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ENERGY MINIMIZATION FOR NETS WITH SLACK*

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1. Introduction. The theory of inextensible networks with slack [1] is a continuum model of the mechanical behavior of networks of fibers that can support tension but not compression. It is postulated that the fibers can grow shorter, but not longer, in a deformation of the network. The governing equations and inequalities of the theory are such that it is not obvious, as a mathematical matter, that boundary-value problems in the theory necessarily have solutions. As a partial indication that boundary-value problems are well set, it was shown [1] that if a problem has a solution, the deformation makes the energy an absolute minimum, and this was used to establish a certain uniqueness result.

In the present note we use the minimum energy principle to show that solutions exist. It is easy to show that there is an energy-minimizing deformation that satisfies the kinematic constraint conditions of the theory. Existence of a field of fiber tensions that satisfies the equations and inequalities of the theory is less easy to prove because the tensions are not directly related to the deformation by constitutive equations and do not appear in the energy functional. They are entirely reactions to one-sided constraints.

In Sec. 2 we pose a standard mixed boundary-value problem in the theory of nets with slack. Leaving this aside, in Sec. 3 we state the constraint on kinematically admissible deformations as a Lipschitz condition and conclude that admissible deformations form a closed, bounded, convex, and equicontinuous family. The energy is a linear functional of the deformation. In Sec. 4 we show that it has a kinematically admissible minimizer.

As a linear functional defined on a convex set, the energy is minimized at a boundary point of the set, where the constraints are in play. The usual method of deriving Euler equations is irrelevant in such cases. In Secs. 5 and 6 we derive the equilibrium equations satisfied by the forces of constraint by using an argument involving dual cones in function space. This kind of argument is familiar to workers in control theory (see Girsanov [2], for example), but is less familiar in continuum mechanics.

The existence proof is defective in that we do not precisely characterize the class of admissible tension fields, but only point out the properties that are being assumed. From experience with particular problems we know that tension fields may exhibit Dirac delta

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singularities that represent finite forces supported by single fibers. We accordingly regard tension fields as generalized functions [3].

2. Boundary value problems for nets with slack. We consider an initially plane sheet or network formed from fibers that are parallel to the x and y axes of a system of Cartesian coordinates. In the initial configuration, the sheet occupies a bounded, connected region S with boundary curve C. The network is treated as a continuum, so that every line x = constant or y = constant is regarded as a fiber. For simplicity, we limit attention to regions S whose boundaries are intersected at most twice by any line x = constant or y = constant.

In a deformation, the particles of both families of fibers that were initially at (x, y) move to the place $\mathbf{r}(x, y)$ in three-dimensional space. The derivatives $\mathbf{a} = \mathbf{r}_x$ and $\mathbf{b} = \mathbf{r}_y$ are tangential to the deformed fibers, and the magnitudes of these vectors represent the ratio of deformed to undeformed length. As a one-sided constraint of inextensibility, we stipulate that these magnitudes cannot exceed unity:

$$d\mathbf{r} = \mathbf{a}dx + \mathbf{b}dy, \quad |\mathbf{a}| \leqslant 1, |\mathbf{b}| \leqslant 1. \tag{2.1}$$

The fiber tension fields T_a and T_b must satisfy the equilibrium equation

$$(T_a\mathbf{a})_x + (T_b\mathbf{b})_y + \mathbf{f} = \mathbf{0}, \tag{2.2}$$

where $\mathbf{f}(x, y)$ is a prescribed body force. We require the tensions to be nonnegative:

$$T_a \geqslant 0, \qquad T_b \geqslant 0. \tag{2.3}$$

In addition, we require the tension to be zero in a fiber that is not fully extended:

$$T_a = 0 \quad \text{if } |\mathbf{a}| < 1, \qquad T_b = 0 \quad \text{if } |\mathbf{b}| < 1. \tag{2.4}$$

The relations (2.1) to (2.4) are to be satisfied in S. Let C_p and C_t be complementary, nonempty parts of the boundary of S. We require that

$$\mathbf{r}(x, y) = \mathbf{r}_0(x, y) \quad \text{on } C_p \tag{2.5}$$

and

$$T_a\mathbf{a}\, dy - T_b\mathbf{b}\, dx = \mathbf{T}\, ds \quad \text{on } C_t, \tag{2.6}$$

where $\mathbf{r}_0$ and $\mathbf{T}$ are prescribed functions and dx, dy, and ds are increments of coordinates and arc length along C_p. We suppose that $\mathbf{r}_0$ is such that at least one deformation satisfies (2.1) and (2.5).

By a *solution*, we mean a set $\mathbf{r}$, T_a, T_b satisfying (2.1) to (2.6).

3. Admissible deformations. We now begin anew, not assuming that any of the relations (2.1) to (2.4) are satisfied. In the present section we consider the kinematic constraint condition and its consequences for the deformation $\mathbf{r}$.

Because of the restriction on the shape of S mentioned in Sec. 2, two points (x_1, y_1) and (x_2, y_2) in S can be connected by a path composed of segments x = constant and y = constant whose total length is $|x_1 - x_2| + |y_1 - y_2|$. The constraint of inextensibility requires that these two particles never be further apart than that length:

$$|\mathbf{r}(x_1, y_1) - \mathbf{r}(x_2, y_2)| \leqslant |x_1 - x_2| + |y_1 - y_2|. \tag{3.1}$$

By an *admissible deformation*, or simply a *deformation*, henceforward we mean a function $\mathbf{r}(x, y)$ that satisfies (3.1) and the boundary condition (2.5).

The constraint (3.1) is a Lipschitz condition, which guarantees that $\mathbf{r}$ is absolutely continuous [4]. In particular, the derivatives $\mathbf{a} = \mathbf{r}_x$ and $\mathbf{b} = \mathbf{r}_y$ exist almost everywhere and satisfy (2.1).

By letting (x_1, y_1) be a running point and choosing (x_2, y_2) to be a point on C_p, where $\mathbf{r}$ is specified, we conclude from (3.1) that $\mathbf{r}$ is bounded, by R, say:

$$|\mathbf{r}(x, y)| \leqslant R. \tag{3.2}$$

From (3.1) it also follows that the deformations form an equicontinuous family. Then every infinite sequence of deformations has a subsequence that converges to an admissible deformation.

Although we do not directly use the fact, it is of interest that the set of admissible deformations is convex. That is, if $\mathbf{r}_1(x, y)$ and $\mathbf{r}_2(x, y)$ satisfy (2.5) and (3.1), then so does

$$m\mathbf{r}_1(x, y) + (1 - m)\mathbf{r}_2(x, y) \quad (0 \leqslant m \leqslant 1). \tag{3.3}$$

4. Energy minimization. The network has no strain energy. The energy of the loads is the linear functional of $\mathbf{r}(x, y)$ defined by

$$E = -\int_{C_t} \mathbf{r} \cdot T\, ds - \iint_S \mathbf{r} \cdot \mathbf{f}\, dx\, dy. \tag{4.1}$$

For abbreviation, later we use the notation

$$\mathbf{X} = \{\mathbf{u}, \mathbf{v}\}, \qquad (\mathbf{z}, \mathbf{X}) = -\int_{C_t} \mathbf{z} \cdot \mathbf{u}\, ds - \iint_S \mathbf{z} \cdot \mathbf{v}\, dx\, dy. \tag{4.2}$$

In this notation the energy is

$$E = (\mathbf{r}, \mathbf{D}), \qquad \mathbf{D} = \{\mathbf{T}, \mathbf{f}\}. \tag{4.3}$$

We note in passing that because the energy is a linear functional defined on a convex set, energy minimization by direct numerical methods would lead to a linear programming problem.

With $|\mathbf{r}| \leqslant R$, the energy is bounded:

$$|E| \leqslant R\int_{C_t} |\mathbf{T}|\, ds + R\iint_S |\mathbf{f}|\, dx\, dy. \tag{4.4}$$

Let E_0 be its greatest lower bound. Let $\{\mathbf{r}_n\}$ be a sequence with $E(\mathbf{r}_n)$ approaching E_0. Then from Sec. 3, $\{\mathbf{r}_n\}$ has a convergent subsequence approaching $\mathbf{r}_0$, say, and $E(\mathbf{r}_0) = E_0$. Thus there is an admissible deformation that minimizes the energy.

5. Admissible variations. Let $\mathbf{r}$ be admissible and suppose that $\mathbf{r} + \varepsilon\delta\mathbf{r}$ is also admissible for all sufficiently small $\varepsilon > 0$. Then $\delta\mathbf{r}$ is an *admissible variation*. Admissible variations satisfy $\delta\mathbf{r} = \mathbf{0}$ on C_p, and from now on we tacitly assume that this is satisfied. We wish to express the constraint condition (2.1) or (3.1) as an integral constraint on $\delta\mathbf{r}$ involving Lagrange multipliers.

Let $\delta\mathbf{a} = (\delta\mathbf{r})_x$ and $\delta\mathbf{b} = (\delta\mathbf{r})_y$. Then from

$$|\mathbf{a} + \varepsilon\delta\mathbf{a}| \leqslant 1 \quad \text{and} \quad |\mathbf{b} + \varepsilon\delta\mathbf{b}| \leqslant 1 \tag{5.1}$$

we find that

$$\begin{aligned} \mathbf{a}\cdot\delta\mathbf{a} &\leqslant 0 \quad \text{if } |\mathbf{a}| = 1 \text{ (in } R_a\text{)}, \\ \mathbf{b}\cdot\delta\mathbf{b} &\leqslant 0 \quad \text{if } |\mathbf{b}| = 1 \text{ (in } R_b\text{)}. \end{aligned} \tag{5.2}$$

These conditions are satisfied if $\delta\mathbf{r}$ is admissible, and only then.

Given $\mathbf{r}$, let R_a and R_b be the regions in which $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 1$, respectively. We say that (T_a, T_b) is a kinematically admissible tension field for $\mathbf{r}$ if T_a and T_b are nonnegative everywhere and are zero except in R_a and R_b, respectively. That is, T_a and T_b satisfy (2.3) and (2.4). Let AT be the set of admissible tensions. We regard AT as a closed set, although we have not specified much about the nature of T_a and T_b. Then AT is a closed convex cone. That is, if (T_a^1, T_b^1) and (T_a^2, T_b^2) are in AT, then so is

$$\left(\alpha T_a^1 + \beta T_a^2, \alpha T_b^1 + \beta T_b^2\right), \quad \alpha \geqslant 0, \beta \geqslant 0. \tag{5.3}$$

From (5.2) and the definition of AT, if $\delta\mathbf{r}$ is admissible then

$$\iint \left[T_a\mathbf{a}\cdot\delta\mathbf{a} + T_b\mathbf{b}\cdot\delta\mathbf{b}\right] dx\, dy \leqslant 0 \tag{5.4}$$

for all (T_a, T_b) in AT. Thus we limit AT to be such that (5.4) makes sense. But we stipulate that AT is a rich enough set that (5.2) can be recovered from (5.4), so that $\delta\mathbf{r}$ is admissible if (5.4) is satisfied.

By integration by parts we find that (5.4) is equivalent to

$$\int_{C_t} \delta\mathbf{r}\cdot(T_a\mathbf{a}y' - T_b\mathbf{b}x')\, ds - \iint_S \delta\mathbf{r}\cdot\left[(T_a\mathbf{a})_x + (T_b\mathbf{b})_y\right] dx\, dy \leqslant 0, \tag{5.5}$$

where $x' = dx/ds$ and $y' = dy/ds$. Using the notation defined in (4.2), (5.5) is

$$(\delta\mathbf{r}, \mathbf{C}) \geqslant 0, \tag{5.6}$$

where

$$\mathbf{C} = \left\{T_a\mathbf{a}y' - T_b\mathbf{b}x', -(T_a\mathbf{a})_x - (T_b\mathbf{b})_y\right\}. \tag{5.7}$$

Because AT is assumed to be a closed convex cone, the set of all $\mathbf{C}$ described by (5.7) is also a convex cone, K say, and we regard TA to be such that K is also closed. Then (5.6) is the statement that $\delta\mathbf{r}$ is in the cone K^* dual to K [2]. The assumption that K is closed implies that the dual of K^* is K rather than merely the closure of K.

6. Equilibrium equations. Let $\mathbf{r}$ be a deformation that minimizes the energy. Then from (4.3),

$$\delta E = (\delta\mathbf{r}, \mathbf{D}) \geqslant 0 \quad \forall\, \delta\mathbf{r} \in K^*, \tag{6.1}$$

where $\delta\mathbf{r} \in K^*$ if and only if

$$(\delta\mathbf{r}, \mathbf{C}) \geqslant 0 \quad \forall\, \mathbf{C} \in K. \tag{6.2}$$

Now, (6.1) implies that **D** is in the cone dual to K^*, namely K, since we have assumed that K is closed. So there is a **C** in K such that **D** = **C**. With (4.3) and (5.7), this means that there is a tension field (T_a, T_b) that satisfies the equilibrium Eq. (2.2) and the traction boundary condition (2.6). We recall that (2.3) and (2.4) are satisfied by definition of T_a and T_b. Thus, the problem specified in Sec. 2 has a solution.

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