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FINITE ELEMENT APPROXIMATION OF A REACTION-DIFFUSION EQUATION
PART I: APPLICATION OF TOPOLOGICAL TECHNIQUES TO THE ANALYSIS OF ASYMPTOTIC BEHAVIOR OF THE SEMIDISCRETE APPROXIMATIONS*

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Abstract. The initial-boundary value problem for a reaction-diffusion equation

$$u_t = u_{xx} + f(u), \qquad f(u) = -u(u-b)(u-1), \quad 0 < b < 1/2, \qquad (*)$$

was analyzed in [4, 17] by using the Conley index. In this paper we study the asymptotic behavior of the semidiscrete finite element approximations, with interpolation of the coefficients in the nonlinear terms. We show that for small h the spectrum of the linearized discrete steady-state problem is a "good" approximation for the spectrum of the linearized continuous steady-state problem. Using the interpretation of the Conley index as the dimension of an unstable manifold of a steady-state solution, we establish that the properties of the semidiscrete approximations are completely analogous to those of the solutions of $(*)$. The asymptotic, as $t \to \infty$, optimal order convergence of the approximate solutions is proved.

1. Introduction. Many phenomena in biology and physiology can be modelled by certain nonlinear reaction-diffusion equations which have several steady-state solutions. The asymptotic state of a solution specifies its ultimate behavior while ignoring transient effects. The stable asymptotic states may be presented by solutions which can be

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perturbed by a uniformly small function without destroying their long-time behavior. These are the solutions generally seen in applied contexts. It is therefore crucial to know that the asymptotic behavior of the approximate solutions mimic those of the exact solutions.

The asymptotic behavior of solutions for the problem

$$u_t = u_{xx} + f(u), \quad |x| < L,\ t > 0; \tag{1.1.a}$$

$$u(x,0) = u^0(x), \quad |x| < L; \tag{1.1.b}$$

$$u(\pm L, t) = 0, \quad t > 0; \tag{1.1.c}$$

where

$$f(u) = -u(u-b)(u-1), \quad 0 < b < 1/2, \tag{1.2}$$

was analyzed in [4] and [17] (see Theorem 2.1 below). In particular, it was shown there that for $L > L_0$ the steady-state problem

$$u'' + f(u) = 0, \quad |x| < L; \tag{1.3.a}$$

$$u(\pm L) = 0; \tag{1.3.b}$$

has exactly three solutions, u_0, u_1, and u_2 with $u_0 \equiv 0 < u_1(x) < u_2(x)$, which are nondegenerate (i.e., zero is not in the spectrum of the linearized problem). u_0 and u_2 are attractors for the associated parabolic problem (1.1.a)–(1.1.c). By this we mean that if the initial data u^0 is sufficiently close (in C) to either u_0 or u_2, then the corresponding solution of (1.1.a)–(1.1.c) converges (in C) to the corresponding solution of (1.3.a)–(1.3.b). Similarly, u_1 is unstable and has a one-dimensional unstable manifold. These results are based on computation of the Conley index (the Morse index of an isolated invariant set) for each of u_0, u_1, and u_2. The Conley index generalizes the classical Morse index of a nondegenerate critical point of a vector field in that the classical index is a nonnegative integer n, where n is the dimension of the unstable manifold to the critical point, and, considered as an isolated invariant set, the homotopy index of the critical point is the homotopy type of a pointed n-sphere.

In this paper we analyze the asymptotic behavior of semidiscrete finite element approximations with the interpolation of coefficients for the evaluation of nonlinear terms, the so-called product approximation. We extend the results in [13], where a simple finite difference scheme has been considered.

We first study the approximate steady-state problem. Convergence of the product approximations for semilinear elliptic problems was recently analyzed in [16] using a continuity argument for the case $f_u \geqslant m > -\pi^2$. However, in our case the nonlinearity f does not satisfy the above inequality. We therefore employ a general method of approximation of nonlinear equations developed in [2]. The results in [2] were extended to cover the case of finite element methods with numerical integration [15]. We follow the approach in [15] with numerical integration replaced by the product approximation. In Theorem 4.2 we prove that for small enough h the discrete stationary problem has, as the continuous one, exactly three solutions $0 \equiv u_0^h < u_1^h < u_2^h$ and establish quasi-optimal convergence of the finite element solutions in H^1 norm.

In Theorem 4.4 we show that the properties of the semidiscrete approximations are completely analogous to those of the solutions of the continuous parabolic problem and establish the asymptotic, as $t \to \infty$, optimal order convergence of the semidiscrete approximations.

Our approach is based on the fact that for small h the spectrum of the linearized discrete steady-state problem is a "good" approximation of the spectrum of the linearized continuous steady-state problem (Theorem 4.3). This gives existence and convergence of the approximate steady-state solutions, and also implies that the Conley index of a rest point u_i^h of the approximate problem is the same as the one of the corresponding rest point $u_i(x)$, $i = 0, 1, 2$, of the exact problem. The latter implies the existence of orbits connecting the rest points of the approximate problem.

In section 2 we formulate the continuous problem in H_0^1.

In section 3 we formulate the approximate problem in a finite element subspace $S_0^h \subset H_0^1$.

In section 4 we analyze the convergence of the approximate solutions and prove our principle result, Theorem 4.4.

Recently, asymptotic convergence of numerical solutions of systems of reaction-diffusion equations to constant and zero rest points was analyzed by several authors. In [10, 11] finite difference approximations were shown to converge with a time-independent error bound by imposing a monotonicity condition on the reaction term or under the conditions that the reaction term is "slowly varying". The conditions imposed guarantee the exponential decay of the exact solution together with its derivatives. The results of [10, 11] were obtained for both the Dirichlet and Neumann problems. For the Neumann problem, similar results were obtained in [7] and in the linear case, using finite-elements, in [12]. In [8] the asymptotic convergence of finite difference approximations to the stable nonconstant steady-state solution of the one-dimensional logistic model (quadratic nonlinearity) was established using the direct analysis of the finite difference operator.

In [14] numerical studies of the problem (1.1.a)–(1.1.c) were carried out, using the finite element discretization with the product approximation, and estimates for the critical length L_0 were obtained. As far as we are aware, the asymptotic convergence of numerical solutions of (1.1.a)–(1.1.c) (in the case that u_{xx} is approximated by second-order finite differences) has first been analyzed by the author [13].

In Part II we intend to establish the error estimates uniform in time for the problem (1.1.a)–(1.1.c).

Throughout this paper, C (possibly, with subscripts) will denote a generic constant, not necessarily the same at different occurrences.

2. Continuous problem. A solution $u_*(x)$ of the steady-state problem (1.3.a)–(1.3.b) is called an attractor for the associated parabolic problem (1.1.a), (1.1.c) if, for the initial data $u(x, 0)$ sufficiently close (in C) to u_*, the corresponding solution of (1.1.a)–(1.1.c) converges (in C) to u_*. We shall use the notations $h(I)$ for the Conley index of an isolated invariant set I, and Σ^k for the pointed k-sphere.

THEOREM 2.1 [17, Theorem 24.13]. Let f be defined by (1.2), and let $L > L_0$. Then there are exactly three steady-state solutions $u_i \in C^\infty$, $i = 0, 1, 2$, of (1.1.a), (1.1.c): $0 \equiv u_0(x) < u_1(x) < u_2(x) \leqslant 1$, $|x| < L$. They are isolated invariant sets, $h(u_0) = h(u_2) = \Sigma^0$, in particular, u_0 and u_2 are attractors for (1.1.a), (1.1.c); and the linearized operators Q_0 and Q_2, where

$$Q_k = d^2/dx^2 + g_k, \qquad g_k(x) = f'(u_k(x)), \quad k = 0, 1, 2,$$

together with the boundary conditions (1.3.b), have only negative eigenvalues. $h(u_1) = \Sigma^1$, in particular, Q_1 has precisely one positive eigenvalue, and u_1 has a 1-dimensional unstable manifold which consists of orbits connecting u_1 to each of the other rest points. Initial data $u(x,0)$ which satisfies $u_1(x) < u(x,0) < u_2(x)$ (resp. $0 < u(x,0) < u_1(x)$) on $|x| < L$ is in the stable manifold of u_2 (resp. 0).

It is convenient to make the change of variables $y = (x/L + 1)/2$. Then (1.1) and (1.3) become

$$u_t = u_{xx} - G(L,u), \quad 0 < x < 1,\ t > 0, \tag{2.1.a}$$

$$u(x,0) = u^0(x), \quad 0 < x < 1, \tag{2.1.b}$$

$$u(0,t) = u(1,t) = 0, \quad t > 0; \tag{2.1.c}$$

$$u_{xx} - G(L,u) = 0, \quad 0 < x < 1, \tag{2.2.a}$$

$$u(0) = u(1) = 0, \tag{2.2.b}$$

where for $f(x)$ defined by (1.2),

$$G(L,u) = -4L^2 f(u) = 4L^2 u(u-b)(u-1), \quad 0 < b < 1/2. \tag{2.3}$$

For $I = [0,1]$, let $H^s = H^s(I)$, $H_0^s = H_0^1 \cap H^s$, for s real, and $L^\infty = L^\infty(I)$ be the usual Sobolev spaces with the norms $\|\cdot\|_s$ and $\|\cdot\|_{0,\infty}$, respectively; if $s = 0$, we write $\|\cdot\|_0 = \|\cdot\|$. The solutions $u(x,t)$ and $u(x)$ of (2.1) and (2.2), respectively, satisfy the weak formulations

$$(u_t, v) = -(u_x, v_x) - (G(L,u), v), \quad t > 0,\ v \in H_0^1, \tag{2.4}$$

$$(u_x, v_x) + (G(L,u), v) = 0, \quad v \in H_0^1. \tag{2.5}$$

Let T: $L^2 \to H_0^1$ be the linear operator defined by setting $u = Tg$ to be a unique solution of

$$(u_x, v_x) = (g, v), \quad g \in L^2,\ v \in H_0^1. \tag{2.6}$$

Then T: $H^1 \to H_0^1$ is compact. Let $\Lambda \subset (L_0, \infty)$ be a compact interval. Clearly, the mapping G: $\Lambda \times H_0^1 \to H^1$ is C^∞. We set

$$F(L,u) = u + TG(L,u). \tag{2.7}$$

Now the weak formulation (2.5) is equivalent to the problem of finding a pair $(L,u) \in \Lambda \times H_0^1$ that is a solution of the equation

$$F(L,u) = 0. \tag{2.8}$$

Note that by the Implicit Function Theorem, $L \to u(L)$ is a C^∞ function from Λ into H_0^1.

3. Approximate problem. Let $\Delta = \{0 = \bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_{N+1} = 1\}$ be a partition of $I = [0,1]$. Set $I_j = [\bar{x}_{j-1}, \bar{x}_j]$, $h_j = \bar{x}_j - \bar{x}_{j-1}$, $j = 1, \ldots, N+1$ and $h = \max_{1 \leqslant j \leqslant N+1} h_j$. It is assumed that as the meshes vary they are quasi-uniform, i.e., $\max_{1 \leqslant j \leqslant N+1} h h_j^{-1} \leqslant \sigma$, for some $\sigma > 1$. Define the finite element space S_0^h by

$$S_0^h = M_0^k(\Delta) \equiv \left\{ v \in C^0(I) \colon v(0) = v(1) = 0,\ v \in P_k(I_j),\ j = 1, \ldots, N+1 \right\},$$

where for any interval $E \subset I$, $P_k(E)$ denotes the space of polynomials of degree $\leqslant k$ restricted to E. Then the continuous time Galerkin approximation $\tilde{u}^h(x,t)$ to the solution of (2.4) is defined as a differentiable map $\tilde{u}^h\colon (0, \infty) \to S_0^h$ such that

$$(\tilde{u}_t^h, \chi) = -(\tilde{u}_x^h, \chi_x) - (G(L, \tilde{u}^h), \chi), \quad t > 0,\ \chi \in S_0^h,\ \tilde{u}^h(\cdot, 0) \text{ given in } S_0^h. \tag{3.1}$$

The Galerkin approximation $\tilde{u}^h \in S_0^h$ to the solution of the stationary problem (2.5) is defined by

$$(\tilde{u}_x^h, \chi_x) + (G(L, \tilde{u}^h), \chi) = 0, \quad \chi \in S_0^h. \tag{3.2}$$

In practice it is necessary to evaluate the integrals arising in (3.1) and (3.2) numerically. Every interval I_j, $j = 1, \ldots, N+1$, is divided into k subintervals $[\bar{x}_{j-1+(i-1)/k}, \bar{x}_{j-1+i/k}]$, $i = 1, \ldots, k$, where the nodes within each I_j are chosen to be the Gauss–Lobatto points. We relabel $\bar{x}_{j+i/k}$ as x_{kj+i}, $j = 0, 1, \ldots, N$, and set $M = (N+1)k - 1$.

Let $Q_h\colon g \in H_0^1 \to Q_h g \in S_0^h$ be the usual interpolation operator defined by

$$(Q_h g)(x_i) = g(x_i), \quad i = 0, \ldots, M+1.$$

We replace the problems (3.1) and (3.2) by the problems

$$(u_t^h, \chi) = -(u_x^h, \chi_x) - (Q_h G(L, u^h), \chi), \quad t > 0,\ \chi \in S_0^h,\ u^h(\cdot, 0) \text{ given in } S_0^h; \tag{3.3}$$

$$(u_x^h, \chi_x) + (Q_h G(L, u^h), \chi) = 0, \quad \chi \in S_0^h, \tag{3.4}$$

respectively. If $\psi_i(x)$, $i = 1, \ldots, M$, is the usual interpolatory basis for S_0^h, then to compute the solutions of (3.3) and (3.4) one has to solve the problems

$$\sum_{i=1}^{M} \dot{u}_i(\psi_i, \psi_j) = -\sum_{i=1}^{M} u_i(\psi_i', \psi_j') = -\sum_{i=1}^{M} G(L, u_i)(\psi_i, \psi_j), \quad j = 1, \ldots, M, \qquad u_i(0), \quad i = 1, \ldots, M \text{ given}; \tag{3.5}$$

$$\sum_{i=1}^{M} u_i(\psi_i', \psi_j') + \sum_{i=1}^{M} G(L, u_i)(\psi_i, \psi_j) = 0, \quad j = 1, \ldots, M. \tag{3.6}$$

By introducing M-dimensional vectors $\mathbf{u} = (u_1, \ldots, u_M)^T$, $\mathbf{G}(\mathbf{u}) = (G(L, u_1), \ldots, G(L, u_M))^T$ and the $M \times M$ matrices S and A with entries (ψ_i', ψ_j') and (ψ_i, ψ_j), the systems (3.5) and (3.6) become

$$A\dot{\mathbf{u}} = -S\mathbf{u} - A\mathbf{G}(\mathbf{u}), \quad \mathbf{u}(0) \text{ given}; \tag{3.7}$$

$$S\mathbf{u} + A\mathbf{G}(\mathbf{u}) = \mathbf{0}, \tag{3.8}$$

respectively.

We next define the linear operator T_h: $g \in H_0^1 \to u^h = T_h g \in S_0^h$ by setting u^h to be a unique solution of

$$\left(u_x^h, \chi_x\right) = \left(Q_h g, \chi\right), \quad g \in H_0^1, \ \chi \in S_0^h. \tag{3.9}$$

Then an equivalent form of the discrete stationary problem (3.4) consists in finding a solution $(L, u^h) \in \Lambda \times S_0^h$ of

$$F_h(L, u^h) \equiv u^h + T_h G(L, u^h) = 0. \tag{3.10}$$

LEMMA 3.1. Let T and T_h be defined by (2.6) and (3.9), respectively. Then we have

$$\|(T - T_h g)\|_1 \leqslant Ch^r \|g\|_r, \quad r = 1, \ldots, k. \tag{3.11}$$

Proof. The error estimate for the finite element method with numerical integration [3, Theorem 4.1.1] is

$$\|Tg - T_h g\|_1 \equiv \|u - u^h\|_1 \leqslant Ch^r \|u\|_{r+1} + \sup_{\chi \in S_0^h} \frac{|(g - Q_h g, \chi)|}{\|\chi\|_1}, \quad r = 0, 1, \ldots, k. \tag{3.12}$$

Now the elliptic regularity implies $\|u\|_{r+1} \leqslant C\|g\|_r$. Together with the interpolation error estimate

$$\|g - Q_h g\|_s \leqslant Ch^{r-s+1} \|g\|_{r+1}, \quad s = 0, 1, \ r = 0, 1, \ldots, k, \ g \in H_0^{k+1}, \tag{3.13}$$

with $s = 0$, this gives (3.11).

4. Convergence.

THEOREM 4.1. Let $\{(L, u(L));\ L \in \Lambda\}$ be a branch provided by Theorem 2.1 of nonsingular solutions of (2.8), i.e.,

$$F(L, u(L)) = 0; \tag{4.1}$$

$$D_u F(L, u(L)) \text{ is an isomorphism of } H_0^1. \tag{4.2}$$

Let also T, T_h, and F_h be defined by (2.6), (3.9), and (3.10), respectively. Then (i) there exists $\varepsilon > 0$ and for $h \leqslant h_0$ small enough, a unique C^∞ function $L \in \Lambda \to u^h(L) \in S_0^h$ such that for all $L \in \Lambda$, $u^h(L)$ is a solution of (3.10) (and (3.4)) and $\|u^h(L) - u(L)\|_1 \leqslant \varepsilon$. (ii) Furthermore, for any integer $m \geqslant 0$ we have the error estimate

$$\left\|\left(u^{(m)}(L)\right) - \left(u^h(L)\right)^{(m)}\right\|_1 \leqslant C_m h^r \|u\|_{r+2}, \quad r = 1, \ldots, k. \tag{4.3}$$

In (4.3), $(u^h)^{(m)}$ and $u^{(m)}$ stand for mth derivatives with respect to L, and C_m does not depend on L.

Proof. From (3.11) with $r = 1$ we get

$$\|T - T_h\|_1 \leqslant Ch, \tag{4.5}$$

and from (3.13) with $s = 1$, $r = 0$ we get that Q_h is bounded in H_0^1: $\|Q_h\|_1 \leqslant C$. Now from [2, Theorem 6] we obtain (i) and the error estimate

$$\left\|(u^h(L))^{(m)} - u^{(m)}(L)\right\|_1 \leqslant C_m \sum_{i=0}^{m} \left\{ \left\|u^{(i)}(L) - Q_h u^{(i)}(L)\right\|_1 + \left\|(T - T_h)\frac{d^i}{dL^i} G(L, u(L))\right\|_1 \right\}. \tag{4.6}$$

Now recall that $L \to u(L)$ is a C^m function from Λ into H_0^{k+1}. Then (4.3) follows from (3.13) with $s = 1$ and (3.11).

In order to shorten the notation, we consider in the sequel the case $m = 0$, and suppress the dependence of u and u^h on L.

Lemma 4.1. Under the assumptions of Theorem 4.1, for any $\delta > 0$ there exists h_0 such that for $h \leqslant h_0$ any solution u^h of (3.10) (and (3.4)) satisfies

$$\|u^* - u^h\|_1 \leqslant \delta, \tag{4.7a}$$

$$\|u^* - u^h\|_{0,\infty} \leqslant C\delta, \tag{4.7b}$$

for some solution $u^*(x)$ of (2.8).

Proof. Let U be the set of all solutions of (2.8), and U^δ be the union of open balls in H_0^1 of radius $< \delta$ about each point $u \in U$. It is clear that the closed set $H_0^1 \setminus U^\delta$ contains no solutions of (2.8), and therefore

$$\alpha_\varepsilon = \inf_{u \in H_0^1 \setminus U^\delta} \|u - TG(L, u)\|_1 > 0.$$

If $\chi \in S_0^h \cap (H_0^1 \setminus U^\varepsilon)$ then, using (4.5),

$$\|\chi - T_h G(L, \chi)\|_1 \geqslant \|\chi - TG(L, \chi)\|_1 - \|(T - T_h)G(L, \chi)\|_1 \geqslant \alpha_\varepsilon - Ch > 0,$$

if h is sufficiently small (say, $h \leqslant h_0$). Therefore $u^h - T^h G(L, u^h) = 0$ implies that for $h \leqslant h_0$ we have $u^h \notin H_0^1 \setminus U^\delta$, i.e., $u^h \in U^\delta$, and thus we have (4.7a) for some $u^* \in U$.

Since for any $u \in H_0^1$ we have $\|u\|_{0,\infty} \leqslant C\|u\|_1$, we arrive at (4.7b).

Theorem 4.2. Let $u_i(x)$, $i = 0, 1, 2$, be the solutions of (2.8) provided by Theorem 2.1. There exists $h_0 > 0$ such that for $h \leqslant h_0$ the approximate stationary problem (3.10) (and (3.4)) has exactly three solutions u_i^h, $i = 0, 1, 2$, satisfying

$$\|u_i - u_i^h\|_1 \leqslant Ch^r \|u_i\|_{r+2}, \quad i = 1, 2;\ r = 1, \dots, k, \tag{4.8}$$

and

$$0 \equiv u_0^h < u_1^h < u_2^h. \tag{4.9}$$

Proof. The existence of u_i^h, $i = 0, 1, 2$, satisfying (4.8) follows from Theorem 4.1 with $u = u_i$ and $u^h = u_i^h$. By Lemma 4.1 for $h \leqslant h_0$, u_i^h, $i = 0, 1, 2$, are the only solutions of (3.10). By Theorem 2.1, $0 \equiv u_0 < u_1 < u_2$ are isolated solutions of (2.8). Together with (4.7b) this gives (4.9) for h_0 sufficiently small.

We next want to show that for $i = 0, 1, 2$, the eigenvalues of the linearized discrete problems

$$-B_h^i(v^h, \chi) \equiv (v_x^h, \chi_x) + (a_h^i v^h, \chi) = \lambda(h)(v^h, \chi),$$
$$a_h^i(x) = Q_h G_u(L, u_i^h(x)), \quad v^h, \chi \in S_0^h, \tag{4.10}$$

are "good" approximations to the corresponding eigenvalues of the linearized continuous problems

$$-B^i(v, w) \equiv (v_x, w_x) + (a^i v, w) = \lambda(v, w), \qquad a^i(x) = G_u(L, u_i(x)), \quad v, w \in H_0^1. \tag{4.11}$$

To shorten the notation we shall sometimes suppress the superscript i in B_h^i, a_h^i, B^i, and a^i. Note that in (4.10) we used the fact that

$$((Q_h G(L, u))_u v^h, \chi) = (Q_h G_u(L, u) v^h, \chi), \quad u \in H_0^1,\ v^h, \chi \in S_0^h.$$

We also assume that $a(x)$ and $a_h(x) > 0$ (we can always achieve this by shifting λ and λ_h in (4.10) and (4.11)).

Let K: $L^2 \to H_0^1$ and K_h: $L^2 \to S_0^h$ be the linear operators defined, respectively, by

$$-B(Kf, w) = (f, w), \quad f \in L^2,\ w \in H_0^1, \tag{4.12}$$

$$-B_h(K_h f, \chi) = (f, \chi), \quad f \in L^2,\ \chi \in S_0^h. \tag{4.13}$$

LEMMA 4.2. Let $f \in H^{r-1}$ and $\phi \in H^s$ with $0 \leqslant s \leqslant k - 1$, $r = 1, \ldots, k$. Then

$$\|(K - K_h) f\|_1 \leqslant C h^r \|f\|_{r-1}, \tag{4.14}$$

$$|(K - K_h) f, \phi| \leqslant C h^r \|f\|_{r-1} \|\phi\|_s. \tag{4.15}$$

Proof. Let $v = Kf$ and $v^h = K_h f$ be the solutions of (4.12) and (4.13), respectively. Then by [3, Theorem 4.1.1] we have

$$\|v - v^h\|_1 \leqslant C\|v - v_h\|_1 + \sup_{\chi \in S_0^h} \frac{|((a - a_h) v_h, \chi)|}{\|\chi\|_1}, \tag{4.16}$$

where $v_h \in S_0^h$ is chosen to satisfy

$$\|v - v_h\| + h\|v - v_h\|_1 \leqslant C h^t \|v\|_t, \quad 1 \leqslant t \leqslant k + 1. \tag{4.17}$$

We first estimate $\|a - a_h\|_{0,\infty}$. From (4.10) and (4.11) we have

$$\begin{aligned} \|a - a_h\|_{0,\infty} &= \|G_u(L, u) - Q_h G_u(L, u^h)\|_{0,\infty} \\ &\leqslant \|(I - Q_h) G_u(L, u)\|_{0,\infty} + \|Q_h(G_u(L, u) - G_u(L, u^h))\|_{0,\infty} \\ &\equiv I_1 + I_2. \end{aligned} \tag{4.19}$$

$$I_1 \leqslant c_1 h^{k+1}, \ c_1 = \|d^{k+1} G_u(L, u(\cdot)/dx^{k+1})\|_{0,\infty}.$$

Since from (4.3) and (4.7b)

$$\|u - u^h\|_{0,\infty} \leqslant C h^r \|u\|_{r+2}, \quad r = 1, \ldots, k, \tag{4.20}$$

then

$$I_2 \leqslant \|Q_h\|_{0,\infty} \|G_u(L,u) - G_u(L,u^h)\|_{0,\infty}$$
$$\leqslant \|Q_h\|_{0,\infty} \|G_{uu}\|_{0,\infty} \|u - u^h\|_{0,\infty} \leqslant c_2 h^k.$$

Substituting the estimates for I_1 and I_2 into (4.19) we arrive at

$$\|a - a_h\|_{0,\infty} \leqslant Ch^k, \tag{4.21}$$

and therefore, using also (4.17) with $t = 1$, we have

$$|((a - a_h)v_h, \chi)| \leqslant \|a - a_h\|_{0,\infty} \|v_h\| \|\chi\| \leqslant Ch^k \|v\|_1 \|\chi\|. \tag{4.22}$$

Combining (4.17) and (4.22), the right-hand side of (4.16) is estimated by

$$\|v - v^h\|_1 \leqslant C_1 h^r \|v\|_{r+1} + C_2 h^k \|v\|_1, \quad r = 0, \ldots, k. \tag{4.23}$$

Using the regularity estimate for K,

$$\|Kf\|_{s+2} \leqslant C_s \|f\|_s, \quad f \in H^s, \tag{4.24}$$

we finally arrive at (4.14).

We next obtain (4.15) by the duality argument. Let $\phi \in H^s$ with $0 \leqslant s \leqslant k - 1$. Then by (4.12) and (4.13)

$$((K - K_h)f, \phi) = B((K - K_h)f, K\phi)$$
$$= B((K - K_h)f, K\phi - \chi) + ((a - a_h)K_h f, \chi)$$

for any $\chi \in S_0^h$. Thus, using (4.14) and (4.21),

$$|((K - K_h)f, \phi)| \leqslant Ch^r \|f\|_{r-1} \|K\phi - \chi\|_1 + Ch^k \|K_h f\| \|\chi\|.$$

Note that by (4.23) $\|K_h f\| \leqslant \|Kf\|_1$. Together with (4.17) and (4.24) this implies

$$|((K - K_h)f, \phi)| \leqslant C_1 h^{r+s+1} \|f\|_{r-1} \|\phi\|_s + C_2 h^k \|f\| \|\phi\|_s,$$

and (4.15) follows.

Remark. In a future publication we shall derive an optimal order L_∞ estimate for $u - u^h$. Then h^r in (4.15) can be replaced by h^{r+1}.

Let us rewrite (3.7) in an equivalent form

$$\dot{\mathbf{u}} = -A^{-1}S\mathbf{u} - \mathbf{G}(\mathbf{u}). \tag{4.25}$$

Define

$$F(u) = \int_0^u G(L,t)\,dt, \tag{4.26}$$

$$\Phi(\mathbf{u}) = \frac{1}{2}\langle A^{-1}S\mathbf{u}, \mathbf{u}\rangle + \sum_{i=1}^{M} F(u_i), \tag{4.27}$$

where $\mathbf{u} = (u_1, \ldots, u_m)^T \in \mathbf{R}^M$ and $\langle \cdot , \cdot \rangle$ is the scalar product in $\mathbf{R}^M$. Then we have

LEMMA 4.3. The system (4.25) is a gradient one with respect to the function Φ, i.e.,

$$\dot{\mathbf{u}} = -\nabla\Phi. \tag{4.28}$$

THEOREM 4.3. Suppose the assumptions of Theorem 4.2 hold. Then for $h > 0$ sufficiently small we have:

(i) for $i = 0, 2$ the linearized discrete problem (4.10) has only negative eigenvalues; for $i = 1$ (4.10) has precisely one positive eigenvalue, and the rest negative, and u_1^h has a 1-dimensional unstable manifold.

(ii) Moreover, let λ be an eigenvalue of algebraic multiplicity m satisfying (4.11). Then the eigenvalues $\lambda_1(h), \ldots, \lambda_m(h)$ of (4.10) which converge to λ as $h \to 0$ satisfy the estimate

$$\left| \lambda - \left(\frac{1}{m} \sum_{j=1}^{M} 1/\lambda_j(h) \right)^{-1} \right| \leqslant Ch^k. \tag{4.29}$$

(iii) u_i^h, $i = 0, 1, 2$, are isolated invariant sets of the problem (3.3), and $h(u_0^h) = h(u_2^h) = \Sigma^0$, $h(u_1^h) = \Sigma^1$.

Proof. (4.15) with $r = 1$ and $s = 0$ yields $\|(K - K_h)f, \phi)| \leqslant Ch\|f\|\,\|\phi\|$ and hence

$$\|K - K_h\| \leqslant Ch. \tag{4.30}$$

Note that λ is an eigenvalue satisfying (4.11) if and only if $\mu = -1/\lambda$ is an eigenvalue of K, and $\lambda(h)$ is an eigenvalue of K_h if and only if $\mu(h) = -1/\lambda(h)$ is an eigenvalue of K_h. Now let $\mu_1, \mu_2, \ldots$ be the nonzero eigenvalues of K ordered by decreasing magnitude taking account of algebraic multiplicities. Then it is well known (cf. [1]) that for each $h > 0$ there is an ordering of the eigenvalues of K_h, $\mu_1(h)$, $\mu_2(h), \ldots,$ such that, for each integer j, $\lim_{h \to 0} \mu_j(h) = \mu_j$. Therefore (i) follows from Theorem 2.1. Equation (4.30) implies that $\lim_{h \to 0} \|T - T_h\| = 0$, and hence [1, Theorem 2] applies. Taking into account (4.15) and repeating the argument in [1, pp. 539–540], this yields (ii).

By (i), u_i^h, $i = 0, 1, 2$, are nondegenerate rest points of (3.3). Since from Lemma 4.3 the system (4.25) is a gradient one, then by [17, pp. 151–152], the Morse index of u_i^h, which is the number of positive eigenvalues of (4.10), is defined and is equal to the Conley index $h(u_i^h)$. Thus by (i), $h(u_0^h) = H(u_2^h) = \Sigma^0$ and $h(u_1^h) = \Sigma^1$. Alternatively, the latter result follows from [17, §4, pp. 503–504]. By [17, p. 468 and Theorem 23.32], u_i^h, $i = 0, 1, 2$, are isolated invariant sets for (3.3).

LEMMA 4.4. For $h > 0$ sufficiently small, the rectangle

$$R = \bigcap_{i=1}^{M} \left\{ \mathbf{u}\colon 0 \leqslant u_i \leqslant 1 + Ch^k \right\} \tag{4.31}$$

is attracting for the problem (3.7) (and (4.25)), i.e., all solutions $\mathbf{u}(t)$ of (3.7) tend to R (in C) as $t \to \infty$.

Proof. By Lemma 4.3 the system (4.25) is a gradient one, and therefore (cf. [9, p. 203, Theorem 4]) all solutions of (4.25) tend to the stationary solutions of (4.25), i.e., solutions of (3.8). By Theorem 2.1 the continuous stationary solutions belong to $[0, 1]$. Together with Theorem 4.2 and (4.7), this implies that the approximate stationary solutions belong to $[0, 1 + Ch^k]$, and (4.31) follows.

Let $u_i^h(x) = \Sigma_{j=1}^M u_j^i \psi_j(x)$, $i = 0, 1, 2$, be the approximate stationary solutions, i.e., the solutions of (3.4). Using Lemmas 4.3 and 4.4 and repeating the proof of [17, Lemma 24.12] with $u_i(x)$ replaced by $\mathbf{u}^i = (u_1^i, \dots, u_M^i)^T$, $i = 0, 1, 2$, we arrive at the next lemma.

LEMMA 4.5. Under the assumptions of Lemma 4.3, there exist solutions $\mathbf{v}^0$ and $\mathbf{v}^2$ of (3.5), which connect $\mathbf{u}^1$ to $\mathbf{u}^0$ and $\mathbf{u}^1$ to $\mathbf{u}^2$, respectively, i.e.,

$$\lim_{t \to -\infty} \mathbf{v}^0(t) = \mathbf{u}^1, \qquad \lim_{t \to \infty} \mathbf{v}^0(t) = \mathbf{u}^0,$$

and

$$\lim_{t \to -\infty} \mathbf{v}^2(t) = \mathbf{u}^1, \qquad \lim_{t \to \infty} \mathbf{v}^2(t) = \mathbf{u}^2.$$

Combining the above results, we have the following

THEOREM 4.4. Let G be defined by (2.3) and $L > L_0$. Then there exists $h_0 > 0$ such that for $h \leqslant h_0$,

(i) the approximate steady-state problem (3.4) has exactly three solutions $u_i^h \in S_0^h$, $i = 0, 1, 2$, satisfying $0 \equiv u_0^h < u_1^h < u_2^h < 1 + Ch^k$ and

$$\|u_i - u_i^h\|_1 \leqslant Ch^r \|u_i\|_{r+2}, \quad i = 1, 2, \ r = 1, \dots, k. \tag{4.32}$$

(ii) u_i^h, $i = 0, 1, 2$, are isolated invariant sets for the associated approximate parabolic problem (3.3). $h(u_0^h) = h(u_2^h) = \Sigma^0$, in particular, u_0^h and u_2^h are the attractors, and the linearized problem (4.10) has only negative eigenvalues for $i = 0, 2$. $h(u_1^h) = \Sigma^1$, in particular, u_1^h has a 1-dimensional unstable manifold which consists of orbits connecting u_1^h to each of the other rest points, and the linearized problem (4.10) has precisely one positive eigenvalue and the rest negative for $i = 1$.

(iii) Let $u(x, t)$ and $u^h(x, t)$ be the solutions of (2.4) and (3.3), respectively. Initial data which satisfies $u_1^h(x) < u^h(x, 0)$ is in the stable manifold of u_2^h, and there holds:

$$\lim_{t \to \infty} \|u(\cdot, t) - u^h(\cdot, t)\|_1 \leqslant Ch^r \|u_2\|_{r+2}, \quad r = 1, \dots, k. \tag{4.33}$$

Initial data which satisfies $u^h(x, 0) < u_1^h(x)$ is in the stable manifold of $u_0^h \equiv 0$, and there holds:

$$\lim_{t \to \infty} \|u^h(\cdot, t)\|_1 = 0. \tag{4.34}$$

Proof. (i) follows from Theorem 4.2. (ii) follows from Theorem 4.3 and Lemma 4.5. (iii) follows from (i), (ii), and Lemma 4.5 by repeating the argument in [17 pp. 535–536] and making use of a comparison principle by Kamke in the form [5]: "Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be solutions of (3.7), defined for $a \leqslant t \leqslant b$. Then $\mathbf{u}(a) \leqslant \mathbf{v}(a)$ implies $\mathbf{u}(b) \leqslant \mathbf{v}(b)$".

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Note. After this paper was accepted the author became familiar with a related work by Hale, Lin, and Raugel [18], where for evolutionary equations with compact attractors conditions were given to ensure the approximate finite-dimensional system has a compact attractor which converges, in the sense of sets, to the original one as the approximation is refined.

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