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PERTURBATION TECHNIQUES IN THE THEORY OF LUBRICATION*

By

GIOVANNI CIMATTI

University of Pisa, Pisa, Italy

Abstract. Perturbation techniques are a very useful tool also in the theory of lubrication. Two typical situations are rigorously discussed in this paper: the bifurcation for the "shoe-bearing" and the asymptotic analysis for a finite gas slider bearing of narrow geometry. Furthermore the mathematical basis of a new application of the Reynolds equation is briefly treated.

Introduction. This paper deals with various perturbation problems arising in the theory of lubrication. The main analytic tool of this branch of fluid mechanics is the Reynolds equation [1], [5] which reads

$$\nabla \cdot \left(\frac{H^3 \rho}{\mu} \nabla P \right) = 6U(\rho H)_X. \tag{1.1}$$

In (1.1), $P(X)$, $X = (x, y)$ is the pressure in a thin film of lubricating fluid of density ρ and viscosity μ. $H(X)$ is the film thickness and U the x-component of the velocity of the moving surface.

A bifurcation problem motivated by the study of the so-called "shoe-bearing" is examined in Section 2. For this kind of bearing the thickness of the film is not fixed because the upper surface (the shoe) is freely pivoted; thus the equilibrium positions under load need not be unique. The occurrence of bifurcation is proved with elementary analytic means.

In Section 3 we are concerned with a singular perturbation problem for the nonlinear Reynolds equation of gas-lubrication. We take into account the width of the occurring boundary layer and prove a result of convergence previously stated without proof by Diprima and Shepherd (see [4]).

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A. B. Tayler has proposed in [9] a new mathematical model based on the theory of lubrication for describing the impregnation of a sheet of absorbent material with a viscous fluid. A theorem of existence and uniqueness is given in Section 4 for the relevant nonlinear elliptic boundary value problem.

2. An elementary problem of bifurcation for the "shoe-bearing". Let Ω be an open and bounded subset of $\mathbf{R}^2$ with a regular boundary $\partial\Omega$. We suppose in this section ρ = constant and take for brevity $6U\mu = 1$; thus equation (1.1) becomes

$$\nabla \cdot (H^3 \nabla P) = H_x \quad in\ \Omega. \tag{2.1}$$

As boundary condition we assume

$$P = 0 \quad on\ \partial\Omega. \tag{2.2}$$

Now suppose

$$H(X; \varepsilon, \alpha) = h_0 + \alpha(x - \varepsilon), \qquad h_0 > 0, \tag{2.3}$$

to be the equation of the pivoted bearing surface free to rotate around the axis $x = \varepsilon$ with ε and α small parameters. By the Lax-Milgram lemma for every α and ε sufficiently small, there exists one and only one solution $P(X; \varepsilon, \alpha)$ of problem (2.1), (2.2). On the other hand, when $\alpha = 0$ (case of the parallel bearings surfaces), we have for every ε the trivial solution

$$P(X; \varepsilon, 0) = 0. \tag{2.4}$$

When $\alpha \neq 0$ the pressure is also different from zero and for a steady situation the following equation of equilibrium, expressing the balance of angular momentum

$$\Phi(\varepsilon, \alpha) = \int_\Omega (x - \varepsilon) P(X; \varepsilon, \alpha)\, dX = 0, \tag{2.5}$$

must hold. Our goal is to study equation (2.5) in a neighbourhood of the point $(0,0)$ in the plane ε, α. We shall prove that under assumptions largely verified in practice, there exists locally a second branch of solutions starting from $(0,0)$ in addition to the trivial ones for $\alpha = 0$. All our arguments shall be quite elementary.

Let us suppose $\partial\Omega \in C^3$, and assume Ω symmetric with respect to the y-axis. Moreover suppose

$$\{(x, y), |x| < |x_0|, y = y_0\} \subset \Omega \tag{2.6}$$

when $(x_0, y_0) \in \partial\Omega$. Define

$$\Omega^+ = \{(x, y) \in \Omega, x > 0\}, \qquad \partial\Omega^+ = \{(x, y) \in \partial\Omega, x > 0\}.$$

Clearly we have

$$\Phi(\varepsilon, 0) = 0 \tag{2.7}$$

for all ε, $|\varepsilon| < \varepsilon_0$. Let us differentiate (2.1) with respect to α and set $\alpha = 0$. Since by (2.4)

$$\nabla P(X; \varepsilon, 0) = 0,$$

with a straightforward calculation we get, recalling (2.2)

$$\Delta P_\alpha(X;\varepsilon,0) = 1/h_0^3 \quad \text{in } \Omega, \tag{2.8}$$

$$P_\alpha(X;\varepsilon,0) = 0 \quad \text{on } \partial\Omega. \tag{2.9}$$

From (2.8), (2.9), it follows that $P_\alpha(X;\varepsilon,0)$ does not depend on ε. Differentiating (2.1) two times with respect to α and then putting $\alpha = \varepsilon = 0$ we get by (2.8)

$$-\Delta P_{\alpha\alpha}(X;0,0) = 6x/h_0^4 + 6(P_{\alpha x}(X;0,0))/h_0 \quad \text{in } \Omega. \tag{2.10}$$

$$P_{\alpha\alpha}(X;0,0) = 0 \quad \text{on } \partial\Omega. \tag{2.11}$$

Further properties of the solutions to problems (2.8), (2.9) and (2.10), (2.11) are given in the following

LEMMA 2.1. If P_α, $P_{\alpha\alpha}$ are respectively the solutions of problems (2.8), (2.9) and (2.10), (2.11) then

$$P_\alpha < 0 \quad \text{in } \Omega, \tag{2.12}$$

$$P_\alpha(x,y) = P_\alpha(-x,y) \quad \text{in } \Omega, \tag{2.13}$$

$$P_{\alpha x}(x,y) = -P_{\alpha x}(-x,y) \quad \text{in } \Omega, \tag{2.14}$$

$$P_{\alpha x} > 0 \quad \text{in } \Omega^+, \tag{2.15}$$

$$P_{\alpha\alpha}(x,y) = -P_{\alpha\alpha}(-x,y) \quad \text{in } \Omega, \tag{2.16}$$

$$P_{\alpha\alpha} > 0 \quad \text{in } \Omega^+. \tag{2.17}$$

Proof. Inequality (2.12) is a consequence of the maximum principle.

Since Ω is symmetric with respect to the y-axis and the left-hand side of equation (2.8) is an even function, (2.13) and (2.14) follow easily.

By (2.14) $P_{\alpha x}(0,y) = 0$. Moreover if $(x_0, y_0) \in \partial\Omega^+$ we obtain by (2.6)

$$P_\alpha(x, y_0) < 0 \quad \text{for } 0 \leqslant x < x_0,$$

hence $P_{\alpha x}(x_0, y_0) \geqslant 0$. Thus $P_{\alpha x}$ satisfies

$$\Delta P_{\alpha x} = 0 \quad \text{in } \Omega, \qquad P_{\alpha x} \geqslant 0 \quad \text{on } \partial\Omega^+. \tag{2.18}$$

This implies (2.15) by the maximum principle.

By (2.14) the right-hand side of equation (2.10) is an odd function in x; taking into account the symmetry of Ω we obtain (2.16). From (2.15) we obtain $6x/h_0^4 + 6P_{\alpha x}/h_0 > 0$ in Ω^+; moreover we have, recalling (2.16), $P_{\alpha\alpha} = 0$ on $\partial\Omega^+$. Again by the maximum principle (2.17) follows. □

Now we have

$$\Phi_\alpha(0,0) = \int_\Omega x P_\alpha(X)\, dX. \tag{2.19}$$

On the other hand by (2.13) $xP_\alpha(X)$ is an odd function in x. Hence recalling the symmetry of Ω

$$\Phi_\alpha(0,0) = 0. \tag{2.20}$$

For every ε sufficiently small we have (2.7), therefore

$$\Phi_\varepsilon(0,0) = 0. \tag{2.21}$$

By (2.16) $xP_{\alpha\alpha}(X)$ is an even function with respect to x. Hence the symmetry of Ω implies

$$\Phi_{\alpha\alpha}(0,0) = \int_\Omega xP_{\alpha\alpha}(X)\,dX = 2\int_{\Omega^+} xP_{\alpha\alpha}(X)\,dX.$$

Therefore from (2.17)

$$\Phi_{\alpha\alpha}(0,0) = m^2 > 0. \tag{2.22}$$

We want to compute

$$\Phi_{\varepsilon\alpha}(0,0) = \int_\Omega \left[xP_{\varepsilon\alpha}(X;0,0) - P_\alpha(X;0,0)\right] dX.$$

Differentiating (2.8) and (2.9) with respect to ε we get

$$\Delta P_{\varepsilon\alpha} = 0 \quad \text{in } \Omega, \qquad P_{\varepsilon\alpha} = 0 \quad \text{on } \partial\Omega,$$

thus $P_{\varepsilon\alpha}(X;0,0) = 0$ in Ω. Hence by (2.12) we have

$$\Phi_{\varepsilon\alpha}(0,0) = l^2 > 0. \tag{2.23}$$

Finally from (2.7) we obtain

$$\Phi_{\varepsilon\varepsilon}(0,0) = 0. \tag{2.24}$$

Using Taylor's theorem we can write

$$\Phi(\varepsilon,\alpha) = (1/2)\alpha(m^2\alpha + 2l^2\varepsilon) + \text{higher order terms}.$$

We conclude that there exists a nontrivial branch of solutions starting from $(0,0)$ with tangent in the ε, α plane given by $\alpha = -2l^2\varepsilon/m^2$.

Physically realistic solutions occur when $\varepsilon > 0$. In fact only in this case is the pressure generated by the bearing positive and can sustain a loading.

3. A boundary layer problem for the nonlinear Reynolds equation of gas-lubrication. R. C. DiPrima and J. J. Shepherd study (see [4]) with a perturbation technique the following boundary value problem for the nonlinear Reynolds equation valid for compressible fluids

$$\varepsilon\left(H^3PP_x\right)_x + \left(H^3PP_y\right)_y = \varepsilon(HP)_x \quad \text{in } R, \tag{3.1}$$

$$P = 1 \quad \text{in } \partial R. \tag{3.2}$$

In (3.1) ε is a small positive parameter and $R = \{(x, y);\ 0 < x < 1,\ |y| < 1\}$.

In this section we try to clarify in which sense the asymptotic development in ε

$$P(X;\varepsilon) = 1 + \varepsilon P_1(X) + \varepsilon^2 P_2(X) + \cdots$$

constructed by DiPrima gives an approximation to the exact solution of problem (3.1), (3.2). The interest of the problem lies in the nonlinearity of equation (3.1) coupled with the boundary layers which occur at the leading and trailing edges $x = 0$ and $x = 1$ of the bearing. Our discussion is valid in the case of the converging wedge bearing, i.e., when

$$H_x(X) < 0 \quad \text{in } \overline{R}. \tag{3.3}$$

In addition to (3.3) we make the standard assumptions

$$H_1 \geqslant H(X) \geqslant H_0 > 0, \qquad H(X) \in C^1(\overline{R}). \tag{3.4}$$

If (3.3) and (3.4) hold true it is possible to prove (see [2], [3]) that there exists a unique solution to problem (3.1), (3.2) which satisfies

$$P > 1 \quad \text{in } R. \tag{3.5}$$

Define $Q = P - 1$. Problem (3.1), (3.2) can be rewritten as

$$\varepsilon\left[H^3(1+Q)Q_x\right]_x + \left[H^3(1+Q)Q_y\right]_y = \varepsilon\left[H(1+Q)\right]_x, \quad \text{in } R, \tag{3.6}$$

$$Q = 0 \quad \text{on } \partial R. \tag{3.7}$$

Let us consider the formal development

$$Q(X;\varepsilon) = Q_0 + \varepsilon Q_1(X) + \cdots. \tag{3.8}$$

Substituting (3.8) in (3.6) and equating terms of equal power in ε, we obtain $Q_0 = 0$ and

$$\left(H^3 Q_{1y}\right)_y = H_x. \tag{3.9}$$

The solution of (3.9) which satisfies the boundary condition $Q_1(x, \pm 1) = 0$ is given by

$$Q_1(x,y) = F_1(x,y) - \frac{F_1(x,1)}{F_2(x,1)} F_2(x,y), \tag{3.10}$$

where

$$F_1(x,y) = \int_{-1}^{y} H^{-3}(x,s)\left[\int_0^s H_x(x,t)\,dt\right] ds,$$

and

$$F_2(x,y) = \int_{-1}^{y} H^{-3}(x,s)\,ds.$$

It is easily seen that $Q_1(x,y)$ in general does not verify the boundary condition (3.7) on $x = 0$ and $x = 1$. On these sides of R we expect to find, when $\varepsilon \to 0$, boundary layers where the pressure adjusts from the value 1 on the boundary to the value given by $1 + \varepsilon Q_1$. We want to make precise this rather vague remark.

First of all we recall the following Poincaré like

LEMMA 3.1. If $u \in H^1(R)$ and $u = 0$ on the side $x = 0$ then

$$\|u\|_{L^2} \leqslant C\|u_x\|_{L^2}.$$

Similar results hold true if u vanishes on the other sides of R.

Let us consider the problem

$$\varepsilon\left(H^3\phi_x\right)_x + \left(H^3\phi_y\right)_y = 0 \quad \text{in } R, \quad (\varepsilon > 0) \tag{3.11}$$

$$\phi = Q_1 \quad \text{on } \partial R. \tag{3.12}$$

We have the following

LEMMA 3.2. When $\varepsilon \to 0$ the solution $\phi(X;\varepsilon)$ of (3.11), (3.12) gives a boundary layer of width ε^α $(0 < \alpha < 1)$ near $x = 0$, $x = 1$. More precisely the following estimates hold true:

$$\varepsilon^\alpha \int_{R_\varepsilon} \phi_x^2\, dX \leqslant C, \qquad \int_{R_\varepsilon} \phi_y^2\, dX \leqslant C\varepsilon^{1-\alpha}, \tag{3.13}$$

$$\int_{R_\varepsilon} \phi^2\, dX \leqslant C\epsilon^{1-\alpha}, \tag{3.14}$$

where $R_\varepsilon = \{(x, y) \in R;\ \varepsilon^\alpha < x < 1 - \varepsilon^\alpha\}$.

Proof. Suppose $\zeta(x) \in C^2[0,1]$, $\zeta(x) = 1$ when $\varepsilon^\alpha \leqslant x \leqslant 1 - \varepsilon^\alpha$ and $0 \leqslant \zeta(x) \leqslant 1$ in $[0,1]$. Moreover let $\zeta(0) = \zeta'(0) = \zeta(1) = \zeta'(1) = 0$. By (3.11) we get

$$\varepsilon\left[H^3(\zeta\phi)_x\right]_x + \left[H^3(\zeta\phi)_y\right]_y = \varepsilon\left(H^3\zeta_x\phi\right)_x + \varepsilon\zeta_x H^3\phi_x. \tag{3.15}$$

Let us multiply (3.15) by $\zeta\phi$ and integrate by parts over R. Recalling that $\phi(x, \pm 1) = 0$ we have

$$\varepsilon\int_R H^3(\zeta\phi)_x^2\, dX + \int_R H^3(\zeta\phi)_y^2\, dX = \varepsilon\int_R H^3\zeta_x^2\phi^2\, dX. \tag{3.16}$$

Define $S_\varepsilon = R - R_\varepsilon$. Applying the maximum principle (3.11) yields

$$0 \leqslant \phi \leqslant \max_{\partial R} Q_1. \tag{3.17}$$

From (3.16) and (3.17) it follows

$$\varepsilon\int_{R_\varepsilon} H^3\phi_x^2\, dX + \int_{R_\varepsilon} H^3\phi_y^2\, dX \leqslant M\varepsilon\int_{S_\varepsilon} \zeta_x^2\, dX, \tag{3.18}$$

where $M = H_1^3 \max_{\partial R}^2 Q_1$. Consider a regular function $\Lambda(z)$, $z \in [0,1]$ such that $\Lambda(0) = 0$, $\Lambda'(0) = 0$, $\Lambda(1) = 1$, and $\Lambda'(1) = 0$. Let

$$\left(\Lambda'(z)\right)^2 = \sum_{k=2}^{\infty} c_k z^k.$$

Set $x = \varepsilon^\alpha z$ and define $\xi(x) = \Lambda(x/\varepsilon^\alpha)$. Whence

$$\left(\xi'(x)\right)^2 = \sum_{k=2}^{\infty} c_k \frac{x^k}{\varepsilon^{\alpha(2+k)}}.$$

Choose in (3.16) $\zeta(x)$ in the following way

$$\zeta(x) = \begin{cases} \xi(x), & 0 \leqslant x \leqslant \varepsilon^\alpha \\ 1, & \varepsilon^\alpha \leqslant x \leqslant 1 - \varepsilon^\alpha \\ \xi(1-x), & 1 - \varepsilon^\alpha \leqslant x \leqslant 1. \end{cases}$$

From (3.18) we obtain

$$\varepsilon\int_{R_\varepsilon} H^3\phi_x^2\, dX + \int_{R_\varepsilon} H^3\phi_y^2\, dX \leqslant 4M\varepsilon\int_0^{\varepsilon^\alpha} \left(\xi'(x)\right)^2 dx \leqslant C\varepsilon^{1-\alpha}, \tag{3.19}$$

where C does not depend on ε. From (3.19) we arrive to (3.13). The estimate (3.14) is a consequence of Lemma 3.1. □

Since the unique solution of problem (3.1), (3.2) satisfies $P \geqslant 1$ we can make the substitution $U = P^2$. We get from (3.1)

$$\varepsilon\left(H^3 U_x\right)_x + \left(H^3 U_y\right)_y = 2\varepsilon\left(H\sqrt{U}\right)_x \text{ in } R, \qquad U = 1 \quad \text{on } \partial R \tag{3.20}$$

and $U \geqslant 1$ in R.

LEMMA 3.3. Let $U(X)$ be the unique solution of (3.20); the following estimates hold true:

$$\|U_y\|_{L^2} \leqslant C\varepsilon, \qquad \|U_x\|_{L^2} \leqslant C\varepsilon^{1/2}, \qquad \|U - 1\|_{L^2} \leqslant C\varepsilon \tag{3.21}$$

$$\|U_{yy}\|_{L^2} \leqslant C\varepsilon, \qquad \|U_{xy}\|_{L^2} \leqslant C\varepsilon^{1/2}, \qquad \|U_{xx}\|_{L^2} \leqslant C. \tag{3.22}$$

In particular if H does not depend on y we have

$$\|U_{xy}\|_{L^2} \leqslant C\varepsilon, \qquad \|U_{xx}\|_{L^2} \leqslant C\varepsilon^{1/2}.^{(1)} \tag{3.23}$$

Proof. Multiplying (3.20) by $U - 1$ and integrating by parts over R we get

$$\|U_x\|_{L^2}^2 + \frac{1}{\varepsilon}\|U_y\|_{L^2}^2 \leqslant C\left(\|U_x\|_{L^2} + \|U_x\|_{L^2}^{3/2}\right), \tag{3.24}$$

where use has been made of the inequality $\|U - 1\|_{L^2} \leqslant C\|U_x\|_{L^2}$. Whence

$$\|U\|_{L^2} \leqslant C, \qquad \|U_x\|_{L^2} \leqslant C, \qquad \|U_y\|_{L^2} \leqslant C\varepsilon^{1/2}. \tag{3.25}$$

Again from (3.20) we have

$$\varepsilon\int_R H^3 U_x^2\, dX + \int_R H^3 U_y^2\, dX = -2\varepsilon\int_R \left(H_x\sqrt{U} + \frac{U_x}{2\sqrt{U}}H\right)(U - 1)\, dX.$$

Since $U \geqslant 1$ and $\|U - 1\|_{L^2} \leqslant C\|U_y\|_{L^2}$, we obtain using (3.25)

$$\varepsilon\|U_x\|_{L^2}^2 + \|U_y\|_{L^2}^2 \leqslant C\varepsilon\left(\|U\|_{L^2}^{1/2} + \|U_x\|_{L^2}\right)\|U_y\|_{L^2} \leqslant C\varepsilon\|U_y\|_{L^2}.$$

This gives (3.21). From (3.20) we get

$$\varepsilon U_{xx} + U_{yy} = -3\varepsilon\frac{H_x}{H}U_x - 3\frac{H_y}{H}U_y + 2\varepsilon\frac{H_x}{H^3}\sqrt{U} + \varepsilon\frac{U_x}{H^2\sqrt{U}}. \tag{3.26}$$

Let us multiply both sides by U_{yy}. After two integrations by parts, recalling that $U_{xx}(x, \pm 1) = U_y(0, y) = U_y(1, y) = 0$, there results

$$\varepsilon\|U_{xy}\|_{L^2}^2 + \|U_{yy}\|_{L^2}^2 \leqslant C\left(\varepsilon\|U_x\|_{L^2} + \|U_y\|_{L^2} + \varepsilon\|U\|_{L^2}^{1/2}\right)\|U_{yy}\|_{L^2}. \tag{3.27}$$

Whence by (3.21)

$$\varepsilon\|U_{xy}\|_{L^2}^2 + \|U_{yy}\|_{L^2}^2 \leqslant C\varepsilon\|U_{yy}\|_{L^2}.$$

Thus $(3.22)_1$ and $(3.22)_2$ follow. In a similar fashion multiplying (3.26) by U_{xx} we get

$$\varepsilon\|U_{xx}\|_{L^2}^2 + \|U_{xy}\|_{L^2}^2 \leqslant C\varepsilon\|U_{xx}\|_{L^2}. \tag{3.28}$$

(1) The various C's denote constants generally different which do not depend on ε.

Therefore $(3.22)_3$ holds true. When $H_y = 0$ the second term on the right-hand side of (3.27) disappears and instead of (3.28) we have the better estimate

$$\varepsilon \|U_{xx}\|_{L^2}^2 + \|U_{xy}\|_{L^2}^2 \leqslant C\varepsilon^{3/2} \|U_{xx}\|_{L^2}. \tag{3.29}$$

This gives (3.23). □

Since $\|U - 1\|_{L^\infty} \leqslant \|U_{xy}\|_{L^2}$ from $(3.22)_2$ we infer

$$\|U - 1\|_{L^\infty} \leqslant C\varepsilon^{1/2}. \tag{3.30}$$

Moreover when $H_y = 0$, $(3.23)_1$ implies

$$\|U - 1\|_{L^\infty} \leqslant C\varepsilon. \tag{3.31}$$

Since $U = (1 + Q)^2$, similar estimates can be derived for Q. Recalling that $Q \geqslant 0$ and

$$Q_x = 2U_x/(1 + Q), \qquad Q_y = 2U_y/(1 + Q)$$

we have from $(3.21)_1$ and $(3.21)_2$

$$\|Q_x\|_{L^2} \leqslant C\varepsilon^{1/2}, \qquad \|Q_y\|_{L^2} \leqslant C\varepsilon. \tag{3.32}$$

Moreover $\|Q\|_{L^\infty} \leqslant \|U - 1\|_{L^\infty}$, thus by (3.30)

$$\|Q\|_{L^\infty} \leqslant C\varepsilon^{1/2} \tag{3.33}$$

and when $H_y = 0$,

$$\|Q\|_{L^\infty} \leqslant C\varepsilon. \tag{3.34}$$

We are now in a position to give the main result of this section.

THEOREM 3.1. Let $Q(X; \varepsilon)$ be the unique solution of problem (3.6), (3.7) and $Q_1(X)$ the solution to the limit problem (3.9) given by (3.10). If $\phi(X; \varepsilon)$ is the solution of the boundary layer equation (3.11), (3.12) and $W = Q - \varepsilon(Q_1 - \phi)$, then

$$\|W_y\|_{L^2} \leqslant C\varepsilon^{3/2}, \qquad \|W_x\|_{L^2} \leqslant C\varepsilon, \qquad \|W\|_{L^2} \leqslant C\varepsilon^{3/2}. \tag{3.35}$$

Moreover when $H_y = 0$

$$\|W_y\|_{L^2} \leqslant C\varepsilon^2, \qquad \|W_x\|_{L^2} \leqslant C\varepsilon, \qquad \|W\|_{L^2} \leqslant C\varepsilon^2. \tag{3.36}$$

Proof. From (3.6), (3.9), and (3.11) we obtain by difference

$$\varepsilon\left(H^3 W_x\right)_x + \left(H^3 W_y\right)_y = \varepsilon(HQ)_x - \varepsilon\left(H^3 Q Q_x\right)_x - \left(H^3 Q Q_y\right)_y - \varepsilon^2\left(H^3 Q_{1x}\right)_x. \tag{3.37}$$

Let us multiply (3.37) by W and integrate by parts over R. Since $W = 0$ on ∂R we get

$$\varepsilon\|W_x\|_{L^2}^2 + \|W_y\|_{L^2}^2 \leqslant C\varepsilon\Big(\|Q\|_{L^2}\|W_x\|_{L^2} + \varepsilon\|Q\|_{L^\infty}\|Q_x\|_{L^2}\|W_x\|_{L^2}$$
$$+ \|Q\|_{L^\infty}\|Q_y\|_{L^2}\|W_y\|_{L^2} + \varepsilon^2\|W\|_{L^2}\Big).$$

Thus by (3.32) and (3.33)

$$\varepsilon\|W_x\|_{L^2}^2 + \|W_y\|_{L^2}^2 \leqslant C\left(\varepsilon^2\|W_x\|_{L^2} + \varepsilon^{3/2}\|W_y\|_{L^2}\right). \tag{3.38}$$

Therefore (3.35) holds true. When $H_y = 0$, instead of (3.38) we have

$$\varepsilon \|W_x\|_{L^2}^2 + \|W_y\|_{L^2}^2 \leqslant C\varepsilon^2 \left(\|W_x\|_{L^2} + \|W_y\|_{L^2} \right) \tag{3.39}$$

and (3.36) follows. □

A regular perturbation problem useful in the applications in which the boundary layer does not arise is the following

$$\nabla \cdot \left[H^3(X; \varepsilon) P \nabla P \right] = \left[H(X; \varepsilon) P \right]_x \quad \text{in } \Omega \tag{3.40}$$

$$P = 1 \quad \text{on } \partial\Omega. \tag{3.41}$$

Now Ω is an open, arbitrary and bounded subset of $\mathbf{R}^2$ with a regular boundary $\partial\Omega$. Suppose

$$H(X; \varepsilon) = H_0 + \varepsilon h(X), \quad H_0 > 0.$$

If ε is sufficiently small, $h(X) \in C^1(\overline{\Omega})$ and $h_x < 0$, problem (3.40), (3.41) has one and only one solution $P(X; \varepsilon) > 1$ in Ω ([2], [3]).

Let us consider the asymptotic development

$$P(X; \varepsilon) = P_0 + \varepsilon P_1(X) + \cdots.$$

By direct substitution in (3.40) we find $P_0 = 1$ whereas $P_1(X)$ is given by the solution of problem

$$\nabla \cdot \left(H_0^3 \nabla P_1 \right) = \left(h_x + H_0 P_1 \right)_x \quad \text{in } \Omega, \tag{3.42}$$

$$P_1 = 0 \quad \text{on } \partial\Omega. \tag{3.43}$$

We have the following result of approximation.

THEOREM 3.2. If $P(X; \varepsilon)$ and $P_1(X)$ are respectively the solutions of (3.40), (3.41) and (3.42), (3.43) then

$$\|P - (1 + \varepsilon P_1)\|_{H^1} \leqslant C\varepsilon^2. \tag{3.44}$$

Proof. Let $Q = P - 1$ and $W = Q - \varepsilon P_1$. From (3.40) and (3.42) we get by difference with an easy calculation

$$\|\nabla W\|_{L^2} \leqslant C\left(\varepsilon \|\nabla Q\|_{L^2} + \|Q\|_{L^\infty} \|\nabla Q\|_{L^2} \right).$$

If we prove

$$\|\nabla Q\|_{L^2} \leqslant C\varepsilon, \qquad \|Q\|_{L^\infty} \leqslant C\varepsilon, \tag{3.45}$$

(3.44) will follow. Define $\phi = P^2 - 1$. Recalling that $P > 1$ in Ω, problem (3.40), (3.41) becomes

$$\nabla \cdot \left[(H_0 + \varepsilon h)^3 \nabla \phi \right] = 2\left[(H_0 + \varepsilon h) \sqrt{\phi + 1} \right]_x \quad \text{in } \Omega, \tag{3.46}$$

$$\phi = 0 \quad \text{on } \partial\Omega. \tag{3.47}$$

From (3.46) we deduce $\|\nabla\phi\|_{L^2} \leqslant C$. Hence $2h_x(\phi + 1)^{1/2}$ is bounded in the L^p norm independently of ε. Moreover because $\phi > 0$ in Ω we have $2(H_0 + \varepsilon h)/(1 + \phi)^{1/2} \leqslant C$ where again C does not depend on ε. Thus by a well-known result (see [6] and [7], page

203) the solution to problem (3.46), (3.47) satisfies

$$\|\phi\|_{H^{2,p}} \leqslant C\varepsilon. \tag{3.48}$$

From (3.48) we get easily (3.45). This completes the proof. □

4. Existence and uniqueness for a new boundary value problem of the theory of lubrication. In an interesting paper [9] A. B. Tayler introduces a modified Reynolds equation to describe the impregnation of a sheet of absorbent material (usually paper) with a viscous fluid. The paper is fed through a couple of rollers rotating in opposite sense using the fluid as a lubricant. A rapid impregnation is produced by the high pressure developed in the gap.

Since the paper is compressible, the film thickness is no longer a given function, but depends on the pressure. Tayler proposes the simple relation

$$H(X, P) = h(X) + cP(X), \tag{4.1}$$

where c is an elastic constant. Moreover the paper is porous. Thus a new term of the form kP (k a porosity coefficient) must be added on the right-hand side of equation (1.1). We obtain a boundary value problem for the modified Reynolds equation which, apart from inessential constants, reads

$$\nabla \cdot \left[(h + P)^3 \nabla P\right] = (h + P)_x + P \quad \text{in } \Omega, \tag{4.2}$$

$$P = 0 \quad \text{on } \partial\Omega. \tag{4.3}$$

Problem (4.2), (4.3) does not seem to have been studied before.

LEMMA 4.1. Let $h(x) \in C^2(\bar{\Omega})$, $h(X) \geqslant h_0 > 0$ and $h_x \leqslant 0$ in Ω. If $P(X) \in C^2(\bar{\Omega})$ is a solution to problem (4.2), (4.3) then

$$P(X) \geqslant 0 \quad \text{on } \bar{\Omega}. \tag{4.4}$$

Proof. We first prove the weaker result $P + h > 0$ in $\bar{\Omega}$. Define $\phi(X) = P(X) + h(X)$ and suppose by contradiction $\phi(\bar{X}) = 0$. Since $\phi = h$ on $\partial\Omega$, we have $\bar{X} = (\bar{x}, \bar{y}) \in \Omega$. Let $\bar{X}$ be such that $\phi(X) > 0$ when $X \in A = \{(x, y) \in \bar{\Omega},\ x^* \leqslant x < \bar{x},\ y = \bar{y}\}$ with $(x^*, \bar{y}) \in \partial\Omega$. If $X \in A$, we can write

$$\phi^2 \Delta P + 3\phi \nabla\phi \cdot \nabla P - P/\phi = \phi_x/\phi.$$

Near $\bar{X}$ we have $P(x, \bar{y}) < 0$ because $P(\bar{X}) = -h(\bar{X}) < 0$. Hence recalling $P \in C^2(\bar{\Omega})$ we get

$$\liminf_{X \in A,\, X \to \bar{X}^-} \left(\phi^2 \Delta P + 3\phi\nabla\phi \cdot \nabla P - P/\phi\right) > -\infty.$$

On the other hand since $\log \phi(x, \bar{y}) \to -\infty$ when $x \to \bar{x}^-$ we have

$$\liminf_{x \to \bar{x}^-} \frac{\phi_x(x, \bar{y})}{\phi(x, \bar{y})} = \liminf_{x \to \bar{x}} (\log \phi(x, \bar{y}))_x = -\infty.$$

It follows

$$P + h > 0 \quad \text{in } \bar{\Omega}. \tag{4.5}$$

Suppose now $P(\tilde{X}) = \inf_\Omega P < 0$, $\tilde{X} \in \Omega$. Setting $X = \tilde{X}$ in (4.2) we get, since $h_x \leqslant 0$ and $\nabla P(\tilde{X}) = 0$,

$$(h + P)^3 \Delta P = h_x + P < 0.$$

But by (4.5) we have $(h + P)^3 \Delta P \geqslant 0$. This contradiction completes the proof. □

Our result of existence and uniqueness is stated in the following

THEOREM 4.1. If $h(x)$ satisfies the hypotheses of Lemma 4.1, there exists one and only one solution $P(X) \in C^{2,\alpha}(\bar{\Omega})$ of problem (4.2), (4.3).

Proof. Uniqueness. Suppose $P_1(X)$ and $P_2(X)$ are both C^2-solutions of (4.2), (4.3). Define $W_i = (P_i + h)^4$, $i = 1, 2$. We have from (4.2)

$$\Delta W_i = 4\nabla \cdot \left(W_i^{3/4} \nabla h\right) + \left(W_i^{1/4}\right)_x + W_i^{1/4} - h \quad \text{in } \Omega, \tag{4.6}$$

$$W_i = h^4 \quad \text{on } \partial\Omega. \tag{4.7}$$

From (4.6) we obtain by difference and integration by parts

$$\begin{aligned} &\int_\Omega \nabla(W_1 - W_2) \cdot \nabla\phi \, dX + 4\int_\Omega \left(W_1^{1/4} - W_2^{1/4}\right)\phi \, dX \\ &\quad = 4\int_\Omega \left(W_1^{3/4} - W_2^{3/4}\right)\nabla h \cdot \nabla\phi \, dX + 4\int_\Omega \left(W_1^{1/4} - W_2^{1/4}\right)\phi_x \, dX, \end{aligned} \tag{4.8}$$

for all $\phi \in H_0^1(\bar{\Omega})$. Let $W = W_1 - W_2$ and set $\phi = (W - \varepsilon)^+/W$, $\varepsilon > 0$ in (4.8). We get

$$\begin{aligned} &\varepsilon\int_\Lambda \frac{|\nabla W|^2}{W^2} dX + 4\int_\Lambda \left(W_1^{1/4} - W_2^{1/4}\right) dX \\ &\quad = 4\varepsilon\int_\Lambda \frac{W_1^{1/4} - W_2^{1/4}}{W} dX \\ &\qquad + 4\varepsilon\int_\Lambda \left(W_1^{3/4} - W_2^{3/4}\right)\nabla h \cdot \frac{\nabla W}{W^2} dX + 4\varepsilon\int_\Lambda \left(W_1^{1/4} - W_2^{1/4}\right)\frac{W_x}{W^2} dX, \end{aligned} \tag{4.9}$$

where $\Lambda = \{X \in \Omega;\ W > \varepsilon\}$. In Λ we have by Lemma 4.1, $W_1 > W_2 \geqslant h_0^4$. Applying the mean-value theorem it follows from (4.9)

$$\begin{aligned} &\varepsilon\int_\Lambda \frac{|\nabla W|^2}{W^2} dX + \int_\Lambda \frac{W}{\xi^{3/4}} dX \\ &\quad = \varepsilon\int_\Lambda \frac{dX}{\xi^{3/4}} + 3\varepsilon\int_\Lambda \frac{\nabla h}{\eta^{1/4}} \cdot \frac{\nabla W}{W} dX + \varepsilon\int_\Lambda \frac{W_x}{\xi^{3/4} W} dX. \end{aligned} \tag{4.10}$$

Since $\xi \geqslant h_0$, $\eta \geqslant h_0$ and

$$\int_\Lambda \frac{W}{\xi^{3/4}} dX > 0, \qquad \int_\Lambda \frac{dX}{\xi^{3/4}} < C,$$

we get from (4.10), using the Cauchy–Schwartz inequality

$$\int_\Lambda \frac{|\nabla W|^2}{W^2} dX \leqslant C\left[\left(\int_\Lambda \frac{|\nabla W|^2}{W^2} dX\right)^{1/2} + 1\right]$$

where the constant C does not depend on ε. It follows

$$\int_\Lambda \frac{|\nabla W|^2}{W^2}\,dX \leqslant C. \tag{4.11}$$

By the Poincaré inequality we have

$$\int_\Lambda \frac{|\nabla W|^2}{W^2}\,dX = \int_\Lambda \left|\nabla \log \frac{W}{\varepsilon}\right|^2 dX \geqslant C\int_\Lambda \left|\log\left(1 + \frac{(W-\varepsilon)^+}{\varepsilon}\right)\right|^2 dX.$$

Combine this and the previous inequality and then let $\varepsilon \to 0$. If Λ has a positive measure when $\varepsilon = 0$, we arrive to a contradiction with (4.11); thus $W^+ = 0$. In a similar way we prove $W^- = 0$.

Existence. Define the following sequence of linear problems: $P_0 = 0$,

$$P \in H_0^1(\Omega), \qquad \nabla \cdot \left[(h + P_{n-1})^3 \nabla P_n\right] = (h + P_n)_x + P_n \quad \text{in } \Omega. \tag{4.12}$$

By the results of the linear theory, $\{P_n(X)\}$ is well-defined. Moreover $P_n(X) > 0$ in Ω by the maximum principle since $h_x \leqslant 0$. According to a theorem of G. Stampacchia (see [8] and [6]) the following estimate holds true

$$\|P_n\|_{H^{1,p}} \leqslant C, \quad (1 < p < \infty) \tag{4.13}$$

where the constant C depends only on the lower bound of $(h + P_{n-1})^3$, Ω, $h(X)$, and $h_x(X)$. Since

$$(h + P_{n-1})^3 \geqslant h_0^3 > 0,$$

we get from (4.13)

$$\|P_n\|_{L^\infty} \leqslant C,$$

where C does not depend on n. Hence $(h + P_{n-1})^3$ is also bounded from above independently of n. We infer, using the usual L^p estimate [7],

$$\|P_n\|_{H^{2,p}} \leqslant C.$$

Then we can extract from $\{P_n\}$ a subsequence weakly convergent to a function $P(X) \in H^{2,p}(\Omega)$. Passing to the limit for $n \to \infty$ in (4.12) we complete the proof. □

References

[1] A. Cameron, *Principles of lubrication*, Longmans Ed., London (1966)

[2] G. Cimatti, *Nonlinear aspects of the theory of lubrication*, Rend. Mat. 3, 399–412 (1983)

[3] G. Cimatti, *On certain nonlinear problems arising in the theory of lubrication*, Appl. Math. Optim. **11**, 227–245 (1984)

[4] R. C. Diprima and J. J. Shepherd, *Asymptotic analysis of a finite gas slider bearing of narrow geometry*, J. Lubrication Technology **105**, 491–495 (1983)

[5] A. W. Gross, *Fluid film lubrication*, John Wiley Ed., New York (1980)

[6] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York (1980)

[7] O. A. Ladyzhenskaya and N. N. Ural'tszva, *Linear and quasilinear elliptic equations*, Academic Press, New York (1968)

[8] G. Stampacchia, *Seminaire sur les equations aux derivées partielles*, College de France (1964)

[9] A. B. Tayler, *Absorbent compressible paper*, Quart. J. Mech. Appl. Math. **4**, 481–495 (1978)