ in such a manner that the modified multipoles, say $\psi_m(\theta; K_0)$, (i) belong to L_B^2 and (ii) satisfy a condition similar to (5.3).

Let us set

$$\psi_m(\theta; K_0) = \varphi_m(\theta; K_0) - A_m^0 \varphi_0(\theta; K_0) - A_m^1 \varphi_1(\theta; K_0), \quad m = 2, 3, \dots, \tag{5.4}$$

and try to determine the constants $A_m^{0,1}$ by means of the relations

$$\begin{aligned} \varphi_{0m}(K_0) - A_m^0 \varphi_{00}(K_0) - A_m^1 \varphi_{01}(K_0) &= 0, \\ \varphi_{1m}(K_0) - A_m^0 \varphi_{10}(K_0) - A_m^1 \varphi_{11}(K_0) &= 0, \end{aligned} \tag{5.5}$$

where $\varphi_{sm}(K_0) = (\cos(s\theta), \varphi_m(\theta; K_0))$. Relations (5.5) are equivalent to the condition $\psi_m(\theta; K_0) \in L_B^2$.

The system (5.5) is uniquely solvable for each m, provided that

$$\Delta(K_0) = \varphi_{11}(K_0)\varphi_{00}(K_0) - \varphi_{10}(K_0)\varphi_{01}(K_0) \neq 0^*. \tag{5.6}$$

(In any case, since $\Delta(K_0)$ is an analytic function of K_0, it may have only isolated zeros in Q. Consequently, we can, if necessary, modify the region Q, excluding from it the zeros of $\Delta(K_0)$). Then, the constants $A_m^{0,1}$ can be expressed in the form

$$A_m^{0,1} = A_m^{0,1}(K_0) = \frac{K_0}{m} \mathscr{P}_m^{0,1}(K_0), \quad m = 2, 3, \dots, \tag{5.7}$$

where

$$\begin{aligned} \mathscr{P}_m^0(K_0) &= (\mathscr{O}_{0m}\varphi_{11}(K_0) - \mathscr{O}_{1m}\varphi_{01}(K_0))/\Delta(K_0), \\ \mathscr{P}_m^1(K_0) &= (-\mathscr{O}_{0m}\varphi_{10}(K_0) + \mathscr{O}_{1m}\varphi_{11}(K_0))/\Delta(K_0), \end{aligned} \tag{5.8}$$

and

$$\mathscr{O}_{sm} = \sum_{l=1}^{\infty} \left\{(-1)^{s+l+m} - 1\right\} \frac{(2-l)c_l(l+m-2)}{(l+m-2)^2 - s^2}. \tag{5.9}$$

The infinite series in (5.9) converges, provided that $\partial D_B \in \mathscr{C}_*^{1,\alpha}$. From Eqs. (5.8) and (5.9) it is easily seen that

$$\left|\mathscr{P}_m^{0,1}(K_0)\right| < B^{0,1}, \quad m = 2, 3, \dots, \quad K_0 \in Q. \tag{5.10}$$

We can now state and prove the following

Lemma 5.2. Suppose $\partial D_B \in \mathscr{C}_*^{1,\alpha}$, $\Delta(K_0) \neq 0$, $K_0 \in Q$, and consider the modified multipoles

$$\psi_m(\theta; K_0) = \cos(m\theta) + K_0 g_m(\theta; K_0), \quad m = 2, 3, \dots, \tag{5.11}$$

where

$$\begin{aligned} g_m(\theta; K_0) = \frac{1}{m} \sum_{l=1}^{\infty} (2-l)c_l \sin(l+m-2)\theta - \frac{1}{m}\mathscr{P}_m^0(K_0)\varphi_0(\theta; K_0) \\ - \frac{1}{m}\mathscr{P}_m^1(K_0)\varphi_1(\theta; K_0). \end{aligned} \tag{5.12}$$

---

*The geometric meaning of this condition is that the projections of the functions $\varphi_0(\theta; K_0)$ and $\varphi_1(\theta; K_0)$ in the subspace L_A^2 form a basis of L_A^2.

Then,

(i)
$$\psi_m(\theta; K_0) \in C([-\pi, 0]) \text{ and}$$
$$|g_m(\theta; K_0)| < D < \infty, \quad m = 2, 3, \ldots, \ \theta \in [-\pi, 0], \ K_0 \in Q, \tag{5.13}$$

(ii)
$$\psi_m(\theta; K_0) \in L_B^2, \ m = 2, 3, \ldots,$$

(iii)
$$\sum_{m=2}^{\infty} \|g_m(\theta; K_0)\|^2 < B < \infty, \tag{5.14}$$

where B is a positive constant independent of K_0.

Proof. The proofs of (i) and (ii) are straightforward, hence will be omitted. To prove (iii) we observe that

$$\|g_m(\theta; K_0)\| \leqslant \frac{1}{m}\left\|\sum_{l=1}^{\infty}(2-l)c_l \sin(l+m-2)\theta\right\| + \frac{1}{m}|\mathscr{P}_m^0(K_0)| \cdot \|\varphi_0(\theta; K_0)\|$$
$$+ \frac{1}{m}|\mathscr{P}_m^4(K_0)| \cdot \|\varphi_1(\theta; K_0)\|.$$

Now using (5.2), (5.10), and Theorem 3.2(iii), we obtain

$$\|g_m(\theta; K_0)\| \leqslant B\frac{1}{m},$$

whereby the desired result (5.14) follows and the proof of the lemma is complete.

Let us recall that the set $Q \subset \mathbb{C}_j$ is a bounded, open, simply connected region containing part of the positive real axis but not containing the origin. Since, however, inequality (5.14) remains valid as $K_0 \to 0$ (see Remark 3.1.2), we can take as Q a region containing a point K_0 such that $|K_0| = B/2$, in which case $B \cdot |K_0| < 1$. Defining $Q \subset \mathbb{C}_j$ in this way and using Lemma 5.2 and Theorem 4.3, we arrive at the following

THEOREM 5.1. Suppose that $\partial D_B \in \mathscr{C}_*^{1,\alpha}$, $\Delta(K_0) \neq 0$, $K_0 \in Q \subset \mathbb{C}_j$. Then, for all $K_0 \in Q \setminus \Sigma$, where Σ is a set of isolated points of Q, the sequence $\{\psi_m(\theta; K_0)\}_2^\infty$ is a (nonorthogonal) Riesz basis of the Hilbert space L_B^2, equivalent to the orthogonal basis $\{\cos(m\theta)\}_2^\infty$.

REMARK 5.1.1. $K_0 \in \Sigma$ if and only if there exists a sequence of scalars $\{\alpha_m\}_2^\infty$ such that

$$0 < \sum_{m=2}^{\infty} |\alpha_m|^2 < \infty \quad \text{and} \quad \sum_{m=2}^{\infty} \alpha_m \psi_m(\theta; K_0) = 0. \tag{5.15}$$

(See Remark 4.3.3.) Clearly (5.15) is a nonuniqueness condition for the radiation problem $\mathscr{P}(K_0)$.

In the remaining part of this paper we shall simplify the notation by neglecting K_0 from the arguments of various functions. Accordingly, we shall write $\varphi_m(\theta)$ instead of $\varphi_m(\theta; K_0)$, φ_{sm} instead of $\varphi_{sm}(K_0)$, and so on.

We can now prove the basis property of the sequence $\{\varphi_m(\theta)\}_0^\infty$ in $L^2(-\pi, 0)$.

THEOREM 5.2. Suppose that $\partial D_B \in \mathscr{C}_*^{1,\alpha}$, $\Delta \neq 0$, and $K_0 \in Q \setminus \Sigma$. Then, the sequence $\{\varphi_m(\theta)\}_0^\infty$ is a (nonorthogonal) Riesz basis of the Hilbert space $L^2(-\pi, 0)$.

Proof. Let $V(\theta)$ be an arbitrary element of $L^2(-\pi, 0)$. In the first place we seek two scalars d_0, d_1 such that

$$U(\theta) \equiv V(\theta) - d_0\varphi_0(\theta) - d_1\varphi_1(\theta) \in L_B^2. \tag{5.16}$$

Condition (5.16) is equivalent to the system

$$\begin{aligned} d_0\varphi_{00} + d_1\varphi_{01} &= V_0, \\ d_0\varphi_{10} + d_1\varphi_{11} &= V_1, \end{aligned} \tag{5.17}$$

$[V_S = (\cos(s\theta), V(\theta))]$, which is always solvable defining uniquely d_0 and d_1, since $\Delta \neq 0$. Since $U(\theta) \in L_B^2$ and the sequence $\{\psi_m(\theta)\}_2^\infty$ is a Riesz basis of L_B^2 (Theorem 5.1), there exists a unique sequence of scalars $\{D_m\}_2^\infty$ such that

$$U(\theta) = \sum_{m=2}^{\infty} D_m\psi_m(\theta) \quad \text{in } L_B^2, \qquad \sum_{m=2}^{\infty} |D_m|^2 < \infty. \tag{5.18}$$

Using (5.16) and (5.18), we obtain the following expansion of $V(\theta)$ in terms of $\varphi_m(\theta)$:

$$V(\theta) = \sum_{m=0}^{\infty} D_m\varphi_m(\theta) \quad \text{in } L^2(-\pi, 0), \qquad \sum_{m=0}^{\infty} |D_m|^2 < \infty, \tag{5.19a}$$

where

$$D_{0,1} = d_{0,1} - K_0 \sum_{m=2}^{\infty} \frac{D_m}{m}\mathscr{P}_m^{0,1}. \tag{5.19b}$$

The series on the right-hand side of (5.19b) is absolutely convergent, since according to (5.10), (5.18), and the Schwarz inequality, we have that ($M > 2$)

$$\sum_{m=2}^{M} \left| \frac{D_m}{m}\mathscr{P}_m^{0,1} \right| \leqslant B^{0,1}\left\{ \sum_{m=2}^{\infty} \frac{1}{m^2} \sum_{m=2}^{\infty} |D_m|^2 \right\}^{1/2} < \infty. \tag{5.20}$$

Thus, the proof of the theorem has been completed.

We shall now state and prove the main result of this work.

THEOREM 5.3 (The weak solvability theorem). Suppose that $\partial D_B \in \mathscr{C}_*^{1,\alpha}$, $\Delta \neq 0$, $u_n(\theta) \in L^2(-\pi, 0)$, and $K_0 \in Q \setminus \Sigma$. [In the physical problem $K_0 \in (0, \infty)$.] Then there exists a unique weak solution of the radiation problem $\mathscr{P}(K_0)$; that is, there exists a unique complex potential $F_\zeta(\zeta)$ such that:

(i) it is continuous throughout $\overline{K} = \{\zeta\colon \mathrm{Im}_i\zeta \leqslant 0, |\zeta| \geqslant 1\}$;

(ii) it satisfies conditions (3.13), (3.14), (3.16), and (3.17); and

(iii) if $F_\zeta^{(1)}(\zeta)$ is its first derivative, the limit

$$\lim_{r \to 1+} F_\zeta^{(1)}(re^{i\theta})$$

exists for almost all $\theta \in [-\pi, 0]$ and defines a function $F_1^{(1)}(\theta) \in L^2(-\pi, 0)$, satisfying the boundary condition (3.15) in the L^2-sense.

REMARK 5.3.1. In fact, the limiting behavior described in (iii) remains valid as ζ approaches $e^{i\theta}$ along any path lying in $\overline{K}$ which is not tangent to the unit circle.

Proof. Under the stated hypotheses the function $V(\theta) = u_n(\theta)|df(e^{i\theta})/d\zeta|$ belongs to $L^2(-\pi, 0)$, and the sequence $\{\varphi_m(\theta)\}_0^\infty$ is a Riesz basis of $L^2(-\pi, 0)$. Therefore, there exists a unique sequence of scalars $\{D_m\}_0^\infty$ such that

$$V(\theta) = \sum_{m=0}^{\infty} D_m\varphi_m(\theta), \qquad \sum_{m=0}^{\infty} |D_m|^2 < \infty. \tag{5.21}$$

It should be noted that the values of the functions $u_n(\theta)$ and $V(\theta)$ as well as the values of the scalars D_m are j-complex numbers. Accordingly,

$$D_m = D'_m + jD''_m; \quad D'_m, D''_m \in \mathbb{R}. \tag{5.22}$$

Now consider the function

$$F_\zeta(\zeta) = D_0G_0(f(\zeta)) + D_1G_1(f(\zeta)) + \sum_{m=2}^{\infty} \frac{D_m}{m} M_m(\zeta). \tag{5.23}$$

(i) $F_\zeta(\zeta)$ is continuous on $\overline{K}$. To establish this fact it suffices to prove the continuity of the function

$$H(\zeta) = \sum_{m=2}^{\infty} \frac{D_m}{m} M_m(\zeta), \quad \zeta \in \overline{K}. \tag{5.24}$$

Since the sequence $M_m(\zeta)$ is uniformly bounded on $\overline{K}$, the series (5.24) is dominated by the series $\sum D_m/m$, which is absolutely convergent [see Eq. (5.20)]. Accordingly, the series (5.24) converges uniformly on $\overline{K}$, defining a continuous function there.

(ii) $F_\zeta(\zeta)$ satisfies conditions (3.13), (3.14), (3.16), and (3.17) because of Theorem 3.1.

(iii) It remains to prove that the function $F_\zeta^{(1)}(\zeta) = dF_\zeta(\zeta)/d\zeta$ converges for almost all $\theta \in [-\pi, 0]$ as $|\zeta| \to 1+$, defining an L^2-function which satisfies the boundary condition (3.15). Differentiating (5.23), we obtain ($|\zeta| > 1$)

$$F_\zeta^{(1)}(\zeta) = D_0\frac{dG_0(f(\zeta))}{d\zeta} + D_1\frac{dG_1(f(\zeta))}{d\zeta} + \frac{dH(\zeta)}{d\zeta}, \tag{5.25}$$

where

$$\frac{dH(\zeta)}{d\zeta} \equiv H^{(1)}(\zeta) = \sum_{m=2}^{\infty} \frac{D_m}{m}\frac{dM_m(\zeta)}{d\zeta}. \tag{5.26}$$

$H^{(1)}(\zeta)$ can be also written in the form

$$H^{(1)}(\zeta) = -\sum_{m=2}^{\infty} D_m\left\{\zeta^{-(m+1)} + \frac{ik_0}{m}\Pi_m(\zeta)\right\}, \tag{5.27}$$

where

$$\Pi_m(\zeta) = \sum_{l=1}^{\infty} (2-l)c_l\zeta^{1-l+m}. \tag{5.28}$$

First, we study the function $H^{(1)}(e^{i\theta})$, $\theta \in [-\pi, \pi]$. According to (5.21) $\sum_{m=2}^{\infty} D_m e^{-i(m+1)\theta} \in L^2(-\pi, \pi)$, while $\sum_{m=2}^{\infty}(D_m/m)\Pi_m(e^{i\theta})$ represents a continuous function, which can be proved in a manner similar to that used to prove the continuity of $H(\zeta)$ in (i). Consequently, $H^{(1)}(e^{i\theta})$ is an L^2-function on the unit circle.

Now we study the limiting behavior of the function $H^{(1)}(\zeta)$ as $|\zeta| \to 1+$. Note that this function is univalent and i-analytic throughout the open domain $\{\zeta\colon |\zeta| > 1\}$. Furthermore, we have

$$|H^{(1)}(\zeta)|^2_{\mathbb{C}_{ij}} \leqslant 2\left|\sum_{m=2}^{\infty} D_m \zeta^{-(m+1)}\right|^2_{\mathbb{C}_{ij}} + 2\left|\sum_{m=2}^{\infty} \frac{D_m}{m}\Pi_m(\zeta)\right|^2_{\mathbb{C}_{ij}}, \tag{5.29}$$

$$\begin{aligned}\left|\sum_{m=2}^{\infty} D_m \zeta^{-(m+1)}\right|^2_{\mathbb{C}_{ij}} &= \left|\sum_{m=2}^{\infty} D'_m \zeta^{-(m+1)}\right|^2_{\mathbb{C}_i} + \left|\sum_{m=2}^{\infty} D''_m \zeta^{-(m+1)}\right|^2_{\mathbb{C}_i} \\ &= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} (D'_m D'_n + D''_m D''_n) r^{-(m+n+2)} e^{i(n-m)\theta},\end{aligned} \tag{5.30}$$

$$\left|\sum_{m=2}^{\infty} \frac{D_m}{m}\Pi_m(\zeta)\right|^2 \leqslant B_1^2\left(\sum_{m=2}^{\infty}\frac{1}{m^2}\right)\left(\sum_{m=2}^{\infty}|D_m|^2\right) = B_2^2 < +\infty, \tag{5.31}$$

where

$$B_1 = \sum_{l=1}^{\infty} |2-l|\cdot|c_l| < +\infty.$$

Using the above relations, we find

$$\int_{-\pi}^{\pi} |H^{(1)}(re^{i\theta})|^2\, d\theta \leqslant 4\pi\left(\sum_{m=2}^{\infty}|D_m|^2 r^{-2(m+1)} + B_2^2\right),$$

whereby, in conjunction with Abel's theorem (see, e.g., Goldberg [19, Sec. 9.6]), we obtain

$$\lim_{v\to 1+}\int_{-\pi}^{\pi} |H^{(1)}(re^{i\theta})|^2\, d\theta \leqslant 4\pi\left(\sum_{m=2}^{\infty}|D_m|^2 + B_2^2\right). \tag{5.32}$$

The last condition characterizes the function $H^{(1)}(\zeta)$ as a Hardy function of class $\mathscr{G}_2$; see Hoffman [20, p. 39] and Walsh [21, Secs. 6.10, 6.11]. Since $H^{(1)}(re^{i\theta})$ is harmonic and satisfies (5.32) and $H^{(1)}(e^{i\theta})$ is an L^2-function of the unit circle, we conclude, applying Fatou's theorem [20, Chapter 3], that the $\lim_{r\to 1+} H^{(1)}(re^{i\theta})$ exists for almost all $\theta \in [-\pi, \pi]$ and the equality

$$\lim_{r\to 1+} H^{(1)}(re^{i\theta}) = H^{(1)}(e^{i\theta}) \tag{5.33}$$

holds almost everywhere in $[-\pi, \pi]$. In fact, (5.33) is valid as the point $\zeta = re^{i\theta}$ approaches $e^{i\theta}$ along any path in the open set $\{\zeta\colon |\zeta| > 1\}$ which is not tangent to the unit circle.

Now using (5.25), (5.26), and (5.33), we see that the $\lim F_\zeta^{(1)}(\zeta)$ exists for almost all $\theta \in [-\pi, 0]$ and

$$\lim F_\zeta^{(1)}(\zeta) = F_\zeta^{(1)}(e^{i\theta}) \in L^2(-\pi, 0)$$

for almost all $\theta \in [-\pi, 0]$ as ζ approaches $e^{i\theta}$ nontangentially.

The function $F_\zeta^{(1)}(e^{i\theta})$ satisfies the boundary condition (3.15) in the L^2-sense since, by construction,

$$\mathrm{Re}_i\left\{e^{i\theta}F_\zeta^{(1)}(e^{i\theta})\right\} = \sum_{m=0}^{\infty} D_m \varphi_m(\theta)$$

and, by the definition of $\{D_m\}_0^\infty$,

$$V(\theta) = \sum_{m=0}^{\infty} D_m \varphi_m(\theta).$$

This completes the proof of the theorem.

6. Regularity of solution. Let us introduce the following geometric assumption:

(A) The body boundary ∂D_B^* is such that the Laurent series of the conformal mapping function $f(\zeta)$ has only a finite number of terms.

In the present section we shall prove, using a method due to Ursell [10], the following regularity result:

THEOREM 6.1. (The regularity theorem). Suppose that ∂D_B^* is a simple, closed curve with continuous tangent, satisfying assumption (A). Suppose also that $u_n(\theta) \in C^2([-\pi, 0])$ and that $\Delta \neq 0$, $K_0 \in Q \setminus \Sigma$. Then, the unique weak solution of the problem $\mathscr{P}(K_0)$ given by expansion (5.23) is actually a classical one; i.e., the velocity field is continuous throughout $D \cup \partial D_F \cup \partial D_B$. The coefficients D_m of expansion (5.23) are of order $O(1/m^2)$.

REMARK 6.1.1. Under the assumptions stated, the curve ∂D_B^* is actually an analytic one.

Proof. According to Theorem 5.2 [see Eq. (5.18)]

$$U(\theta) = \sum_{m=2}^{\infty} D_m \psi_m(\theta) \quad \text{in } L_B^2, \tag{6.1}$$

where $U(\theta) = V(\theta) - d_0\varphi_0(\theta) - d_1\varphi_1(\theta)$, $V(\theta) = u_n(\theta)|df(e^{i\theta})/d\zeta|$, and d_0, d_1 are given by (5.17). Equation (6.1) is equivalent to

$$\frac{2}{\pi} U_s = D_s + \frac{2K_0}{\pi} \sum_{m=2}^{\infty} g_{sm} D_m, \quad s = 2, 3, \ldots, \tag{6.2}$$

where $U_s = (\cos(s\theta), U(\theta))$ and $g_{sm} = (\cos(s\theta),\ g_m(\theta))$. Multiplying (6.2) by s we obtain

$$\frac{2}{\pi} sU_s = sD_s + \frac{2K_0}{\pi} \sum_{m=2}^{\infty} \left(\frac{sg_{sm}}{m}\right)(mD_m), \quad s = 2, 3, \ldots. \tag{6.3}$$

It can be proved, with the aid of assumption (A), that

$$\sum_{m=2}^{\infty} \left|\frac{sg_{sm}}{m}\right|^2 < \frac{C}{s^2} \tag{6.4}$$

(see appendix). Moreover, since ∂D_B^* is analytic and $u_n(\theta) \in C^2([-\pi, 0])$, $U(\theta)$ has a continuous second derivative, from which it follows that $U_s = O(1/s^2)$. Thus $\{sU_s\} \in l^2$.

Now consider (6.3) as a functional equation in l^2. Because of (6.4), the solution of Eq. (6.3) must be in l^2, that is,

$$\sum_{m=2}^{\infty} |mD_m|^2 < B < \infty. \tag{6.5}$$

Using (6.3), (6.4), (6.5), and Schwarz's inequality, we obtain

$$\left|\frac{2}{\pi}sU_s - sD_s\right|^2 \leqslant \frac{4K_0^2}{\pi^2}\sum_{m=2}^{\infty}\left|\frac{sg_{sm}}{m}\right|^2\sum_{m=2}^{\infty}|mD_m|^2 \leqslant \frac{4K_0^2CB}{\pi^2}\frac{1}{s^2},$$

from which it is concluded that $D_s = O(1/s^2)$. The termwise differentiability of the expansion (5.23) and the continuity of $dF_\zeta(\zeta)/d\zeta$ throughout $D \cup \partial D_F \cup \partial D_B$ now follow at once. Thus, the proof of the theorem is complete.

REMARK 6.1.1. Ursell [10] has studied the existence, uniqueness, and regularity of the solution of the radiation (heaving) problem for the semicircular boundary based directly on Eqs. (6.2) and (6.3), i.e., without resorting to the basis properties of the water-wave multipoles, developed in Sec. 5. This more effective approach can be easily extended for body boundaries satisfying assumption (A), and this was essentially done in the present section. However, the author has not succeeded in proving inequality (6.4) when the nonzero conformal mapping coefficients are infinite.

Appendix. Here we shall sketch the proof of inequality (6.4). Since, in the present case,

$$g_m(\theta) = \frac{1}{m}\sum_{l=1}^{N}(2-l)c_l\sin(l+m-2)\theta - \frac{1}{m}\mathscr{P}_m^0\varphi_0(\theta) - \frac{1}{m}\mathscr{P}_m^1\varphi_1(\theta),$$

it follows that

$$\frac{sg_{sm}}{m} = \sum_{\substack{l=1\\ l\neq s-m+2}}^{N}\left\{(-1)^{s+l+m}-1\right\}\frac{(2-l)c_l(l+m-2)s}{m^2(l+m+s-2)(l+m-s-2)} - \mathscr{P}_m^0\frac{s\varphi_{s0}}{m^2} - \mathscr{P}_m^1\frac{s\varphi}{m} \tag{A1}$$

Using (5.10) and the inequality

$$\left|\sum_{n=1}^{N}a_n\right|^2 \leqslant N\sum_{n=1}^{N}|a_n|^2,$$

we obtain

$$\left|\frac{sg_{sm}}{m}\right|^2 \leqslant A(N)\sum_{\substack{l=1\\ l\neq s-m+2}}^{N}\frac{|2-l|^2|c_l|^2(l+m-2)^2s^2}{m^4(l+m+s-2)^2(l+m-s-2)^2} + K\left|\frac{s\varphi_{s0}}{m^2}\right|^2 + \Lambda\left|\frac{s\varphi_{s1}}{m^2}\right|^2, \tag{A2}$$

where $A(N)$, K, and Λ are positive constants. Moreover, $\varphi_0(\theta)$ and $\varphi_1(\theta)$ have continuous second derivatives; hence $\varphi_{s\alpha} = O(1/s^2)$, $\alpha = 0, 1$, and consequently

$$\sum_{m=2}^{\infty}\left|\frac{s\varphi_{s\alpha}}{m^2}\right|^2 \leqslant \frac{A_\alpha}{s^2}\sum_{m=2}^{\infty}\frac{1}{m^4} = \frac{B_\alpha}{s^2}, \quad \alpha = 0, 1. \tag{A3}$$

Each term in the sum appearing on the right-hand side of (A2) is dominated by the quantity

$$B_l\frac{s^2}{m^2(m-s+a_l)^2(m+s+a_l)^2},$$

where $a_l = l - 2$. Following Ursell [10, p. 295] we can easily show that

$$\sum_{\substack{m=2 \\ m \neq s - a_l}}^{\infty} \frac{s^2}{m^2(m - s + a_l)^2(m + s + a_l)^2} \leqslant \frac{B_l'}{s^2}. \tag{A4}$$

Now using (A2), (A3), and (A4), we obtain

$$\sum_{m=2}^{\infty} \left| \frac{s g_{sm}}{m} \right|^2 \leqslant \frac{A_1(N)}{s^2},$$

which is the required inequality. Since $A_1(N) \to \infty$ as $N \to \infty$, this proof breaks down if the number of nonzero coefficients c_l is infinite.

Note added in proof. A fairly general uniqueness theorem for the two-dimensional radiation problem has been recently established by Simon and Ursell [28].

Acknowledgment. The author is grateful to Professor F. Ursell and to the reviewer for their suggestions concerning the possibility of obtaining regularity results by applying Ursell's method. The author would also like to express his deepest gratitute to his friend and colleague P. D. Kaklis, who gave much of his time to reading a preliminary draft of this paper and pointing out errors and deficiencies and suggesting improvements.

References

[1] G. A. Athanassoulis, *An expansion theorem for water-wave potentials*, Journal of Engineering Mathematics **18**, 181–194 (1984)

[2] F. John, *On the motion of floating bodies*, II, Comm. Pure Appl. Math. **3**, 45–101 (1950)

[3] J. T. Beale, *Eigenfunction expansions for objects floating in an open sea*, Comm. Pure Appl. Math. **30**, 283–313 (1977)

[4] M. Lenoir and D. Martin, *An application of the principle of limiting absorption to the motions of floating bodies*, J. Math. Anal. Appl. **79**, 370–383 (1981)

[5] M. Lenoir and D. Martin, *Étude théorique et numérique du problème linéarisé du mouvement sur la houle tridimensionnelle*, École National Supérieure de Techniques Avancées, Paris, Report No. 124 (1980)

[6] M. Lenoir, *Méthod de couplage en hydrodynamique naval et application à la resistance de vagues bidimensionnelles*, École National Supérieure de Techniques Avancées, Paris, Report No. 164, chapter III (1982)

[7] F. Ursell, *Surface waves on deep water in the presence of a submerged circular cylinder*, II, Proc. Camb. Philos. Soc. **46**, 153–158 (1950)

[8] F. Ursell, *The expansion of water-wave potentials at great distances*, Proc. Camb. Philos. Soc. **64**, 811–826 (1968)

[9] F. Ursell, *Short surface waves due to an oscillating immersed body*, Proc. Roy. Soc. London Ser. A **220**, 90–103 (1953)

[10] F. Ursell, *A problem in the theory of water waves*, in *Numerical solution of integral equations*, Clarendon Press, Oxford (1974)

[11] Y. S. Yu and F. Ursell, *Surface waves generated by an oscillating circular cylinder on water of finite depth: Theory and experiment*, J. Fluid Mech. **11**(4), 529–551 (1961)

[12] J. J. Stoker, *Water waves. The mathematical theory with applications*, Interscience, New York (1957)

[13] A. Zygmund, *Trigonometric series*, Vol. I, Cambridge Univ. Press, London (1959)

[14] I. Singer, *Bases in Banach spaces*, Springer-Verlag, Berlin (1970)

[15] J. R. Higgins, *Completeness and basis properties of sets of special functions*, Cambridge Univ. Press, London (1977)

[16] N. Bary, *Sur les systèmes complets des fonctions orthogonales*, Mat. Sb. **14** (56), 51–108 (1944)

[17] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin (1976)

[18] V. I. Smirnov, *A course of higher mathematics*, vol. V, *Integration and functional analysis*, Pergamon Press, Oxford (1964)

[19] R. R. Goldberg, *Methods of real analysis*, Blaisdell, New York (1964)

[20] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J. (1962)

[21] J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, 5th ed., Colloq. Publ. vol. XX, Amer. Math. Soc., Providence, R. I. (1969)

[22] D. F. Harazov, *On the spectrum of completely continuous operators depending analytically on a parameter, in topological linear spaces* (in Russian), Acta Sci. Math. (Szeged) **23**, 38–45 (1962)

[23] I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Transl. Math. Mono. **18**, Amer. Math. Soc., Providence, R. I. (1969)

[24] A. E. Taylor, *Introduction to functional analysis*, Wiley, New York (1957)

[25] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, rev. ed., Colloq. Publ. vol. XXXI, Amer. Math. Soc., Providence, R. I. (1957)

[26] F. Ursell, *On the rolling motion of cylinders in the surface of a fluid*, Quart. J. Mech. Appl. Math. **2**, 335–353 (1949)

[27] D. Euvrard, A. Jami, M. Lenoir, and D. Martin, *Recent progress towards an optimal coupling between finite elements and singularity distribution procedures*, Proc. 3rd Internat. Conf. Numerical Ship Hydrodynamics, Paris (1981)

[28] M. J. Simon and F. Ursell, *Uniqueness in linearized two-dimensional water-wave problems*, J. Fluid Mech. **148**, 137–154 (1984)