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REMARKS ON SOME NONCONSERVATION PROBLEMS*

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Abstract. A horizontal channel of uniform cross section has an impervious channel bed to the left of $x = 0$ and allows infiltration at a constant rate to the right of $x = 0$. Initially there is water at constant depth and zero velocity. There are left and right moving interfaces and, between them, water with positive velocity. At a certain time there will be a water edge to the right of which there is no water in the channel. The time history of this water edge is a free boundary. The solution of this problem, which is nonconservation because mass and momentum are carried away by infiltration, is discussed below. A single equation, which is also nonconservation, has an explicit solution. The characteristics of this single equation have a geometry similar to that the C_2 characteristics of the channel problem.

1. Introduction. Let $u(x, t)$ and $h(x, t)$ be the velocity and depth of the cross section at x at time t, $h(x,0) = h_0$, $c = (gh)^{1/2}$, $c_0 = (gh_0)^{1/2}$, and α the infiltration rate (in volume/area time). There is an interface $x = -c_0t$ moving to the left separating moving from stationary water, and an interface $x = s(t)$ given by

$$s'(t) = \left(c_0^2 - \lambda t\right)^{1/2}, \quad s(0) = 0, \lambda = g\alpha$$

moving to the right separating moving from stationary but subsiding water. Thus

$$s(t) = (2/3\lambda)\left[c_0^3 - \left(c_0^2 - \lambda t\right)^{3/2}\right], \quad 0 \leqslant t \leqslant t_0,$$

where $t_0 = h_0/\alpha = c_0^2/\lambda$. At time t_0 there is no water to the right of $s(t_0)$. Let $x = s^*(t)$, a free boundary,·be the time history of this water edge, and let $\lambda(x) = 0$ or λ according as $x < 0$ or $x > 0$. Then the formulation of the problem is [1, 2]

$$\begin{aligned}
&(u + 2c)_t + (u + c)(u + 2c)_x = -\lambda(x)/c,\\
&(u - 2c)_t + (u - c)(u - 2c)_x = \lambda(x)/c,\\
&\qquad u(-c_0t, t) = 0, \qquad c(-c_0t, t) = c_0, \\
&\qquad u(s(t), t) = 0, \qquad c(s(t), t) = \left(c_0^2 - \lambda t\right)^{1/2}, \quad 0 \leqslant t \leqslant t_0,\\
&\qquad c(s^*(t), t) = 0, \quad t > t_0, s^*(t_0) = s(t_0) = 2c_0^3/3\lambda.
\end{aligned} \tag{1.1}$$

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$u = 0$, $c = c_0$ if $x \leqslant -c_0 t$, and $u = 0$, $c = (c_0^2 - \lambda t)^{1/2}$ if $x \geqslant s(t)$, $0 \leqslant t \leqslant t_0$. We may change to dimensionless variables:

$$\xi = \lambda x / c_0^3,\ \tau = t/t_0,\ c^* = c/c_0,\ u^* = u/c_0,\ \lambda^*(x) = \lambda(x)/\lambda.$$

Replacing ξ, τ, c^*, u^*, λ^* by x, t, c, u, λ, we get

$$\begin{aligned}
&(u + 2c)_t + (u + c)(u + 2c)_x = -\lambda(x)/c,\\
&(u - 2c)_t + (u - c)(u - 2c)_x = \lambda(x)/c,\\
&u(-t, t) = 0, \qquad c(-t, t) = 1,\\
&u(\sigma(t), t) = 0, \qquad c(\sigma(t), t) = (1 - t)^{1/2}, \quad 0 \leqslant t \leqslant 1,\\
&c(\sigma^*(t), t) = 0, \quad t \geqslant 1,\ \sigma^*(1) = 2/3.
\end{aligned} \tag{1.2}$$

In (1.2) $\lambda(x) = 0$ or 1 according as $x < 0$ or $x > 0$ and

$$\sigma(t) = (2/3)\left[1 - (1 - t)^{3/2}\right]. \tag{1.3}$$

The free boundary is $x = \sigma^*(t)$. The characteristics of (1.2) are

$$C_1\colon dx/dt = u + c, \qquad C_2\colon dx/dt = u - c.$$

In Sec. 3 we discuss the solution of (1.2).

The C_2 characteristics of (1.2) have a geometry similar to that of the characteristics of

$$u_t - uu_x = -f(x), \quad u(x,0) = 1 \tag{1.4}$$

where $f(x)$ is 0 or 1 according as $x < 0$ or $x > 0$. Let R_1 and R_0 be the domains in the second quadrant above and below $x = -t$, and R_2 and R_3 the domains in the first quadrant above and below the upper branch of $x = (t - 1)^2/2$. Then the solution of (1.4) is

$$R_0\colon u = 1, \qquad R_1\colon u = -(t - 1)/2 + (1/2)\left[(t - 1)^2 - 4x\right]^{1/2}$$

$$R_2\colon u = -(2x)^{1/2}, \qquad R_3\colon u = -t + 1.$$

The solution is continuous but $u_x(0+, t) = -\infty$, $t > 1$. The characteristics $dx/dt = -u$ in R_1 are a diverging fan of lines issuing from $0 \leqslant t \leqslant 1$ on the t-axis, At $t = 1$, $u = 0$, so the characteristic originating at this point is the t axis. In fact, from each point $(0, t_0)$, $t_0 \geqslant 1$, there issue two characteristics: $x = 0$ and, as is clear from the solution in R_2, $x = (t - t_0)^2/2$, $t \geqslant t_0$. In Sec. 2 we examine (1.4) with a more general $f(x)$, which is still 0 when $x < 0$ but is continuous, nondecreasing, and positive when $x > 0$. This analysis shows the conditions under which we do, or do not, get the behavior of characteristics in the special case $f(x) = 1$, $x > 0$, namely a diverging fan of lines in the second quadrant originating on a finite interval on the t-axis and bounded by $x = -t$ and $x = 0$.

If in (1.4) we change the minus on the left to plus and retain $f(x) = 0$ or 1 according as $x < 0$ or $x > 0$ then, as is shown explicitly in [1], the solution has a shock originating at $(1/2, 1)$.

2. The Equation (1.4). We consider (1.4) for the more general $f(x)$ described above. We have, on characteristics,

$$dx/dt = -u, \qquad du/dt = -f(x), \tag{2.1}$$

so the characteristics are lines in the second quadrant. Between the negative x-axis and $x = -t$ these are the lines on which $dx/dt = -1$. Between $x = -t$ and the t-axis, $dx/dt = -u(0, t_0)$ on a characteristic line which intersects the t-axis at t_0. In the first quadrant we get the characteristics by solving (2.1) subject to $x(0) = x_0$, $u(0) = 1$. Thus

$$x'' - f(x) = 0, \qquad x(0) = x_0, \qquad x'(0) = -1,$$

so the characteristic originating at $(x_0, 0)$ is

$$t = \int_x^{x_0} [1 - 2F(x_0) + 2F(\xi)]^{-1/2}\, d\xi, \qquad F(x) = \int_0^x f(\xi)\, d\xi. \tag{2.2}$$

This characteristic intersects the t-axis at

$$t_0 = \int_0^{x_0} [1 - 2F(x_0) + 2F(\xi)]^{-1/2}\, d\xi.$$

On the characteristic (2.2),

$$u = [1 - 2F(x_0) + 2F(x)]^{1/2},$$

so that at $(0, t_0)$, $u = [1 - 2F(x_0)]^{1/2}$. On the t-axis u is a decreasing function of t which is 1 at $t = 0$ and is 0 for that $t_0 = t_0^*$ for which the corresponding x_0^* is such that $1 - 2F(x_0^*) = 0$. Since F is an increasing function of x and $F(\infty) = \infty$ there will be such an x_0^*. The corresponding t_0^* is

$$t_0^* = \int_0^{x_0^*} [2F(\xi)]^{-1/2}\, d\xi. \tag{2.3}$$

t_0^* may be finite or infinite. Then there is a diverging fan of lines issuing from $0 \leqslant t \leqslant t_0^*$ on the t-axis into the second quadrant bounded by $x = -t$ and, if t_0^* is finite, the t-axis above t_0^*. Thus the convergence or divergence of the integral

$$\int_0^{\epsilon} [F(\xi)]^{-1/2}\, d\xi$$

distinguishes the two possibilities t_0^* finite or infinite. If, for example, $f(x) = ax^{\gamma}$, $\gamma \geqslant 0$, in the neighborhood of $x = 0$ then $F(x) = bx^{\gamma+1}$ in the neighborhood of $x = 0$ and the integral converges or diverges according as $0 \leqslant \gamma < 1$ or $\gamma \geqslant 1$. In (1.4) $\gamma = 0$ so t_0^* is finite. If $f(x) = \alpha^2 x$, so $\gamma = 1$, then $t_0^* = \infty$ and $x_0^* = \alpha^{-1}$. The characteristic issuing from $(x_0, 0)$ is

$$x = x_0 \cosh \alpha t - \alpha^{-1} \sinh \alpha t.$$

These characteristics intersect the t-axis if $x_0 < \alpha^{-1}$ and do not if $x_0 \geqslant \alpha^{-1}$. The characteristic with $x_0 = \alpha^{-1}$ is $x = \alpha^{-1} e^{-\alpha t}$, which is asymptotic to the t-axis. In the first quadrant $u = -\alpha x \tanh \alpha t + (\cosh \alpha t)^{-1}$.

3. The Subsidence Problem (1.2). In [2] we obtained u and c to quadratic terms for small t in the first and second quadrants, and we also obtained u_x, c_x, and h_x on $x = -t$ and on $x = \sigma(t)$. We describe briefly these results: between $x = -t$ and $x = \sigma(t)$, h is a decreasing function of x ($h = c^2$ in the dimensionless variables) with a discontinuity in h_x at $x = 0$; also $h_{xx} < 0$ for small t. On $x = -t$, $h_x = -2/(4 + 3t)$ and $u_x = 2/(4 + 3t)$, valid for all t. On $x = \sigma(t)$, $h_x = -1/2(1 - t)^{1/2}$ and $u_x = -1/2(1 - t)$, valid for $0 < t < 1$. Thus the water edge is initially ($t = 1$) vertical.

From (1.2) we see that $u + 2c = \text{constant}$ on C_1 characteristics in the second quadrant. Since $u + 2c = 2$ on $x = -t$, we get $u + 2c = 2$ in the second quadrant. Let $\phi(t) = c(0, t)$. Then we obtain the left and right derivatives u_x and c_x at $x = 0$ from the first two equations of (1.2):

$$\begin{aligned} u_x(0+, t) &= \Delta^{-1}[-1 - 2\phi'(2 - \phi)], \\ c_x(0+, t) &= (\phi\Delta)^{-1}[1 - \phi + \phi\phi'(2 - \phi)], \\ u_x(0-, t) &= -2\phi'/(3\phi - 2), \\ c_x(0-, t) &= \phi'/(3\phi - 2), \end{aligned} \tag{3.1}$$

where $\Delta = (3\phi - 2)(2 - \phi)$. We make the plausible assumption that $\phi'(t) \leqslant 0$. Then, since $\phi(0) = 1$, we have the two possibilities: (1) $\phi(\infty) \geqslant 2/3$ and (2) $\phi(t^*) = 2/3$ for finite t^*. In case (1), assuming $\phi'(t) \to 0$ as $t \to \infty$, the second equation of (3.1) implies that $c_x(0+, t)$ has a positive limit as $t \to \infty$. Then $h_x(0+, t)$ is positive for large t, which is physically implausible. In case (2) we have, for the bracketed expressions in (3.1) at $t = t^*$, $-\omega$ and $\omega/3$, where $\omega = 1 + (8/3)\phi'(t^*)$. Here $-\infty \leqslant \phi'(t^*) \leqslant 0$, so $-\infty \leqslant \omega \leqslant 1$. We have three cases: (2a) $\omega < 0$, (2b) $\omega = 0$, (2c) $\omega > 0$. In case (2c) we get $c_x(0+, t) \to +\infty$, $t \to t^*$, which, as above, is implausible physically. In case (2a) we get, when $t \to t^*$,

$$\begin{aligned} u_x(0-, t) &\to +\infty, \qquad c_x(0-, t) \to -\infty, \\ u_x(0+, t) &\to +\infty, \qquad c_x(0+, t) \to -\infty. \end{aligned} \tag{3.2}$$

In case (2b) $u_x(0+, t)$ and $c_x(0+, t)$ are indeterminate when $t \to t^*$, but

$$u_x(0-, t) \to +\infty, \qquad c_x(0-, t) \to -\infty.$$

We have $u - c = 2 - 3c$ in the second quadrant. On the segment $0 \leqslant t \leqslant t^*$ of the t-axis, $2 - 3c = 2 - 3\phi$ is a function increasing from -1 to 0 so the C_2 characteristics issuing from this segment into the second quadrant constitute a diverging fan of lines coinciding with $x = -t$ when $t = 0$ and with the t-axis above t^* when $t = t^*$ (because $dx/dt = 0$ when $t = t^*$). Since the water profile is vertical at $x = 0$, $t = t^*$, as indicated by (3.2), there is the possibility of shock formation beginning at this point. This possibility, suggested in [1], is not supported by an examination of the characteristic directions on the line $t = t^*$. As noted above, the C_2 characteristics constitute a diverging fan along this line in the second quadrant. From (3.2) we see that $u_x(0+, t) - c_x(0+, t) \to +\infty$, $t \to t^*$, which suggests that the C_2 characteristic directions along the line $t = t^*$ continue to diverge as we continue past $x = 0$. The C_1 characteristic directions $u + c = 2 - c$ in the second quadrant also diverge along the line $t = t^*$. From (3.1) we get

$$u_x(0+, t) + c_x(0+, t) = (\phi\Delta)^{-1}[1 - 2\phi - \phi\phi'(2 - \phi)]. \tag{3.3}$$

The bracketed quantity in (3.3) goes to $-\omega/3$ as $t \to t^*$, which is positive in case (2a). This suggests, as above, that the C_1 characteristic directions along the line $t = t^*$ continue to diverge as we continue past $x = 0$. The divergence of both characteristic directions along the line $t = t^*$ on both sides of $x = 0$ do not support the possibility of a shock originating at $x = 0$, $t = t^*$.

We can now give a plausible description of the qualitative aspects of the solution of (1.2). The solution is continuous. The interface $x = \sigma(t)$ is given by (1.3); there is stationary but subsiding water, $h = 1 - t$, to the right of this interface. The water profile h is, for fixed t, a decreasing function of x, $-t \leqslant x \leqslant \sigma(t)$, with discontinuous derivatives at $x = -t$, $x = 0$, and $x = \sigma(t)$. At $t = 1$ there is no water to the right of $x = \sigma(1) = 2/3$; the water profile is vertical at this edge, and $\sigma'(1) = 0$. The free boundary $x = \sigma^*(t)$ begins at $(2/3, 1)$ and is a decreasing function of t; i.e., the water edge retreats. At some finite time t^* we get $u(0,t) = 2/3$, $c(0,t) = 2/3$, $h(0,t) = 4/9$, $t \geqslant t^*$. There is a stationary solution as $t \to \infty$ which can be obtained by deleting the time derivative terms in (1.2). For $x < 0$ we have $u = 2/3$, $c = 2/3$, $h = 4/9$, and, for $x > 0$,

$$(u + c)(u + 2c)_x = -c^{-1}, \qquad (u - c)(u - 2c)_x = c^{-1},$$

from which we get

$$u_x = (u^2 - c^2)^{-1}, \qquad c_x = -u\left[2c(u^2 - c^2)\right]^{-1}.$$

Then

$$uu_x + 2cc_x = 0, \qquad c^2 u_x + 2ucc_x = -1$$

and, since $u(0) = 2/3$, $c(0) = 2/3$,

$$u^2/2 + c^2 = 2/3, \qquad uc^2 = 8/27 - x.$$

Since $h = c^2$,

$$u^2/2 + h = 2/3, \qquad uh = 8/27 - x. \tag{3.4}$$

When $x = 0$ the parabola and hyperbola in (3.4) are tangent; as x increases to $8/27$, $h(x)$ decreases to 0 and u increases to $2/\sqrt{3}$. From (3.4) we get

$$h^3 - 2h^2/3 + (8 - 27x)^2/1458 = 0$$

so

$$h'(x) = \frac{8 - 27x}{9h(9h - 4)} = \frac{\sqrt{6}\,(2 - 3h)^{1/2}}{9h - 4}.$$

Thus $h'(0) = -\infty$ and $h'(8/27) = -\sqrt{3}/2$. The second derivative $h''(x) > 0$, $0 \leqslant x \leqslant 8/27$. The free boundary is asymptotic to $x = 8/27$. A rough numerical calculation indicates that t^* lies between 1.0 and 1.1. The C_2 characteristics that originate on $x = \sigma(t)$ either (a) intersect the t-axis on the segment $0 \leqslant t \leqslant t^*$ and are diverging lines in the second quadrant or (b) remain in the first quadrant and terminate on the free boundary $x = \sigma^*(t)$. It seems reasonable, in analogy with (1.4), that for $t \geqslant t^*$ there are characteristics originating on the t-axis and terminating on $x = \sigma^*(t)$. Some further evidence for this can be seen by obtaining (approximately) the C_2 characteristics $dx/dt = u - c$ for large t by replacing $u - c = \xi$ by the function of x defined by (3.4). This is

$$27x = 8 - 2\xi^3 - (4 - 5\xi^2)(4 - 2\xi^2)^{1/2}. \tag{3.5}$$

Here x is an increasing function of ξ, $0 \leqslant \xi \leqslant 2/3^{1/2}$ and $0 \leqslant x \leqslant 8/27$. For small x we get, from (3.5), $\xi^2 = 9x/4$, so $dx/dt = 3x^{1/2}/2$ for small x. This gives the two characteristics $x = 0$ and $x = (9/16)(t - t_0)^2$ issuing from $(0, t_0)$.

It will be necessary to perform numerical calculations to support, or deny, the proposed solution described above. These calculations will also give the value of t^*.

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