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UNSYMMETRIC DEFORMATION OF THE CIRCULAR MEMBRANE*

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Abstract. If the Föppl equations for the deformation of a plane circular membrane under normal pressure are linearized about the radially symmetric solution, it is shown that the resulting linear theory has infinitely many nontrivial angle-dependent solutions. If the prescribed normal pressure and the prescribed (angle-independent) boundary stress are allowed to approach zero in the appropriate way, these nontrivial solutions are retained. Thus the linear theory indicates that, in addition to the solution with radial symmetry, there are infinitely many angle-dependent solutions for arbitrarily small values of the prescribed pressure and prescribed boundary stress.

1. Introduction. In 1907 Föppl [1] suggested an approximate theory for the deformation of plane membranes under normal pressure. In polar coordinates these equations can be written

$$\begin{aligned}\frac{\phi_{\theta\theta}w_{rr} - 2\phi_{\theta r}w_{\theta r} + \phi_{rr}w_{\theta\theta}}{r^2} + \frac{\phi_r w_{rr}}{r} + \frac{w_r\phi_{rr}}{r}& \\ + \frac{2\phi_\theta w_{\theta r}}{r^3} + \frac{2w_\theta\phi_{\theta r}}{r^3} - \frac{2\phi_\theta w_\theta}{r^4} &= -\frac{P}{h},\end{aligned} \tag{1.1a}$$

$$\Delta^2\phi = E\left[\frac{w_{\theta r}^2 - w_{\theta\theta}w_{rr}}{r^2} - \frac{w_r w_{rr}}{r} - 2\frac{w_\theta w_{\theta r}}{r^2} + \frac{w_\theta^2}{r^4}\right], \tag{1.1b}$$

where ϕ is the Airy stress function, w is the normal displacement, P is the normal pressure, h is the thickness of the membrane, and E is the Young's modulus. The independent variables r and θ are the radial and angular variables. The boundary conditions on (1.1) are

$$w_r(0,\theta) = w(1,\theta) = 0, \tag{1.2}$$

$$\phi_r(0,\theta) = \frac{\partial}{\partial r}\Delta\phi\Big|_{r=0} = 0, \tag{1.3a}$$

$$\phi(1,\theta) = 0, \phi_r(1,\theta) = S, \tag{1.3b}$$

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where

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

S is a prescribed radial stress at the boundary (we assume the radius of the undeformed membrane is $r = 1$). The radial stress and circumferential stress are related to Φ by (cf. [2])

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r}, \qquad \sigma_\theta = \frac{\partial^2\phi}{\partial r^2}.$$

If P is independent of θ it makes sense to look for solutions which are independent of θ. If ϕ and w are independent of θ the Eqs. (1.1) reduce to

$$\frac{d}{dr}\left(\frac{d\phi}{dr}\frac{dw}{dr}\right) = -\frac{Pr}{h}, \tag{1.4a}$$

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\right)\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\right)\phi = \frac{-E}{r}\frac{dw}{dr}\frac{d^2w}{dr^2}. \tag{1.4b}$$

In the constant pressure case this problem has been studied in detail (cf. [3, 4, 5, 6]). In these papers it is shown that there is a solution which is independent of θ when $S > 0$ and in fact there are θ independent solutions for some values of $S < 0$ (cf. [6]). Even so it seems plausible to expect that in some circumstances there may also be solutions which depend on θ. Indeed it is likely that in the presence of circumferential compressions the solution which is independent of θ may be physically unstable ([cf. [4, 11]).

2. A special solution. Although the constant pressure problem has been treated in some detail and good qualitative and quantitative information on the solution is available (cf. [4]) an explicit solution is not known. We will consider the zero pressure case as a limiting situation and there is no apparent reason to begin with the constant pressure state. It is more convenient to begin with a problem which we can solve explicitly.

It is a consequence of (1.4a) and the boundary conditions that

$$w_r = -\frac{1}{h\phi_r}\int_0^r tP(t)\,dt. \tag{2.1}$$

In addition (1.4b) can be integrated once to obtain

$$r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{d\phi}{dr} = -\frac{E}{2}w_r^2. \tag{2.2}$$

Introducing the radial stress σ_r as the dependent variable (cf. (1.5)), (2.1) and (2.2) can be written

$$\frac{dw}{dr} = -\frac{1}{hr\sigma_r}\int_0^r tP(t)\,dt, \tag{2.3}$$

$$\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r^2\sigma_r = -\frac{E}{2r}\left(\frac{dw}{dr}\right)^2. \tag{2.4}$$

It is easily verified that the following functions are solutions of (2.3) and (2.4),

$$\sigma_r = A^2 - B^2 r^2, \tag{2.5}$$

$$\sigma_\theta = A^2 - 3B^2 r^2, \tag{2.6}$$

$$w = \frac{2B}{\sqrt{E}}(1 - r^2), \tag{2.7}$$

if

$$P = \frac{8Bh}{\sqrt{E}}(A^2 - 2B^2 r^2), \tag{2.8}$$

where A and B are arbitrary constants. Since we would like the radial stress to be nonnegative throughout the membrane we will require that

$$A^2 - B^2 > 0. \tag{2.9}$$

The circumferential stress may be compressive in a portion of the membrane depending on the relative values of A and B. The boundary condition (1.3b) requires that

$$A^2 - B^2 = S. \tag{2.10}$$

3. The linearized problem. In order to determine whether θ dependent solutions bifurcate from the exact solution described in §2, the Eqs. (1.1) will be linearized about the solutions (2.5) and (2.7). The function ϕ is determined from (2.5) to be

$$\phi = \frac{A^2 r^2}{2} - \frac{B^2 r^4}{4}. \tag{3.1}$$

We look for solutions of (1.1) of the form

$$\phi(r, \theta) = \frac{A^2 r^2}{2} - \frac{B^2 r^4}{4} + \varepsilon\psi(r, \theta), \tag{3.2a}$$

$$w(r, \theta) = \frac{2B}{\sqrt{E}}(1 - r^2) + \varepsilon\omega(r, \theta). \tag{3.2b}$$

ε is to be a small parameter and we shall only keep terms which are linear in ε. The boundary conditions on $\psi(r, \theta)$ and $\omega(r, \theta)$ are

$$\omega_r(0, \theta) = \omega(1, \theta) = 0, \tag{3.3a}$$

$$\psi_r(0, \theta) = \frac{\partial}{\partial r}\Delta\psi(0, \theta) = 0, \tag{3.3b}$$

$$\psi(1, \theta) = \psi_r(1, \theta) = 0. \tag{3.3c}$$

Placing (3.2) in Eqs. (1.1) and keeping only the linear terms, it may be shown that the linearized equations for $\phi(r, \theta)$ and $\omega(r, \theta)$ are

$$(A^2 - B^2 r^2)\omega_{rr} + (A^2 - 3B^2 r^2)\frac{\omega_r}{r} + (A^2 - 3B^2 r^2)\frac{\omega_{\theta\theta}}{r^2} = \frac{4B}{\sqrt{E}}\left(\psi_{rr} + \frac{\psi_r}{r} + \frac{\psi_{\theta\theta}}{r^2}\right), \tag{3.4a}$$

$$\Delta^2\psi = 4B\sqrt{E}\left(\omega_{rr} + \frac{\omega_r}{r} + \frac{\omega_{\theta\theta}}{r^2}\right). \tag{3.4b}$$

The θ dependence in Eqs. (3.4) may be removed by assuming solutions of the form

$$\omega(r,\theta) = \omega(r)\cos n\theta, \qquad \psi(r,\theta) = \psi(r)\cos n\theta. \tag{3.5}$$

The Eqs. (3.4) become ordinary differential equations for the determination of $\omega(r)$ and $\psi(r)$. In particular

$$(A^2 - B^2r^2)\omega'' + (A^2 - 3B^2r^2)\frac{\omega'}{r} - n^2(A^2 - 3B^2r^2)\frac{\omega}{r^2} \tag{3.6a}$$

$$= \frac{4B}{\sqrt{E}}\left(\psi'' + \frac{\psi'}{r} - n\frac{2\psi}{r^2}\right),$$

$$\Delta_n^2\psi = 4B\sqrt{E}\left(\omega'' + \frac{\omega'}{r} - n^2\frac{\omega}{r^2}\right) \tag{3.6b}$$

where $\omega' = d/dr$ and

$$\Delta_n = \frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{n^2}{r^2}. \tag{3.7}$$

The boundary conditions on $\omega(r)$ and $\psi(r)$ are

$$\omega'(0) = \omega(1) = 0, \tag{3.8a}$$

$$\psi'(0) = \frac{d}{dr}\Delta_n\psi\ \Big|_{r=0} = 0, \tag{3.8b}$$

$$\psi(1) = \psi'(1) = 0. \tag{3.8c}$$

The Eqs. (3.6) can be reduced to a single equation for the determination of ω. The Eqs. (3.6) may be rewritten

$$A^2\Delta_n\omega - B^2rU2\left(\omega'' + \frac{3\omega'}{r} - 3n\frac{2\omega}{r^2}\right) = \frac{4B}{\sqrt{E}}\Delta_n\psi, \tag{3.9a}$$

$$\Delta_n^2\psi = 4B\sqrt{E}\,\Delta_n\omega. \tag{3.9b}$$

Equation (3.9b) may be simplified by noting that

$$\Delta_n\left(\Delta_n\psi - 4B\sqrt{E}\,\omega\right) = 0 \tag{3.10}$$

so that (cf. (3.7))

$$\Delta_n\psi = 4B\sqrt{E}\,\omega + C_1r^n + C_2/r^n. \tag{3.11}$$

The boundary condition (3.8) implies $C_2 = 0$. It is also the case that if $n = 1$ then $C_1 = 0$ (cf. (3.8)). However, as we shall see below, $n = 1$ is not a possibility.

It is convenient to solve (3.11) for ψ. This is easily done by introducing the Green's function

$$g(r,\tau) = \begin{cases} \dfrac{1}{2n}\tau^{n+1}(r^n - r^{-n}), & 0 \leqslant \tau \leqslant r, \\ \dfrac{1}{2n}r^n(\tau^{n+1} - \tau^{-n=1}), & r \leqslant \tau \leqslant 1, \end{cases}$$

so that (3.11) can be rewritten

$$\psi(r) = \int_0^1 g(r, \tau)\left(4B\sqrt{E}\,\omega(\tau) + C_1\tau^n\right) d\tau. \tag{3.13}$$

The function $\psi(r)$ satisfies the conditions $\psi'(0) = \psi(1) = 0$. The last condition $\psi'(1) = 0$ requires

$$\int_0^1 \tau^{n+1}\left(4B\sqrt{E}\,\omega(\tau) + C_1\tau^n\right) d\tau = 0. \tag{3.14}$$

This result is obtained by differentiating (3.13) and evaluating the result at $r = 1$. In any case (3.14) implies that

$$C_1 = -8B\sqrt{E}(n+1)\int_0^1 \tau^{n+1}\omega(\tau)\, d\tau. \tag{3.15}$$

The Eqs. (3.9) can be reduced to a single equation for the determination of ω by combining (3.9a) and (3.11) to find

$$A^2\Delta_n\omega - B^2r^2\left(\omega'' + \frac{3\omega'}{r} - 3n^2\omega/r^2\right) = 16B^2\omega + \frac{4BC_1}{\sqrt{E}}r^n$$

where C_1 is given by (3.15) and ω is to satisfy the boundary conditions (3.8a).

The parameters A and B occurring in (3.16) can be removed by introducing the change of dependent variable

$$\rho = B_r/A. \tag{3.17}$$

Equation (3.16) becomes

$$(1-\rho^2)\ddot{\omega} + (1-3\rho^2)\frac{\dot{\omega}}{\rho} - \left(n^2 + (16-3n^2)\rho^2\right)\frac{\omega}{\rho^2} = K_1\rho^n \tag{3.18}$$

$(\cdot = d/d\rho)$ where

$$K_1 = \frac{4B}{\sqrt{E}}\frac{A^n}{B^{n+2}}C_1. \tag{3.19}$$

Combining (3.19) with (3.15) we find

$$K_1 = -32(n+1)\frac{A^{2n+2}}{B^{2n+2}}\int_0^{B/A} \tau^{n+1}\omega(\tau)\, d\tau \tag{3.20}$$

where $\omega(\rho)$ is the solution of (3.18). The boundary conditions on (3.18) are

$$\dot{\omega}(0) = \omega(B/A) = 0. \tag{3.21}$$

If $n = 1$, so that $K_1 = 0$ (cf. (3.19)), it is easily verified that the only solution of (3.18) satisfying (3.21) is $\omega \equiv 0$. Thus $n = 1$ is not a possibility.

4. Solution of the linearized problem. It remains to decide whether Eq. (3.18) with boundary conditions (3.21) has nontrivial solutions. For this purpose it is convenient to transform the equation into a more tractable form. Introduce the change of variable

$$t = \rho^2. \tag{4.1}$$

Equation (3.18) becomes

$$t(1-t)\omega'' + (1-2t)\omega' - \left(\frac{n^2}{4} + \frac{16-3n^2}{4}t\right)\frac{\omega}{t} = \frac{K_1}{4}t^{n/2} \tag{4.2}$$

($' = d/dt$). In terms of the new independent variable, K_1 becomes

$$K_1 = -16(n+1)\left(\frac{A}{B}\right)^{2n+2}\int_0^{(B/A)^2} \tau^{n/2}\omega(\tau)\,d\tau, \tag{4.3}$$

where $\omega(t)$ is the solution of (4.2). The boundary conditions on (4.2) are

$$\omega'(0) = \omega\big((B/A)^2\big) = 0. \tag{4.4}$$

In order to further simplify our problem, introduce the change of dependent variable

$$\omega = t^{n/2}V. \tag{4.5}$$

We obtain a new equation for $V(t)$,

$$t(1-t)V'' + ((n+1)-(n+2)t)V' + \frac{n^2-n-8}{2}V = \frac{K_1}{4}, \tag{4.6}$$

where

$$K_1 = -16(n+1)\left(\frac{A}{B}\right)^{2n+2}\int_0^{(B/A)^2} \tau^n V(\tau)\,d\tau \tag{4.7}$$

and the boundary conditions are: $V(t)$ should be bounded at $t = 0$ and

$$V\big((B/A)^2\big) = 0. \tag{4.8}$$

The solution of (4.6) which is bounded at $t = 0$ is

$$V(t) = CF(\alpha, \beta, \gamma, t) + \frac{K_1}{2(n^2-n-8)}, \tag{4.9}$$

where $F(\alpha, \beta, \gamma, t)$ is the hypergeometric function (cf. [7]) with

$$\alpha = \frac{n+1+\sqrt{3}\sqrt{n^2-5}}{2}, \quad \beta = \frac{n+1-\sqrt{3}\sqrt{n^2-5}}{2}, \quad \gamma = n+1 \tag{4.10}$$

and C and K_1 are to be determined from (4.7) and (4.8). In the case $n = 2$ the parameters α and β in (4.10) are complex. However, since they are conjugates of each other, $F(\alpha, \beta, \gamma, t)$ is real even in this case (cf. [7]). For some purposes it is convenient to rewrite (4.6) in the form

$$\frac{d}{dt}\left(t^{n+1} - t^{n+2}\right)\frac{dV}{dt} + \frac{n^2-n-8}{2}t^n V = t^n K_1/4. \tag{4.11}$$

In order to determine K_1 multiply (4.9) by t^n and integrate from $t = 0$ to $t = (B/A)^2$. Thus

$$\int_0^{(B/A)^2} t^n V\,dt = C\int_0^{(B/A)^2} t^n F(\alpha, \beta, \gamma, t)\,dt - \frac{8}{n^2-n-8}\int_0^{(B/A)^2} t^n V\,dt. \tag{4.12}$$

Equivalently

$$\int_0^{(B/A)^2} t^n V(t)\,dt = \frac{C(n^2-n-8)}{n(n-1)} \int_0^{(B/A)^2} t^n(F(\alpha,\beta,\gamma,t))\,dt \tag{4.13}$$

so that

$$K_1 = \frac{-16C(n+1)(n^2-n-8)}{n(n-1)}\left(\frac{A}{B}\right)^{2n+2} \int_0^{(B/A)^2} t^n F(\alpha,\beta,\gamma,t)\,dt. \tag{4.14}$$

Combining (4.14) with (4.9) we find

$$V(t) = CF(\alpha,\beta,\gamma,t) - \frac{8C(n+1)}{n(n-1)}\left(\frac{A}{B}\right)^{2n+2} \int_0^{(B/A)^2} \tau^n(\alpha,\beta,\gamma,t)\,d\tau. \tag{4.15}$$

It is a consequence of (4.15) that Eq. (3.18) with boundary conditions (3.21) has a nontrivial solution iff

$$F\left(\alpha,\beta,\gamma,(B/A)^2\right) - \frac{8(n+1)}{n(n-1)}\left(\frac{A}{B}\right)^{2n+2} \int_0^{(B/A)^2} \tau^n F(\alpha,\beta,\gamma,\tau)\,d\tau = 0. \tag{4.16}$$

The relation (4.16) may be simplified by noting that since $F(\alpha, \beta, \gamma, t)$ is the solution of

$$\frac{d}{dt}\left(t^{n+1} - t^{n+2}\right)\frac{du}{dt} + \frac{n^2-n-8}{2} t^n u = 0 \tag{4.17}$$

satisfying $u(0) = 1$ (u analytic at $t = 0$), we find by integrating from $t = 0$ to $t = (B/A)^2$ that

$$\int_0^{(B/A)^2} t^n F(\alpha,\beta,\gamma,\tau)\,d\tau = \frac{-2}{n^2-n-8}\left(\frac{B}{A}\right)^{2n+2}\left(1-(B/A)^2\right)F'\left(\alpha,\beta,\gamma,(B/A)^2\right). \tag{4.18}$$

Thus (4.16) can be written

$$H_n\left((B/A)^2\right) = 1 + \frac{16(n+1)}{n(n-1)(n^2-n-8)}\left(1-(B/A)^2\right)\frac{F'\left(\alpha,\beta,\gamma,(B/A)^2\right)}{F\left(\alpha,\beta,\gamma,(B/A)^2\right)} = 0. \tag{4.19}$$

It remains to decide whether Eq. (4.19) has solutions for $0 < B/A < 1$.

The function $H_n(t)$ can be evaluated explicitly at $t = 0$ and $t = 1$. $F(\alpha, \beta, \gamma, t)$ is an analytic function of t for $|t| < 1$ and $F(\alpha, \beta, \gamma, 0) = 1$. Therefore Eq. (4.17) implies that

$$F'(\alpha,\beta,\gamma,0) = \frac{n^2-n-8}{2(n+1)} \tag{4.20}$$

so that

$$\lim_{t\to 0} H_n(t) = 1 - \frac{8}{n(n-1)}. \tag{4.21}$$

In order to study the behavior of $H_n(t)$ as $t \to 1$ we note that Eq. (4.17) can be written

$$t(1-t)u'' + ((n+1)-(n+2)t)u' + \frac{n^2-n-8}{2}u = 0. \tag{4.22}$$

The equation has a regular singular point (cf. [8]) at $t = 1$ and the indicial equation at $t = 1$ is simply

$$r^2 = 0. \tag{4.23}$$

Thus the general solution of (4.22) valid for $0 < 1 - t < 1$ is

$$u(t) = C_1 u_1(t) + C_2 u_2(t), \tag{4.24}$$

where

$$u_1(t) = 1 + \sum_{j=1}^{\infty} a_j (t-1)^j, \tag{4.25a}$$

$$u_2(t) = u_1(t)\ln(1-t) + \sum_{j=1}^{\infty} b_j (t-1)^j. \tag{4.25b}$$

The constants are determined from the fact that $u(t) = F(\alpha, \beta, \beta, t)$. However, as we shall see, an explicit value of C_1 and C_2 will not be required for our purposes. In any case

$$u_1(t) = 1 + O(|t-1|), \tag{4.26a}$$

$$u_2(t) = \ln(1-t) + O(|t-1|), \tag{4.26b}$$

so that

$$u(t) = C_1 + C_2 \ln(1-t) + O(|t-1|). \tag{4.27}$$

The functions $u_1(t)$ and $u_2(t)$ may be differentiated to show

$$u_1'(t) = a_1 + O(|t-1|), \tag{4.28a}$$

$$u_2'(t) = \frac{1}{t-1} + a_1 \ln(1-t) + b_1 + O(|t-1|). \tag{4.28b}$$

We conclude that

$$(1-t)u'(t) = -1 + O(|t-1|). \tag{4.29}$$

Equivalently

$$\lim_{t \to 1} \frac{(1-t)u'(t)}{u(t)} = 0, \tag{4.30}$$

so that

$$\lim_{t \to 1} H_n(t) = 1. \tag{4.31}$$

It is a consequence of (4.21) that $H_n(0) < 0$ when $n = 2$ or $n = 3$. This fact combined with (4.31) implies that (4.19) has at least one solution when $n = 2$ or $n = 3$. Equation (4.19) was solved numerically ($F(\alpha, \beta, \gamma, t)$ was obtained by solving Eq. (4.17) using a finite difference scheme). The numerical results indicate that in both cases there is exactly one solution. In particular the solution of $H_2(t) = 0$ is $t \approx .99$ and $H_3(t) = 0$ has the solution $t \approx .95$.

If $n \geqslant 4$ then $H_n(0) > 0$ and the above argument cannot be used. Nonetheless it is possible to estimate the number of zeros of $H_n(t)$. Define a function $E(q)$ to be the integer part of q if $q > 0$ and $E(q) = 0$ if $q \leqslant 0$.

THEOREM. If $n \geqslant 4$ the function $H_n(t)$ has at least

$$E\left(\left(\sqrt{3}\sqrt{n^2-5}-n+1\right)/2\right)$$

zeros in the interval $0 < t < 1$.

Proof. $H_n(t)$ has the same zeros as

$$J_n(t) = F(\alpha,\beta,\gamma,t) + \frac{16(n+1)(1-t)}{n(n-1)(n^2-n-8)}F'(\alpha,\beta,\gamma,t) \tag{4.32}$$

(cf. (4.19)). Assume $F(\alpha,\beta,\gamma,t)$ has zeros at $0 < t_1 < t_2 < \cdots < t_j \leqslant 1$. $F'(\alpha,\beta,\gamma,t)$ alternates sign at successive zeros of $F(\alpha,\beta,\gamma,t)$. It follows that $J_n(t_1) < 0$, $J_n(t_2) > 0, \ldots$. Therefore $J_n(t)$ has at least one zero between each zero of $F(\alpha,\beta,\gamma,t)$. In addition, since $J_n(0) > 0$ for $n \geqslant 4$ (cf. (4.20)), there is at least one zero between $t = 0$ and $t = t_1$. In any case there are at least as many solutions of $H_n(t) = 0$ as there are zeros of $F(\alpha,\beta,\gamma,t)$ in the interval $0 < t \leqslant 1$. The number of zeros of $F(\alpha,\beta,\gamma,t)$ in the interval has been found by Klein [9] and Van Vleck [10]. We quote the result in the form obtained by Van Vleck.

THEOREM. Assume α,β,γ are real and $1-\gamma < 0$. Let $\lambda_1 = |1-\gamma|$, $\lambda_2 = |\gamma-\alpha-\beta|$, $\lambda_3 = |\alpha-\beta|$. The number of zeros of $F(\alpha,\beta,\gamma,t)$ between 0 and 1 inclusive is $E((\lambda_3 - \lambda_2 - \lambda_1 + 1)/2)$.

Combining this result with (4.10) we obtain the result. Q.E.D.

It is a consequence of the theorem that $H_4(t)$ and $H_5(t)$ have at least one zero and $H_6(t)$ and $H_7(t)$ have at least two zeros. They were found numerically to be $t \approx .89$ when $n = 4$, $t \approx .82$ when $n = 5$, $t \approx .76$ and $t \approx .99$ when $n = 6$, and $t \approx .72$ and $t \approx .95$ when $n = 7$. In any case the above results guarantee that Eq. (3.18) with boundary conditions (3.21) has at least one nontrivial solution when $n = 2$ or $n = 3$ and at least $E((\sqrt{3}\sqrt{n^2 = 5} - n + 1)/2)$ nontrivial solutions when $n \geqslant 4$.

Denote the zeros of $H_n(t)$ by h_{nj}. If we consider the situation where $A^2 \to 0$ and $B^2 \to 0$ but $B^2/A^2 = h_{nj}$ we see that there are nontrivial solutions bifurcating from the zero stress solution. There are infinitely many such solutions. We note that when n is sufficiently large ($n \geqslant 6$) there is more than one solution with the same number of ripples (i.e., with the same n). Indeed $\omega(r,\theta)$ may be written

$$\omega(r,\theta) = r^n\left[F\left(\alpha,\beta,\gamma,h_{nj}r^2\right) - K_{nj}\right]\cos n\theta, \tag{4.33}$$

where

$$K_{nj} = \frac{-16(n+1)(n^2-n-8)}{n(n+1)h_{nj}^{n+1}}\int_0^{h_{nj}} \tau^n F(\alpha,\beta,\gamma,\tau)\,d\tau \tag{4.34}$$

for $n \geqslant 2$.

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